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ON EXISTENCE OF BEST PROXIMITY PAIRS AND A GENERALIZATION OF NASH EQUILIBRIUM

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Abstract. We consider a constrained m-person game, in which each player has two strategy spaces and two pay-off functions, namely a manufacturing pay-off function and a selling pay-off function. In this paper, we give sufficient conditions for the existence of an equilibrium pair which minimizes the manufacturing pay-off and maximizes the selling pay-off for each player. To prove the existence of such an equilibrium, we introduce a notion of relatively upper semi-continuous mapping and therein prove the existence of a best proximity pair.

1. Introduction and preliminaries

Nash equilibrium is a fundamental notion and one of the most widely used methods, for predicting the outcome of a strategic interaction in game theory. An n-person game consists of a set of n players with a strategy set and a payoff (or utility) function for each player. The payoff functions represent each player's preference over action profiles, where an action profile is simply a list of actions, one for each player. A *Nash equilibrium* is an event(or an action profile) with the property that no single player can obtain an optimal payoff by deviating unilaterally from this profile.

John Nash in his articles ([6, 7]) defined a mixed-strategy Nash equilibrium for a game with a finite set of actions and therein proved the existence of a (mixed-strategy) Nash equilibrium for such a game. From this standpoint, we would like to consider an economical abstract situation, such as, in an m-person game, let us assume every player has a manufacturing unit with an action/strategy set X_i and a selling unit with an action/strategy set Y_i. Two payoff functions (say, manufacturing profile and selling profile) are associated with each player. Our objective is to find an equilibrium that optimizes both payoff functions in the economical sense. To achieve such an equilibrium, we use the theory of best proximity pairs.

Let A and B be two non-empty subsets of a metric space X. Suppose T is a cyclic map on $A \cup B$ (*i.e.*, $T(A) \subseteq B$ and $T(B) \subseteq A$). A pair (x, Tx) is said to be best proximity pair of T if $d(x, Tx) = \operatorname{dist}(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$. In this article, by introducing a notion of relatively upper semi-continuous mappings, we prove the existence of best proximity pairs for such multivalued mappings. We also establish some properties of such a mapping in the setting of a strictly convex Banach space. Finally we use such best proximity pairs to establish the equilibrium for the considered abstract economics. We give examples to illustrate our results.

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Now we recall the notion of upper semi-continuous mappings and quote few related results, which we use in the sequel.

Definition 1.1 ([12]). Let X be a metric space. A multivalued map T on X is said to be *upper semi-continuous (usc)* if $\{u \in X : T(u) \cap V \neq \emptyset\}$ is closed in X for every closed subset V of X.

A multivalued map T on X is said to have closed graph, if for any sequence $\{x_n\}$ in X that converges to some x and for each $n \in \mathbb{N}$, $u_n \in T(x_n)$ such that $\{u_n\}$ converges to u in X, we have $u \in Tx$. The following lemma is a characterization of upper semicontinuous maps.

Lemma 1.2 ([12]). *Let A and B be non-empty subsets of a metric space X. Suppose B is compact. Then every multivalued map T from A to B is usc if and only if T has closed graph.*

Theorem 1.3 (Fan-Glicksberg fixed point theorem, [12]). *Let K be a non-empty, compact and convex subset of a normed linear space X. Suppose $T : K \to K$ is a closed and convex valued usc mapping. Then T has a fixed point.*

2. Relatively upper semi-continuous mappings

Let (X, d) be a metric space and $A,\ B$ be two non-empty subsets of X. We now fix notations for the proximal subsets of (A, B), $A_0 := \{x \in A : \exists\, y \in B \text{ with } d(x, y) = \text{dist}(A, B)\}$ and $B_0 := \{y \in B : \exists\, x \in A \text{ with } d(x, y) = \text{dist}(A, B)\}$. Suppose T is a cyclic multivalued map on $A \cup B$, that is for $x \in A, T(x) \subseteq B$ and for $y \in B, T(y) \subseteq A$. In this section, we introduce a notion called relatively upper semi-continuous map on $A \cup B$ and establish basic properties of such a map.

Definition 2.1. Let A, B be two non-empty subsets of a metric space (X, d). A multivalued cyclic map T on $A \cup B$ is said to be *relatively upper semi-continuous* (*rusc*) if $\{x_n\}$, $\{y_n\}$ are two sequences in A, B respectively such that $x_n \to x,\ y_n \to y$ with $d(x, y) = \text{dist}(A, B)$, further if $v_n \in T(x_n),\ u_n \in T(y_n)$ for all $n \in \mathbb{N}$ with $u_n \to u$ in $A,\ v_n \to v$ in B, then we have

$$\text{dist}(T(x), u) = \text{dist}(A, B) = \text{dist}(T(y), v).$$

Let A and B be non-empty closed subsets of a metric space. It is to be noted that if $\text{dist}(A, B) = 0$ and either A or B is compact, then every closed valued rusc map is upper semi-continuous. To see this, choose any closed subset V of the compact set $A \cap B$ and set $U := \{u \in A \cap B : T(u) \cap V \neq \emptyset\}$. Suppose $\{x_n\}$ is a sequence in U that converges to x in $A \cap B$. Then there exists $u_n \in T(x_n) \cap V$, for $n \in \mathbb{N}$. As V is compact, there exists a sub-sequence $\{u_{n_k}\}$ of $\{u_n\}$, that converges in V, say to u. As T is rusc, we have $d(T(x), u) = \text{dist}(A, B)$ and hence $u \in T(x)$. Therefore $x \in U$.

Let $(X, \|\cdot\|)$ be a normed linear space and $A,\ B$ be closed convex subsets of X. A projection map onto A, is the map $P_A : B \to A$ defined by, for $y \in B$, $P_A(y) := \{x \in A : \|x - y\| = \text{dist}(y, A) := \inf\{\|a - y\| : a \in A\}$. We first show that the composite map $P_A T$ is a convex valued map on A_0.

Proposition 2.2. *Let A, B be two non-empty closed convex subsets of a normed linear space $(X, \|.\|)$. If T is a closed convex valued rusc mapping on $A \cup B$, then for any $z \in A_0$, $P_A T(z)$ is a convex subset of A_0.*

Proof. For $z \in A_0$, suppose $x, y \in P_A T(z)$ and $\lambda \in [0,1]$. Then there are $u,\ v \in T(z)$ such that $x \in P_A(u)$ and $y \in P_A(v)$. Hence there exists $u' \in A$ such that $d(u, u') = dist(A, B)$. Since Tz is convex, we have $\lambda u + (1-\lambda)v \in T(z)$. Now $\operatorname{dist}(A, B) \leq \|x - u\| = \operatorname{dist}(A, u) \leq \|u' - u\| = \operatorname{dist}(A, B)$. That is $\|x - u\| = \operatorname{dist}(A, B)$. In a similar way, one can prove $\|y - v\| = \operatorname{dist}(A, B)$. Now

$$\begin{aligned}\operatorname{dist}(A, \lambda u + (1-\lambda)v) &\leq \|(\lambda x + (1-\lambda)y) - (\lambda u + (1-\lambda)v)\| \\ &\leq \lambda\|x - u\| + (1-\lambda)\|y - v\| \\ &= \lambda \operatorname{dist}(A, B) + (1-\lambda)\operatorname{dist}(A, B) \\ &= \operatorname{dist}(A, B) \leq \operatorname{dist}(A, \lambda u + (1-\lambda)v).\end{aligned}$$

That is $\lambda x + (1-\lambda)y \in P_A(\lambda u + (1-\lambda)v)$ and hence $P_A T(z)$ is convex. □

A non-empty subset A of a metric space (X, d) is said to be approximatively compact ([2]) if $\{x_n\}$ is a sequence in A and $y \in X$ such that $d(x_n, y)$ converging to $\operatorname{dist}(y, A)$, then $\{x_n\}$ has a convergent subsequence in A. Suppose B is a compact subset of X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B respectively with $d(x_n, y_n) \to \operatorname{dist}(A, B)$. As B is compact, there exists $y \in B$, such that $d(x_n, y) \to \operatorname{dist}(A, B)$. By using approximatively compactness of A, there exists $x \in A$ with $d(x, y) = \operatorname{dist}(A, B)$. Thus we have:

Lemma 2.3. *Let A, B be two non-empty subsets of a metric space X. Suppose B is compact and A is approximatively compact. Then A_0, B_0 are non-empty compact subsets of X.*

Also for $x \in A_0$ and $v \in T(x)$, choose $y \in B_0$ such that $d(x, y) = \operatorname{dist}(A, B)$ and $u \in T(y)$. By fixing $x_n = x,\ y_n = y, u_n = u$ and $v_n = v$, we can conclude $\operatorname{dist}(T(x), u) = \operatorname{dist}(A, B) = \operatorname{dist}(T(y), u)$. So we have:

Lemma 2.4. *Let A, B and X be as in Lemma 2.3. If T is a rusc map on $A \cup B$ then $T(x) \cap B_0$, $T(y) \cap A_0$ are non-empty, for each $x \in A_0$ and $y \in B_0$.*

Thus, if A, B and T are as in Lemma 2.4, then $T\Big|_{A_0 \cup B_0}$ defined by

$$T\Big|_{A_0 \cup B_0}(x) := \begin{cases} T(x) \cap B_0 & \text{if } x \in A_0, \\ T(x) \cap B_0 & \text{if } x \in B_0, \end{cases}$$

is a well-defined rusc map on $A_0 \cup B_0$. Hereafter, without ambiguity we denote, $T\Big|_{A_0 \cup B_0}$ by T, when we consider the map on $A_0 \cup B_0$. Now we prove the following theorem, which will be useful in the sequel.

Theorem 2.5. *Let X be a normed linear space and A, B be as in Lemma 2.3. If T is a closed valued rusc map on $A \cup B$, then $P_A T$ is upper semi-continuous on A_0.*

Proof. Let $\{x_n\}$ be a sequence in A_0 that converges to x in A_0. Suppose $z_n \in P_A T(x_n)$ for each $n \in \mathbb{N}$ and $z_n \to z$. Get y_n in B such that $d(x_n, y_n) = \operatorname{dist}(A, B)$ and $y_n \to y$, for some $y \in B$. Choose $u_n \in T(x_n)$ for all $n \in \mathbb{N}$ with $u_n \to u$.

Set $v_n = z_n$ for all $n \in \mathbb{N}$. As T is rusc, we have $\operatorname{dist}(T(x), z) = \operatorname{dist}(A, B) = \operatorname{dist}(T(y), u)$. Now $\operatorname{dist}(A_0, Tx) \leq \operatorname{dist}(Tx, z) = \operatorname{dist}(A, B) \leq \operatorname{dist}(A_0, Tx)$. As $T(x)$ is compact, there exists a $w \in T(x)$ such that $d(z, w) = \operatorname{dist}(A_0, z)$. Hence $z \in P_A T(x)$. In view of Lemma 1.2, $P_A T$ is upper semi-continuous on A_0. □

We conclude this section by giving a natural example to illustrate the notion of rusc mappings on a normed linear space.

Proposition 2.6. *Let A, B and X be as in Lemma 2.3. Then the projection map P on $A_0 \cup B_0$ is rusc, where*

$$P(x) := \begin{cases} P_B(x) & \text{if } x \in A_0, \\ P_A(x) & \text{if } x \in B_0. \end{cases}$$

Proof. Let $x \in A_0$ and $z \in P(x) = P_B(x)$. Therefore there exists $y \in B_0$ such that $d(x, y) = \operatorname{dist}(A, B)$ and $d(x, z) = \operatorname{dist}(x, B)$. Now $\operatorname{dist}(A, B) \leq d(x, z) = \operatorname{dist}(x, B) \leq d(x, y) = \operatorname{dist}(A, B)$. This implies $z \in B_0$. Hence $P(A_0) \subseteq B_0$. In a similar fashion one can prove that $P(B_0) \subseteq A_0$. Without loss of generality we assume $A_0 = A$ and $B_0 = B$. So for any $u \in A$, $\operatorname{dist}(u, B) = \operatorname{dist}(A, B)$. Suppose $\{x_n\}$ is a sequence in A and $\{y_n\}$ is a sequence in B such that $x_n \to x$, $y_n \to y$ with $d(x, y) = \operatorname{dist}(A, B)$. Further assume that $u_n \in P_B(x_n)$, $v_n \in P_A(y_n)$ for all $n \in \mathbb{N}$ with $u_n \to u$, $v_n \to v$. Therefore $y \in P_B(x)$, $x \in P_A(y)$ and $d(u_n, x_n) = \operatorname{dist}(x_n, B)$, $d(v_n, y_n) = \operatorname{dist}(y_n, A)$. Now, for any $n \in \mathbb{N}$, $\operatorname{dist}(A, B) \leq d(u_n, x_n) = \operatorname{dist}(x_n, B) = \operatorname{dist}(A, B)$. Hence $d(u_n, x_n) = \operatorname{dist}(A, B)$, for all $n \in \mathbb{N}$. In a similar fashion one can have $d(v_n, y_n) = \operatorname{dist}(A, B)$, for all $n \in \mathbb{N}$. Hence $\operatorname{dist}(u, P_A(y)) = d(u, x) = \operatorname{dist}(A, B)$. In a similar fashion one can prove that $\operatorname{dist}(v, P_B(x)) = \operatorname{dist}(A, B)$. Hence P is rusc. □

3. Existence of best proximity pairs

Let A and B be two non-empty subsets of a metric space X. We say that (A, B) is semisharp proximinal pair if for every $x \in A$(respectively in B), there exists atmost one y in B(respectively in A) such that $d(x, y) = \operatorname{dist}(A, B)$. If such a y exists, we denote this y as x'. Hereafter we use the above notation, throughout this manuscript, if it exists. These pairs naturally occur in the case of A and B being closed convex weakly compact subsets of a uniformly convex Banach space ([8, 12]). Now we prove that every rusc map commutes with the projection map P, defined in Proposition 2.6.

Theorem 3.1. *Let (A, B) be a non-empty semisharp proximinal pair in a metric space (X, d). Suppose B is compact and A is approximatively compact. If T is a rusc map on $A \cup B$, then T commutes with P on $A_0 \cup B_0$, where P is the projection map on $A \cup B$.*

Proof. Suppose $w \in A_0$ and $v \in TP(w)$. As (A, B) is semisharp proximinal pair, $v \in T(w')$. For a fixed $u \in T(w)$, set $x_n = w$, $y_n = w'$, $u_n = u$ and $v_n = v$ for all $n \in \mathbb{N}$. As T is rusc, we have $\operatorname{dist}(v, Tw) = \operatorname{dist}(A, B)$. Hence there is a $k \in T(w)$ such that $d(k, v) = \operatorname{dist}(A, B)$. Hence $v' = k$ and so $v' \in T(w)$. Therefore $v = (v')' \in P_A T(w)$. As $v \in TP(w)$, was chosen arbitrarily, we have $TP(w) \subseteq PT(w)$. This completes the proof. □

The following example shows that Theorem 3.1 fails to hold if (A,B) is not a semisharp proximinal pair.

Example 3.2. Let $X = (\mathbb{R}^2, \|.\|_\infty)$. Set $A := \{(0,x) \in X : 0 \leq x \leq 2\}$ and $B := \{(1,x) \in X : 0 \leq x \leq 2\}$. Define T on $A \cup B$ by, for $(a,x) \in A \cup B$,

$$T(a,x) := \begin{cases} \{(1,y) : 0 \leq y \leq x,\ y \in \mathbb{Q}\} & \text{if } a = 0 \text{ and } x \in \mathbb{Q} \\ (1,1) & \text{if } a = 0 \text{ and } x \in \mathbb{Q}^c \\ & \\ \{(0,y) : 0 \leq y \leq x,\ y \in \mathbb{Q}\} & \text{if } a = 1 \text{ and } x \in \mathbb{Q} \\ (0,1) & \text{if } a = 1 \text{ and } x \in \mathbb{Q}^c. \end{cases}$$

It is easy to see that, $A_0 = A$, $B_0 = B$ and $\operatorname{dist}(A,B) = 1$. We first show that T is a rusc map. For this choose any two sequences $\{(0,x_n)\} \in A$ and $\{(1,y_n)\} \in B$, that converge to $(0,x)$ and $(1,y)$ respectively with $d((0,x),(1,y)) = \operatorname{dist}(A,B) = 1$. Then $\{x_n\}$ converges to x and $\{y_n\}$ converges to y and $|x-y| \leq 1$. Suppose $(1,u_n) \in T(0,x_n)$ and $(0,v_n) \in T(1,y_n)$ for $n \in \mathbb{N}$ with

$$\lim_{n\to\infty}(1,u_n) = (1,u) \text{ and } \lim_{n\to\infty}(0,v_n) = (0,v).$$

It is to be noted that, if $\{y_n\}$ contains an irrational subsequence, then $\{v_n\}$ contains a constant sub-sequence $\{1\}$. Hence $v = 1$. In this case we have $\operatorname{dist}(T(0,x),(0,v)) = 1 = \operatorname{dist}(A,B)$. Also if x is irrational, then we have $T(0,x) = (1,1)$. Therefore $\operatorname{dist}(T(0,x),(0,v)) = \|(1,1)-(0,v)\|_\infty = 1 = \operatorname{dist}(A,B)$. Hence, without loss of generality we assume that $\{(0,y_n)\}$ is a rational sequence and x is a rational number. As $(0,v_n) \in T(1,y_n)$, we have $0 \leq v_n \leq y_n$, for $n \in \mathbb{N}$ and so $0 \leq v \leq y$. It is to be observed that, if $x \geq y$, then $(1,v) \in T(0,x)$. Hence $\operatorname{dist}(T(0,x),(0,v)) = 1 = \operatorname{dist}(A,B)$. Also if $x \leq y$, choose $z = \min\{0, v-1\}$. As $|x-y| \leq 1$ and $0 \leq v \leq y$ we have $(1,z) \in T(0,x)$. Hence $\operatorname{dist}(T(0,x),(0,v)) = 1 = \operatorname{dist}(A,B)$. In a similar fashion one can show that $\operatorname{dist}(T(1,y),(1,u)) = 1 = \operatorname{dist}(A,B)$. Therefore T is a rusc map. Now for $(0,1) \in A_0$, we have

$$T(0,1) = \{(1,x) : 0 \leq x \leq 1,\ x \in \mathbb{Q}\}.$$

Hence

$$\begin{aligned} PT(0,1) &= \bigcup_{0\leq x\leq 1,\ x\in\mathbb{Q}} P((1,x) \\ &\supseteq P(1,1) \\ &= \{(0,x) \in A : \|(0,x)-(1,1)\|_\infty = 1\} \\ &= \{(0,x) : 0 \leq x \leq 2\} = A. \end{aligned}$$

Also $PT(0,1) \subseteq A$, so that $PT(0,1) = A$.

On the other hand, $P(0,1) = \{(1,x) : 0 \leq x \leq 2\} = B$. Now

$$\begin{aligned} TP(0,1) &= \cup\{T(1,x) : 0 \leq x \leq 2\} \\ &= \Big[\cup\{T(1,x) : 0 \leq x \leq 2,\ x \in \mathbb{Q}\}\Big] \cup \Big[\cup\{T(1,x) : 0 \leq x \leq 2,\ x \in \mathbb{Q}^c\}\Big] \\ &= \{(0,x) : 0 \leq x \leq 2,\ x \in \mathbb{Q}\} \cup \{(0,1)\} \\ &= \{(0,x) : 0 \leq x \leq 2,\ x \in \mathbb{Q}\} \neq A. \end{aligned}$$

Hence $PT(0,1) \neq TP(0,1)$.

The following theorem guarantees the existence of a best proximity pair.

Theorem 3.3. *Let A, B and X be as in Theorem 3.1. If T is a closed convex valued rusc map on $A \cup B$, then there exist x in A, y in B such that $dist(x, Tx) = dist(A, B) = dist(y, Ty)$ and $d(x, y) = dist(A, B)$.*

Proof. By Theorem 2.5, the multivalued map $P_A T : A \to A$ is upper semicontinuous. Hence by Theorem 1.3, there exists an element $x \in A$ such that $x \in P_A T(x)$. Since $T(x)$ compact, there exists an element $u \in T(x)$ such that $x \in P_A(u)$. Hence $\|x - u\| = \operatorname{dist}(A, B)$. Therefore $\operatorname{dist}(x, Tx) = \operatorname{dist}(A, B)$. Also by Theorem 3.1, we get $x \in T(P_A(x))$. By setting $y = P_A(x)$, we have $\operatorname{dist}(y, Ty) = \operatorname{dist}(A, B)$. This completes the proof. □

As a consequence of the above theorem we get the following:

Corollary 3.4. *Let A and B be two non-empty convex subsets of a strictly convex Banach space X. Suppose B is compact and A is approximatively compact. If T is a closed convex valued rusc mapping on $A \cup B$, then there exist x in A, y in B such that $dist(x, Tx) = dist(A, B) = dist(y, Ty)$ and $d(x, y) = dist(A, B)$.*

We conclude this section by giving an example to illustrate Theorem 3.3.

Example 3.5. Let $A = \{(x, y, 0) : 0 \le x, y \le 2\}$ and $\{(x, y, 1) : 0 \le x, y \le 2\}$ in the Euclidean space $\left(\mathbb{R}^3, \|\cdot\|_2\right)$. It is easy to see that $\operatorname{dist}(A, B) = 1$. Suppose $f : [0, 2] \to [1, 2]$ be a continuous function such that $t \le f(t)$ for all $t \in [0, 2]$. Define a cyclic multivalued map on $A \cup B$ by

$$T(x, y, z) := \begin{cases} \{(2 - x, a, 1) : 0 \le a \le f(y)\} & \text{if } (x, y, z) \in A \\ \{(2 - x, c, 0) : 0 \le c \le f(y)\} & \text{if } (x, y, z) \in B. \end{cases}$$

Suppose $\{\alpha_n = (x_n, y_n, 0)\}$ and $\{\beta_n = (u_n, v_n, 1)\}$ are two sequences that converge to $\alpha = (x, y, 0)$ and $\beta = (u, v, 1)$ in A and B respectively with $\|(x, y, 0) - (u, v, w)\| = 1$. Then we have

$$x = u \text{ and } \; y = v. \tag{3.1}$$

Further assume that, for $n \in \mathbb{N}$, $\{(\rho_n, \gamma_n, 1)\}$ and $\{(\zeta_n, \eta_n, 0)\}$ are two sequences in $T(\alpha_n)$ and $T(\beta_n)$ respectively, that converge to $(\rho, \gamma, 1)$ and $(\zeta, \eta, 0)$. Then,

$$\rho_n = 2 - x_n, \; \zeta_n = 2 - u_n, \; 0 \le \gamma_n \le f(y_n) \text{ and } 0 \le \eta_n \le f(v_n) \tag{3.2}$$

From (3.1), (3.2) and continuity of f, we have

$$\rho = \zeta = 2 - x \text{ and } 0 \le \gamma, \eta \le f(y).$$

Hence $(2 - x, \eta, 1) \in T(x, y, 0)$. Therefore $\operatorname{dist}(T(x, y, 0), (\zeta, \eta, 0)) = \operatorname{dist}(A, B)$. Also as $0 \le \gamma \le f(y)$ and $y = v$, we have $(2 - u, \gamma, 0) \in T(u, v, 1)$. Therefore $\operatorname{dist}(T(u, v, 1), (\rho, \gamma, 1)) = \operatorname{dist}(A, B))$. Thus A, B and T satisfy the assumptions of Theorem 3.3. Also the pair $((1, 1, 0), (1, 1, 1)) \in A \times B$ satisfies the conclusions of Theorem 3.3.

4. Generalization of Nash equilibrium

Let $I = \{1, 2, \ldots, m\}$ be a set of players. A non-cooperative Nash game of normal form, $(X_i; p_i)_{i\in I}$ is an ordered $2m$-tuple $(X_1, ..., X_m; p_1, ..., p_m)$, where the non-empty set X_i is the strategy space and $p_i : X = \prod_{i\in I} X_i \to \mathbb{R}$ is the payoff function, for each player $i \in I$. The set X, joint strategy space, is the Cartesian product of the individual strategy sets X_i. An element of X is called a strategy (see [12, 6, 7, 1, 9, 11]). A strategy $x^* = (x_1^*, x_2^*, \ldots, x_m^*) \in X$ is called a Nash equilibrium for the game if the following system of inequalities hold:

$$p_i(x^*) \leq p_i(x_i, x_{-i}^*), \text{ for all } \ x_i \in X_i, \ i \in I$$

where $(x_i, x_{-i}^*) = (x_1^*, x_2^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, ..., x_m^*)$.

In [10], the authors considered an n-person game in which each player has two strategy sets and then, by using best proximity pair theorems, established an equilibrium pair for a constrained game. Later many authors proved the existence of equilibrium pairs in similar lines using best proximity pair theorems for multivaled mappings [10, 5, 4, 3].

We consider an economical situation, with m players and a manufacturing unit strategy space X_i and a selling unit strategy space Y_i associated with it. Also it is to be assumed that the goods from X_i are transformed to Y_i, $i = 1, 2, \ldots, m$, in this case $d(x_i, y_i)$ is the transportation cost for the i^{th} player. We denote $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. Knowing the manufacturing strategies $x_{-i} \in X_{-i} := \prod_{i\in I-\{i\}} X_i$ and selling strategies $y_{-i} \in Y_{-i} := \prod_{i\in I-\{i\}} Y_i$ of all other players, the i^{th} player has to choose his / her manufacturing profile, say x_i, and selling profile, say y_i. In this case the pay-off functions for the i^{th} player are defined as follows:

$$\begin{aligned} f_i &: X_i \times Y_{-i} \to \mathbb{R}, \\ g_i &: Y_i \times X_{-i} \to \mathbb{R}. \end{aligned}$$

Here f_i and g_i represent the manufacturing profile and the selling profile of the i^{th} player, respectively, by knowing both manufacturing and selling profiles of all the other players. We denote (X, Y, F, G) is the normal form of such an abstract economy, where $F = (f_i, f_2, \ldots, f_m)$ and $G = (g_1, g_2, \ldots, g_m)$.

We say that a pair $(x^*, y^*) \in X \times Y$ is an *equilibrium pair* for such an abstract economy (X, Y, F, G), if for each $i = 1, 2, \ldots, m$,

$$\begin{aligned} f_i(x_i^*, y_{-i}^*) &= \inf_{a_i \in X_i} f_i(a_i, y_{-i}^*), \\ g_i(y_i^*, x_{-i}^*) &= \sup_{b_i \in Y_i} g_i(b_i, x_{-i}^*) \ \text{ and} \\ \|x_i^* - y_i^*\| &= \operatorname{dist}(X_i, Y_i). \end{aligned}$$

The equilibrium is a pair of action profiles with the property that every player can obtain optimal payoffs (optimal manufacturing cost and profit) with minimum transportation cost.

Remark 4.1. It is to be observed that, if we assume the manufacturing profile is less than the selling profile, that is $f_i \leq g_i$, for all $i = 1, 2, \ldots, m$, then an equilibrium pair (x^*, y^*) of the aforementioned economy (X, Y, f, G) turns out to be a natural generalization of equilibrium:

$$g_i(y_i^*, x_{-i}^*) - f_i(x_i^*, y_{-i}^*) = \sup_{(a_i,b_i)\in X_i\times Y_i} g_i(b_i, x_{-i}^*) - f_i(a_i, y_{-i}^*),\ i = 1, 2, \dots, m.$$

Consider the abstract economy with the normal form (X, Y, F, G). We say that (F, G) satisfies the property (A) if for any $x = (x_1, x_2, \dots, x_m) \in X,\ y = (y_1, y_2, \dots, y_m) \in Y$ with $d(x, y) = \mathrm{dist}(X, Y)$ and for $(u_1, u_2, \dots, u_m) \in X$ satisfying

$$f_i(u_i, y_{-i}) \le f_i(a_i, y_{-i}) \text{ for all } a_i \in X_i,\ 1 \le i \le m,$$

then there exists $(v_1, v_2, \dots, v_m) \in Y$ such that

$$g_i(v_i, x_{-i}) \ge g_i(b_i, x_{-i}), \forall\ b_i \in Y_i \text{ and } d(u_i, v_i) = \mathrm{dist}(X_i, Y_i),\ 1 \le i \le m.$$

The Property (A) can be interpreted naturally as, when a player produces a low cost product, then he/she may get maximum profit by selling the same. Now we prove the existence of an equilibrium pair.

Theorem 4.2. *Let (X, Y, F, G) be a normal form of an abstract economy as defined above, with (X, Y) a non-empty compact convex semisharp proximinal pair in a normed linear space H (for example, (X, Y) can be taken to be non-empty closed, bounded and convex subsets of $\mathbb{R}^m$, with respect to Euclidean norm). Suppose $f_i,\ g_i$ are continuous and (F, G) has the property (A). If $f_i(\cdot, y_{-i})$ is convex, $g_i(\cdot, x_{-i})$ is concave, for all $(x, y) \in X \times Y$ and $i \in I$, then (X, Y, F, G) admits an equilibrium pair.*

Proof. For fixed $x \in X$ and $y \in Y$, define
$T_{f_i}(y) := \{u_i \in X_i : f_i(u_i, y_{-i}) \le f_i(a_i, y_{-i}) \text{ for all } a_i \in X_i\}$ and
$T_{g_i}(x) := \{v_i \in Y_i : g_i(v_i, x_{-i}) \ge g_i(b_i, x_{-i}) \text{ for all } b_i \in Y_i\}$, for $i = 1, 2, \dots m$. As $f_i(\cdot, y_{-i})$ and $f_i(\cdot, x_{-i})$ are continuous functions on compact sets, $T_{f_i}(y)$ and $T_{g_i}(x)$ are non-empty subsets of X_i and Y_i respectively, for $i = 1, 2, \dots m$. Define $\psi : X \cup Y \to X \cup Y$ by

$$\psi(x) := \begin{cases} T_{f_1}(x) \times T_{f_2}(x) \times ... \times T_{f_m}(x), & \text{if } x \in Y \\ T_{g_1}(x) \times T_{g_2}(x) \times ... \times T_{g_m}(x), & \text{if } x \in X. \end{cases}$$

It is to be noted that $\psi(\eta) \in Y$, for $\eta \in X$ and $\psi(\zeta) \in X$, for $\zeta \in Y$. As $f_i(\cdot, y_{-i})$ is a convex continuous function and $g_i(\cdot, x_{-i})$ is a concave continuous function, we have that $\psi(\cdot)$ is a closed and convex valued map.
Assertion. ψ is rusc.
Proof of Assertion. Let $\{x^n\} \subseteq X,\ \{y^n\} \subseteq Y$ be such that $x^n \to x,\ y^n \to y$ with $d(x, y) = \mathrm{dist}(X, Y)$ and $w^n \in \psi(x^n),\ u^n \in \psi(y^n)$ for all $n \in \mathbb{N}$ with $u^n \to u,\ w^n \to w$. That is

$$w_i^n \in T_{g_i}(x^n) \text{ and } u_i^n \in T_{f_i}(y^n) \text{ for } i \in \{1, 2, \dots, m\},\ n \in \mathbb{N}.$$

Hence $f_i(u_i^n, y_{-i}^n) \le f_i(a_i, y_{-i}^n)$, for all $a_i \in X_i,\ i \in \{1, 2, \dots, m\}, n \in \mathbb{N}$. Therefore $f_i(u_i, y_{-i}) \le f_i(a_i, y_{-i})$, for all $a_i \in X_i,\ i \in I$. As (F, G) satisfies the property (A), there exists $v_i \in Y_i$ such that $g_i(v_i, x_{-i}) \ge g_i(b_i, x_{-i})$, for all $b_i \in Y_i$ and $\|u_i - v_i\| = \mathrm{dist}(X_i, Y_i)$ for all $i \in I$. Hence $v_i \in T_{g_i}(x)$ for all $i \in \{1, 2, \dots, m\}$. Therefore $v = (v_1, v_2, \dots, v_m) \in \psi(x)$ and $\|u - v\| = \mathrm{dist}(X, Y)$. Now $\mathrm{dist}(X, Y) \le$

$\mathrm{dist}(u,\psi(x)) \le \|u-v\| = \mathrm{dist}(X,Y)$. Hence $\mathrm{dist}(u,\psi(x)) = \mathrm{dist}(X,Y)$. In a similar fashion one can prove that $\mathrm{dist}(w,\psi(y)) = \mathrm{dist}(X,Y)$. This establishes the assertion. Now by Theorem 3.3, ψ admits a best proximity pair (x^*,y^*) in $X\times Y$ such that $\|x^*-y^*\| = \mathrm{dist}(X,Y)$. Hence $\mathrm{dist}(x_i^*,T_{g_i}(x^*)) = \mathrm{dist}(X_i,Y_i) = \mathrm{dist}(y_i^*,T_{f_i}(y^*))$ and $\|x_i^*-y_i^*\| = \mathrm{dist}(X_i,Y_i)$ for all $i\in I$. Since $T_{g_i}(x^*)$ is compact, there exists $z_i\in T_{g_i}(x^*)$ such that $\|x_i^*-z_i\| = \mathrm{dist}(X_i,Y_i)$ for all $i\in I$. By uniqueness of best approximation, we have $y_i^* = z_i$, for all $i\in I$. Hence $y_i^*\in T_{g_i}(x^*)$, for all $i\in I$. Equivalently,

$$g_i(y_i^*,x_{-i}^*) \ge g_i(b_i,x_{-i}^*), \text{for all } b_i\in Y_i,\ i=1,2,\ldots,m.$$

In a similar fashion one can prove that $x_i^*\in T_{f_i}(y^*)$, for all $i\in I$. Equivalently,

$$f_i(x_i^*,y_{-i}^*) \le f_i(a_i,y_{-i}^*), \text{for all } a_i\in X_i,\ i=1,2,\ldots,m.$$

This completes the proof. $\square$

As a consequence of the above theorem we obtain the following standard Nash equilibrium.

Corollary 4.3. *Let (X,Y,F,G) be a normal form of an abstract economy, as in Theorem 4.2.*

(a.) *If $f_i(x_i,y_{-i}) \le g_i(y_i,x_{-i})$ for all $x\in X,\ y\in Y$ with $d(x,y) = dist(X,Y)$, then there exists $(x^*,y^*)\in X\times Y$ such that*

$$g_i(y_i^*,x_{-i}^*) - f_i(x_i^*,y_{-i}^*) = \sup_{\substack{(x,y)\in X\times Y\\ \|x_i-y_i\| = dist(X_i,Y_i)}} g_i(y_i,x_{-i}) - f_i(x_i,y_{-i}),\ i=1,2,\ldots,m.$$

(b.) *If $f_i(x_i,y_{-i}) \ge g_i(y_i,x_{-i})$ for all $x\in X,\ y\in Y$ with $d(x,y) = dist(X,Y)$, then there exists $(x^*,y^*)\in X\times Y$ such that*

$$f_i(x_i^*,y_{-i}^*) - g_i(y_i^*,x_{-i}^*) = \inf_{\substack{(x,y)\in X\times Y\\ \|x_i-y_i\| = dist(X_i,Y_i)}} f_i(x_i,y_{-i}) - g_i(y_i,x_{-i}),\ i=1,2,\ldots,m.$$

Proof. (a). If $f_i(u_i,v_{-i}) \le g_i(v_i,u_{-i})$ for all $u\in X,\ v\in Y$ with $d(u,v) = \mathrm{dist}(X,Y)$, then,

$$g_i(y_i,x_{-i}) - f_i(x_i,y_{-i}) = \sup_{(a_i,b_i)\in X_i\times Y_i} g_i(b_i,x_{-i}) - f_i(a_i,y_{-i}),\ 1\le i\le m. \tag{4.1}$$

Now set $A := \{(x,y)\in X\times Y : \|x_i-y_i\| = \mathrm{dist}(X_i,Y_i),\ 1\le i\le m$ and satisfies (4.1)$\}$. We have, by Theorem 4.2, A is a non-empty subset of $X\times Y$. As f_i, g_i are continuous, for all i, and A is a closed subset of the compact set of $X\times Y$, there exists $(x^*,y^*)\in X\times Y$ such that for each $i=1,2,\ldots,m$,

$$g_i(y_i^*,x_{-i}^*) - f_i(x_i^*,y_{-i}^*) = \sup_{\substack{(x,y)\in X\times Y\\ \|x_i-y_i\| = \mathrm{dist}(X_i,Y_i)}} g_i(y_i,x_{-i}) - f_i(x_i,y_{-i}).$$

(b). Employing the same techniques in the case when $f_i(x_i,y_{-i}) \ge g_i(y_i,x_{-i})$ for all $x\in X,\ y\in Y$ with $d(x,y) = \mathrm{dist}(X,Y)$, we have $(x^*,y^*)\in X\times Y$ such that for each $i=1,2,\ldots,m$,

$$f_i(x_i^*,y_{-i}^*) - g_i(y_i^*,x_{-i}^*) = \inf_{\substack{(x,y)\in X\times Y\\ \|x_i-y_i\| = \mathrm{dist}(X_i,Y_i)}} f_i(x_i,y_{-i}) - g_i(y_i,x_{-i}).$$

□

Remark 4.4. It is to be observed (in the above proof) that if $f_i(\cdot, y_{-i})$ is not a convex function, for some $y \in Y$, then $\psi(x)$ may not be a convex valued map. To see this, set $X_1 = Y_1 = [0,1]$, $X_2 = \frac{-1}{2}$, $Y_2 = \frac{1}{2}$ and define

$$\begin{aligned}
f_1\Big(x_1, \frac{1}{2}\Big) &= \frac{\min\{x_1, (1-x_1)\}}{2}, \text{ for } x_1 \in X_1,\\
g_1\Big(y_1, \frac{-1}{2}\Big) &= \frac{\min\{y_1, (1-y_1)\}}{2}, \text{ for } y_1 \in Y_1,\\
f_2\Big(\frac{-1}{2}, y_1\Big) &= \frac{-y_1}{2}, \text{ for } y_1 \in Y_1 \text{ and}\\
g_2\Big(x_1, \frac{1}{2}\Big) &= \frac{x_1}{2}, \text{ for } x_1 \in X_1.
\end{aligned}$$

Then it is easy to see that f_i, g_i are continuous, $i = 1, 2$. But for any $y \in Y$, we have $T_{f_1}(y) = \{0, 1\}$. Therefore $T_{f_1}(\cdot)$ is not a convex set and hence so is ψ. In a similar fashion, if $g_i(\cdot, x_{-i})$ is not a concave function, for some $x \in X$, then $\psi(\cdot)$ may not be a convex valued map.

Finally we give an example to illustrate Theorem 4.2, by using the following fact.

Fact 4.5. Let (a_1, a_2), $(b_1, b_2) \in \mathbb{R}^2$. If $\min\{a_1, a_2\} \leq \min\{b_1, b_2\}$ and $a_1 + a_2 = b_1 + b_2$, then $\max\{a_1, a_2\} \geq \max\{b_1, b_2\}$.

Example 4.6. Let $X_1 = Y_1 = [-1, 1]$, $X_2 = Y_2 = [0, 1]$, $X_3 = -\frac{1}{2}$, $Y_3 = \frac{1}{2}$ and $X = X_1 \times X_2 \times X_3$, $Y = Y_1 \times Y_2 \times Y_3$. Let us consider that X and Y are subsets of the Euclidean space $\mathbb{R}^3$ (with respect to the Euclidean norm). It is to be observed that $\text{dist}(X, Y) = 1$. Also, if $x = (x_1, x_2, x_3) \in X$, $y = (y_1, y_2, y_3) \in Y$ satisfy $d(x, y) = \text{dist}(X, Y)$, then $x_1 = y_1$, $x_2 = y_2, x_3 = -\frac{1}{2}$ and $y_3 = \frac{1}{2}$. Define $f_i : X_i \times Y_{-i} \to \mathbb{R}$ and $g_i : Y_i \times X_{-i} \to \mathbb{R}$ (for $i = 1, 2, 3$) by

$$\begin{aligned}
f_1(u_1, v_2, v_3) &= u_1^2(v_2 + v_3),\\
f_2(v_1, u_2, v_3) &= \max\{u_2, 1 - u_2\} + v_1 + v_3,\\
f_3(v_1, v_2, u_3) &= v_1 + v_2 + u_3,\\
g_1(v_1, u_2, u_3) &= \big(1 - v_1^2\big)(u_2 + u_3),\\
g_2(u_1, v_2, u_3) &= \min\{v_2, 1 - v_2\} + u_1 + u_3,\\
g_3(u_1, u_2, v_3) &= u_1 + u_2 + v_3,
\end{aligned}$$

for all $(u_1, u_2, u_3) \in X$ and $(v_1, v_2, v_3) \in Y$. It is easy to see that, f_i, g_i are continuous. Also for each fixed $(x, y) \in X \times Y$, $f_i(\cdot, y_{-i}) : X_i \to \mathbb{R}$ is convex and $g_i(\cdot, x_{-i}) : Y_i \to \mathbb{R}$ is concave, for $i = 1, 2, 3$. Let $x = (x_1, x_2, x_3) \in X$ and $y = (y_1, y_2, y_3) \in Y$ be such that $d(x, y) = \text{dist}(X, Y)$. Then,

$$x_1 = y_1,\ x_2 = y_2,\ x_3 = \frac{-1}{2},\ y_3 = \frac{1}{2} \tag{4.2}$$

Suppose $u = (u_1, u_2, u_3) \in X$ satisfies

$$f_i(u_i, y_{-i}) \leq f_i(a_i, y_{-i}), \text{ for all } a_i \in X_i\ (i = 1, 2, 3).$$

Elementary numerical calculations show that $g_1(v_1, x_{-i}) \geq g_i(b_1, x_{-i})$ for all $b_1 \in Y_1$, $1 \leq i \leq 3$, where $v = (v_1, v_2, v_3) = (u_1, u_2, \frac{1}{2})$. Hence (F, G) satisfies the property (A), where $F = (f_1, f_2, f_3)$ and $G = (g_1, g_2, g_3)$. Also, by using basic numerical calculation, it is easy to see that $(x^*, y^*) = \Big((0, \frac{1}{2}, \frac{-1}{2}), (0, \frac{1}{2}, \frac{1}{2})\Big) \in X \times Y$ is an equilibrium pair for (X, Y, F, G). Also we have

$$g_i(y_i^*, x_{-i}^*) - f_i(x_i^*, y_{-i}^*) = \sup_{\substack{(x,y) \in X \times Y \\ \|x_i - y_i\| = \operatorname{dist}(X_i, Y_i)}} g_i(y_i, x_{-i}) - f_i(x_i, y_{-i}), \text{ for } i = 1, 2, 3.$$

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