# WEIGHTED CORNER SPACES 

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#### Abstract

This article contains new elements of the pseudo-differential analysis on specific classes of stratified spaces. We study structures appearing in parametrices of elliptic operators and we establish the interplay between corner degenerate symbols of operators and weighted Sobolev spaces on the underlying spaces $M$ with the singular geometry of cones, edges or corners. In particular, we establish an accessible approach to corner Sobolev spaces. On the smooth part of $M$ the pseudo-differential operators locally refer to the Fourier transform. Close to the singularities we employ different variants of Mellin pseudo-differential operators, in order to control phenomena up to the singularities.


## Introduction

Partial differential equations (PDEs) in the framework of elliptic boundary value problems (BVPs) on a manifold with smooth boundary can be embedded into suitable algebras of pseudo-differential operators such that not only the operators themselves but also their parametrices belong to the algebras, cf., the work of Boutet de Monvel [1] and several monographs or original papers on this topic, e.g., [15, 18, 29]. In particular, elliptic BVPs are Fredholm between Sobolev spaces, for instance, when the underlying configuration is compact, and we have elliptic regularity of solutions. The arguments on BVPs are of a similar structure as those on a closed smooth manifold $M$ and the results are analogous. The corresponding pseudodifferential machinery in the closed case is standard and much easier than for BVPs. In any case the expected conclusions depend on the smoothness of $M$, in particular, on the well-known coordinate invariance of pseudo-differential operators. However, numerous applications give rise to models where the underlying configuration is not smooth, e.g., with a non-complete Riemannian metric, induced from an ambient space which is smooth. This may concern piecewise smooth domains embedded in $\mathbb{R}^{n}$, manifolds with conical singularities or edges, higher corners, or interfaces with such singularities. The corresponding techniques also suggest considering manifolds with conical exits to infinity. It is obvious in such cases that basic methods from the "smooth" pseudo-differential analysis are no longer available, and new tools have to be established in order to control the operators together with their symbol structures as well as distribution spaces, adapted to the specific nature of singularities. It is a particularly delicate task to manage the interplay between symbols and associated operators or to recognize algebras containing the parametrices of elliptic elements. The "singular analysis", represented by numerous research teams worldwide, is developed in rather different directions. On the one hand, the focus may

[^0]lie on explicit answers to concrete PDE-problems in models of physics, geometry, or other applications. On the other hand one might strive for all elliptic problems at the same time when they share common properties, such that it is reasonable to look for algebras of operators in the above-mentioned sense, like on closed smooth manifolds, or in Boutet de Monvel's calculus where the operators have the transmission property at the boundary. For instance, given an infinite straight cone rather than a smooth manifold, we might ask for a kind of cone algebra containing not only elliptic differential operators, degenerate in stretched coordinates at the tip of the cone, with the axial variable $r>0$ (such as Laplace-Beltrami operators associated with a cone-metric) and with $r \rightarrow \infty$ as a conical exit to infinity. Corresponding cone algebras do exist, ideed, cf., examples below and other results on edge algebras, created in [35], see also [2-4, 12-14, 20, 21, 23, 32-34, 36, 37]. Clearly several approaches are not really disjoint, but not always coordinated. Degenerate differential operators, partly in a pseudo-differential context, have been investigated by many authors, see also, $[11,22,26-28]$. In the present paper we develop fuctional analytic properties of weighted corner spaces, cf. Theorem 3.1 which extends a corresponding result for weighted edge spaces, added here for completeness in Section 4 as Proposition 4.2 with a more transparent proof, compared with that of [37, Proposition 3.1.21].

## 1. Mellin-edge operators

In the following we refer to the terminology from [3, 4] on categories $\mathfrak{M}_{k}, k \in$ $\mathbb{N}=\{0,1,2, \ldots$,$\} of corner spaces M$ of singularity order $k$, see also [7, Section 1]. Roughly speaking, the elements of $\mathfrak{M}_{k}$ are stratified spaces, obtained by repeatedly forming cones

$$
\begin{equation*}
X^{\Delta}:=\overline{\mathbb{R}}_{+} \times X /(\{0\} \times X) \tag{1.1}
\end{equation*}
$$

or wedges

$$
\begin{equation*}
X^{\Delta} \times \mathbb{R}^{q} \tag{1.2}
\end{equation*}
$$

$k$ times, combined with globalizations, where $X$ in the first step is a smooth manifold, often assumed to be compact. $\mathfrak{M}_{0}$ is a category of oriented smooth manifolds with diffeomorphisms as isomorphisms, and we assume that $M \in \mathfrak{M}_{k}$ entails $\Omega \times M, M \times \Omega \in \mathfrak{M}_{k}$ for every $\Omega \in \mathfrak{M}_{0}$ and all $k$. The analysis will take place on the respective open stretched cones $X^{\wedge}:=\mathbb{R}_{+} \times X$ and wedges $X^{\wedge} \times \mathbb{R}^{q}$ in the splitting of variables $(r, x)$ and $(r, x, y)$, respectively. The variable $\mathbb{R}_{+}$is interpreted as the (local) cone-axis variable, and $X$ as the base (link) of the respective cone, or of the model cone of the wedge when $q>0$. In the Appendix below we recall more terminology in this context.

Example 1. Let $X$ be a closed manifold, endowed with a Riemannian metric $g_{X}$. Then $X^{\wedge}=\mathbb{R}_{+} \times X$ in the variables $(r, x)$ is an infinite (stretched) cone with the metric $d r^{2}+r^{2} g_{X}$ and the associated Laplace-Beltrami operator is a special degenerate operator of the form

$$
\begin{equation*}
A_{\text {cone }}=r^{-\mu} \sum_{j=0}^{\mu} a_{j}(r)\left(-r \frac{\partial}{\partial r}\right)^{j} \tag{1.3}
\end{equation*}
$$

for coefficients $a_{j} \in C^{\infty}\left(\overline{\mathbb{R}}_{+}, \operatorname{Diff}^{\mu-j}(X)\right)$, in this case for $\mu=2$, with $\operatorname{Diff}^{\nu}(X)$ being the Fréchet space of differential operators of order $\nu$ on $X$ with smooth coefficients. Sometimes, differential operators like (1.3) will be called to be of Fuchs type. Those are formulated here for arbitrary $\mu \in \mathbb{R}$, since we are talking about algebras, generated by such operators, and we reach operators of order $-\mu$ when we talk about ellipticity and express parametrices. More generally, on $X^{\wedge} \times \Omega, \Omega \subseteq \mathbb{R}^{q}$ open, in the coordinates $(r, x, y)$, we have so-called edge-degenerate differential operators

$$
\begin{equation*}
A_{\text {edge }}=r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j, \alpha}(r, y)\left(-r \frac{\partial}{\partial r}\right)^{j}\left(r D_{y}\right)^{\alpha} \tag{1.4}
\end{equation*}
$$

for coefficients $a_{j, \alpha} \in C^{\infty}\left(\overline{\mathbb{R}}_{+} \times \Omega\right.$, $\left.\operatorname{Diff}^{\mu-(j+|\alpha|)}(X)\right)$. Special cases for $\mu=2$ are Laplace-Beltrami operators belonging to $d r^{2}+r^{2} g_{X}+d y^{2}$. Operators of the form (1.4) belong to the edge algebra, to be considered below, and parametrices in the elliptic case are of opposite order.

Remark 1.1. Cones $X^{\Delta}$ and wedges $X^{\Delta} \times \Omega$ are special examples of elements in $\mathfrak{M}_{1}$. For any $M \in \mathfrak{M}_{1}$ we have the singular stratum $s_{1}(M)$ which is equal to a single point in $X^{\Delta}$, the vertex of the cone, and for $M=X^{\Delta} \times \Omega$ we have $s_{1}(M)=\Omega$, the edge of $M$. More generally, an element $M \in \mathfrak{M}_{1}$ is called a manifold with conical singularities when $\operatorname{dim} s_{1}(M)=0$, otherwise, a manifold with edge $s_{1}(M)$ when its dimension is equal to $q>0$.

For the pseudo-differential analysis on spaces $M \in \mathfrak{M}_{k}$ for $k>1$ it is important to refer to information from the case $k=1$. For a space in $E \in \mathfrak{M}_{1}$ with $\operatorname{dim} s_{1}(E)=0$ we often assume that $s_{1}(E)$ is a single point; otherwise it would be a discrete subset of $E$, and a space with edge $Y$ of dimension $>0$ will be denoted by $B$. While $E$ is locally close to $s_{1}(E)$ modelled on $X^{\Delta}$ for some compact $X \in \mathfrak{M}_{0}$, a space $B \in \mathfrak{M}_{1}$ with edge $Y=S_{1}$ of dimension $q>0$ is locally near $Y$ modelled on $X^{\Delta} \times \mathbb{R}^{q}$; without loss of generality we often refer to local coordinates $y \in \mathbb{R}^{q}$.

Spaces $E$ with conical singularity $s_{1}(E)$ are considered in terms of the stratification

$$
\begin{equation*}
s(E):=\left(s_{0}(E), s_{1}(E)\right) \tag{1.5}
\end{equation*}
$$

for $s_{0}(E):=E \backslash s_{1}(E)$. The principal symbol hierarchy of operators $A_{\text {cone }}$ on $E$ has two components

$$
\begin{equation*}
\sigma\left(A_{\text {cone }}\right):=\left(\sigma_{0}\left(A_{\text {cone }}\right), \sigma_{1}\left(A_{\text {cone }}\right)\right) \tag{1.6}
\end{equation*}
$$

consisting of the interior homogeneous principal symbol $\sigma_{0}\left(A_{\text {cone }}\right)$ of $A_{\text {cone }}$ as an element of $\mathrm{Diff}^{\mu}\left(E \backslash s_{1}(E)\right)$ and the operator-valued conormal symbol $\sigma_{1}\left(A_{\text {cone }}\right)$ of $A_{\text {cone }}$. For differential operators (1.3) we have the principal conormal symbol

$$
\begin{equation*}
\sigma_{1}\left(A_{\text {cone }}\right)(w):=\sum_{j=0}^{\mu} a_{j}(0) w^{j} \tag{1.7}
\end{equation*}
$$

which is operator-valued as a family of continuous operators between Sobolev spaces on $X$

$$
\begin{equation*}
\sigma_{1}\left(A_{\text {cone }}\right)(w): H^{s}(X) \rightarrow H^{s-\mu}(X) \tag{1.8}
\end{equation*}
$$

with $w$ being the complex Mellin covariable, dual to $-r \partial_{r}$ varying on $\Gamma_{(n+1) / 2-\beta_{\mu}}$ for some weight $\beta_{\mu} \in \mathbb{R}$, cf., notation (1.25) below.

Spaces $B \in \mathfrak{M}_{1}$ with edge singularities $Y=s_{1}(B)$ of dimension $q>0$ have an analogous stratification

$$
\begin{equation*}
s(B):=\left(s_{0}(B), s_{1}(B)\right) \tag{1.9}
\end{equation*}
$$

for $s_{0}(B):=M \backslash s_{1}(B)$ and the edge $s_{1}(B)$. The principal symbol hierarchy of operators $A_{\text {edge }}$ on $B$ has the form

$$
\begin{equation*}
\sigma\left(A_{\text {edge }}\right):=\left(\sigma_{0}\left(A_{\text {edge }}\right), \sigma_{1}\left(A_{\text {edge }}\right)\right), \tag{1.10}
\end{equation*}
$$

with the interior homogeneous principal symbol $\sigma_{0}\left(A_{\text {edge }}\right)$ interpreted as an element $A_{\text {edge }} \in \operatorname{Diff}^{\mu}\left(B \backslash s_{1}(B)\right)$. Moreover, we have the operator-valued principal edgesymbol $\sigma_{1}\left(A_{\text {edge }}\right)(y, \eta)$ of $A_{\text {edge }}$ as a family of continuous operators between weighted Kegel spaces $\mathcal{K}^{s, \beta}\left(X^{\wedge}\right)$ of smoothness $s$ and weight $\beta$ on the open stretched cone $X^{\wedge}$, cf. notation (1.20) below,

$$
\begin{equation*}
\sigma_{1}\left(A_{\text {edge }}\right)(y, \eta): \mathcal{K}^{s, \beta}\left(X^{\wedge}\right) \rightarrow \mathcal{K}^{s-\mu, \beta-\mu}\left(X^{\wedge}\right) \tag{1.11}
\end{equation*}
$$

which is for differential operators (1.4) of the form

$$
\begin{equation*}
\sigma_{1}\left(A_{\text {edge }}\right)(y, \eta)=r^{-\mu} \sum_{j+|\alpha| \leq \mu} a_{j, \alpha}(0, y)\left(-r \frac{\partial}{\partial r}\right)^{j}(r \eta)^{\alpha} \tag{1.12}
\end{equation*}
$$

for all $y \in \mathbb{R}^{q}$ and $\eta \in \mathbb{R}^{q} \backslash\{0\}$. Recall that in this case there is a "subordinate" $y$-dependent conormal symbol belonging to the conical singularity of the model cone $X^{\Delta}$, here denoted by

$$
\begin{equation*}
\sigma_{\mathrm{c}} \sigma_{1}\left(A_{\text {edge }}\right)(y, w)=\sum_{j=0}^{\mu} a_{j, 0}(0, y) w^{j}: H^{s}(X) \rightarrow H^{s-\mu}(X), \tag{1.13}
\end{equation*}
$$

for $w \in \Gamma_{(n+1) / 2-\beta_{\mu}}$. Recall from [35] that one of the basic observations of homogeneous principal edge symbols is the relation

$$
\begin{equation*}
\sigma_{1}\left(A_{\text {edge }}\right)(y, \delta \eta)=\delta^{\mu} \kappa_{\delta}^{-1} \sigma_{1}\left(A_{\text {edge }}\right)(y, \eta) \kappa_{\delta} \tag{1.14}
\end{equation*}
$$

for every $\delta \in \mathbb{R}_{+}$, where $\kappa_{\delta}$ is the transformation $\left(\kappa_{\delta} u\right)(r, x):=\delta^{(1+n) / 2} u(\delta r, x)$, acting on $\mathcal{K}^{s, \beta}\left(X^{\wedge}\right)$-spaces for arbitrary $s, \beta$, cf. also notation (1.32), below.

The edge calculus, or its parameter-dependent analogue

$$
\begin{equation*}
L^{\mu}(B, \boldsymbol{g}), \quad \text { and } \quad L^{\mu}\left(B, \boldsymbol{g} ; \mathbb{R}_{\lambda}^{d}\right) \tag{1.15}
\end{equation*}
$$

respectively, for weight data $\boldsymbol{g}:=(\beta, \beta-\mu)$, with $\lambda \in \mathbb{R}^{d}$ being a parameter, will be systematically employed in the considerations below, cf., Definition 1.2, and it is necessary to recall some of its structures. First of all, a part of the definition of a manifold $B$ with edge $Y$ is that $B \backslash Y$ belongs to $\mathfrak{M}_{0}$. Moreover, $Y$ has a neighborhood $V \subset B$ with the structure of an $X^{\Delta}$-bundle over $Y$ for a compact $X \in \mathfrak{M}_{0}$, where the transition maps between the fibres $X^{\Delta}$ are induced by isomorphisms

$$
\begin{equation*}
\mathbb{R} \times X \rightarrow \mathbb{R} \times X \tag{1.16}
\end{equation*}
$$

in $\mathfrak{M}_{0}$ such that $(t, x) \rightarrow(\tilde{t}, \tilde{x})$ restricts to continuous maps $\overline{\mathbb{R}}_{+} \times X \rightarrow \overline{\mathbb{R}}_{+} \times X$ and isomorphisms $X \rightarrow X$ in $\mathfrak{M}_{0}$ and satisfies $(\delta t, x) \rightarrow(\delta \tilde{t}, \tilde{x})$ for every $\delta \in \mathbb{R}_{+}$.

Operators $A \in L^{\mu}(B, \boldsymbol{g})$ for weight data $\boldsymbol{g}:=(\beta, \beta-\mu)$ are pseudo-differential in general, and because of their specific form near the edge $Y$ they are called edgedegenerate. In particular, by

$$
\begin{equation*}
\operatorname{Diff}_{\operatorname{deg}}^{\mu}(B) \tag{1.17}
\end{equation*}
$$

for $\mu \in \mathbb{N}$ we denote the space of all differential operators on the smooth manifold $B \backslash Y$ of order $\mu$ which are close to $Y$ in local variables $(r, x, y), y \in \Omega$ of the form (1.4). In case of ellipticity the space $L^{-\mu}\left(B, \boldsymbol{g}^{-1}\right)$ for $\boldsymbol{g}^{-1}:=(\beta-\mu, \beta)$ just contains parametrices of operators in (1.17). Similar notation may be used including parameters $\lambda \in \mathbb{R}^{d}$, and we write

$$
\begin{equation*}
\operatorname{Diff}_{\mathrm{deg}}^{\mu}\left(B ; \mathbb{R}^{d}\right) . \tag{1.18}
\end{equation*}
$$

Operators in (1.17) or (1.18) are connected with their own natural weight data $\boldsymbol{g}:=(\beta, \beta-\mu)$ where the choice of $\beta$ appears in connection with ellipticity.

On the open stretched cone

$$
\begin{equation*}
X^{\wedge}:=\mathbb{R}_{+} \times X \tag{1.19}
\end{equation*}
$$

we now recall some notation on distribution spaces, namely, weighted Kegel spaces

$$
\begin{equation*}
\mathcal{K}^{s, \beta}\left(X^{\wedge}\right):=\left\{u=\omega u_{0}+(1-\omega) u_{\infty}: u_{0} \in \mathcal{H}^{s, \beta}\left(X^{\wedge}\right), u_{\infty} \in H_{\text {cone }}^{s}\left(X^{\wedge}\right)\right\} . \tag{1.20}
\end{equation*}
$$

Here $\omega=\omega(r)$ is a cut-off function on the positive $r$ half-axis. Moreover, $H_{\text {cone }}^{s}\left(X^{\wedge}\right)$ is an analogue of the standard Sobolev space $H^{s}\left(\mathbb{R}^{1+n}\right)$ in variables $\tilde{x}=\left(\tilde{x}_{0}, \tilde{x}^{\prime}\right)$, based on the Fourier transform in $\tilde{x}$. For the definition of $\mathcal{H}^{s, \beta}$-spaces involved in (1.20) are first consider

$$
\begin{equation*}
\mathcal{H}^{s, \beta}\left(\mathbb{R}_{+} \times \mathbb{R}^{n}\right) \tag{1.21}
\end{equation*}
$$

where we employ the weighted Mellin transform with respect to $r \in \mathbb{R}_{+}$together with the Fourier transform in $x$. The space (1.21) is the completion of $C_{0}^{\infty}\left(\mathbb{R}_{+} \times \mathbb{R}^{n}\right)$ with respect to the norm

$$
\begin{equation*}
\|u\|_{\mathcal{H}^{s, \beta}\left(\mathbb{R}_{+} \times \mathbb{R}^{n}\right)}:=\left\{\int_{\mathbb{R}^{n}} \int_{\Gamma_{\frac{n+1}{2}-\beta}}\langle\xi, w\rangle^{2 s}\left|\left(F_{x \rightarrow \xi} M_{r \rightarrow w} u\right)(\xi, w)\right|^{2} d w d \xi\right\}^{\frac{1}{2}} \tag{1.22}
\end{equation*}
$$

where

$$
\begin{gather*}
M:\left.C_{0}^{\infty}\left(\mathbb{R}_{+}\right) \longrightarrow \mathcal{A}\left(\mathbb{C}_{w}\right)\right|_{\Gamma_{\frac{n+1}{2}-\beta}},  \tag{1.23}\\
M u(w)=\int_{0}^{\infty} r^{w} u(r) \frac{d r}{r} \tag{1.24}
\end{gather*}
$$

is the weighted Mellin transform and $\mathcal{A}\left(\mathbb{C}_{w}\right)$ the (Fréchet) space of holomorphic functions in $w$,

$$
\begin{equation*}
\Gamma_{\alpha}=\{w \in \mathbb{C}: \operatorname{Re} w=\alpha\} \tag{1.25}
\end{equation*}
$$

a weight line, here for $\alpha=\frac{n+1}{2}-\beta$, and $M$ applied to $C_{0}^{\infty}$ - functions in $r \in \mathbb{R}_{+}$, taking values in $C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$. From (1.21) we then pass to spaces $\mathcal{H}^{s, \beta}\left(X^{\wedge}\right)$ using an
open covering of $X$ by coordinate neighborhoods $\left\{U_{1}, \ldots, U_{N}\right\}$ and a subordinate partition of unity $\left\{\varphi_{1}, \ldots, \varphi_{N}\right\}$. Concerning more details, especially functional analytic properties, see, e.g., [37]. The space $H_{\text {cone }}^{s}\left(X^{\wedge}\right)$ occurring in (1.20) treats the "cylinder" (1.19) as a manifold with conical exit to $\infty$, which is an important property for the correct meaning of edge spaces below.

For the definition we first choose a coordinate neighborhood on the unit sphere $S^{n}$ in $\mathbb{R}^{1+n} \ni\left(\tilde{x}_{0}, \tilde{x}^{\prime}\right)$ where we assume $V \subset\left\{\tilde{x}_{0}>0\right\}$ and we form

$$
\begin{equation*}
V^{\wedge}:=\left\{\tilde{x} \in \mathbb{R}^{1+n} \backslash\{0\}: \frac{\tilde{x}}{|\tilde{x}|} \in V\right\} \tag{1.26}
\end{equation*}
$$

For any cut-off function $\omega(r)$ and $s \in \mathbb{R}$ we set

$$
\begin{equation*}
H_{\text {cone }}^{s}\left(V^{\wedge}\right):=\left\{\left.u(\tilde{x}) \in H_{\mathrm{loc}}^{s}\left(\mathbb{R}^{1+n}\right)\right|_{\mathbb{R}_{+} \times V}:\left.(1-\omega(|\tilde{x}|)) u \in H^{s}\left(\mathbb{R}^{1+n}\right)\right|_{\mathbb{R}_{+} \times V}\right\} \tag{1.27}
\end{equation*}
$$

Choose a system of diffeomorphisms $\chi_{j}: U_{j} \longrightarrow V$, and set

$$
\begin{equation*}
\chi_{j}^{\wedge}: \mathbb{R}_{+} \times U_{j} \longrightarrow V^{\wedge}, \quad \chi_{j}^{\wedge}(r, x)=r \chi_{j}(x)=r \chi_{j}(x)=\tilde{x} \tag{1.28}
\end{equation*}
$$

and

$$
\begin{equation*}
H_{\text {cone }}^{s}\left(\mathbb{R}_{+} \times U_{j}\right):=\left\{u(r, x)=\left(\left(\chi_{j}\right)_{*}^{-1} v\right)(r, x): v \in H_{\text {cone }}^{s}\left(V^{\wedge}\right)\right\} \tag{1.29}
\end{equation*}
$$

Then

$$
\begin{equation*}
H_{\text {cone }}^{s}\left(X^{\wedge}\right):=\left\{\sum_{j=1}^{N} \varphi_{j} u_{j}: u_{j} \in H_{\text {cone }}^{s}\left(\mathbb{R}_{+} \times U_{j}\right), j=1, \ldots, N\right\} \tag{1.30}
\end{equation*}
$$

Recall that the spaces (1.20) are independent of the choice of $\omega$. For $s=\beta=0$ we have the relation

$$
\begin{equation*}
\mathcal{K}^{0,0}\left(X^{\wedge}\right)=\mathcal{H}^{0,0}\left(X^{\wedge}\right)=r^{-n / 2} L^{2}\left(\mathbb{R}_{+} \times X\right) \tag{1.31}
\end{equation*}
$$

where $L^{2}$ refers to $d r d x$ and $d x$ to a fixed Riemannian metric on $X$.
Another important class of spaces are the weighted edge spaces which make sense in abstract form in terms of a (separable) Hilbert space $H$ endowed with a group action

$$
\begin{equation*}
\kappa=\left\{\kappa_{\delta}\right\}_{\delta \in \mathbb{R}_{+}} \quad \text { for isomorphisms } \quad \kappa_{\delta}: H \longrightarrow H \tag{1.32}
\end{equation*}
$$

such that $\kappa_{\delta} \kappa_{\delta^{\prime}}=\kappa_{\delta \delta^{\prime}}$ and $\kappa_{1}=\mathrm{id}_{H}$. Moreover, we ask strong continuity, i.e., for every fixed $h \in H$ the function $\left\{\kappa_{\delta} h: \delta \in \mathbb{R}_{+}\right\}$belongs to $C\left(\mathbb{R}_{+}, H\right)$. Let us set, cf. [35],

$$
\begin{equation*}
\|u\|_{\mathcal{W}^{s}\left(\mathbb{R}^{q}, H\right)}:=\left\{\int[\eta]^{2 s}\left\|\kappa_{[\eta]}^{-1} \hat{u}(\eta)\right\|_{H}^{2} d \eta\right\}^{\frac{1}{2}} \tag{1.33}
\end{equation*}
$$

which is finite on $\mathcal{S}\left(\mathbb{R}^{q}, H\right)$ for every real $s$. Here $\eta \rightarrow[\eta]$ is any strictly positive function in $C^{\infty}\left(\mathbb{R}^{q}\right)$ such that $[\eta]=|\eta|$ for $|\eta|>c$ for some $c>0$. An equivalent norm is obtained when we replace $[\eta]$ by $\langle\eta\rangle$. This gives us by completion the abstract edge space $\mathcal{W}^{s}\left(\mathbb{R}^{q}, H\right)$ which is an analogue of the scalar Sobolev spaces. In analogous meaning later on we also employ notation $[\eta, \lambda]$ where $\lambda \in \mathbb{R}^{d}$ treated as an extra
component of the corresponding covariable $(\eta, \lambda)$. In this sense we also have spaces $\mathcal{H}^{s, \gamma}\left(\mathbb{R}_{+} \times \mathbb{R}^{q}, H\right)$ for some real $\gamma$ obtained by completing $C_{0}^{\infty}\left(\mathbb{R}_{+} \times \mathbb{R}^{q}, H\right)$ with respect to the norm

$$
\begin{equation*}
\|f\|_{\mathcal{H}^{s, \gamma}\left(\mathbb{R}_{+} \times \mathbb{R}^{q}, H\right)}:=\left\{\int_{\mathbb{R}^{q}} \int_{\Gamma_{\frac{n+2}{2}-\gamma}}[v, \eta]^{2 s}\left\|\kappa_{[v, \eta]}^{-1}\left(F_{y \rightarrow \eta} M_{t \rightarrow v} f\right)(v, \eta)\right\|_{H}^{2} d v d \eta\right\}^{\frac{1}{2}} \tag{1.34}
\end{equation*}
$$

The choice of the weight line $\Gamma_{\frac{n+2}{2}-\gamma}$ in (1.34) depends on the specific meaning of the space $H$ and then $n=n(H)$. For instance, we may have $H=\mathcal{K}^{s, \beta}\left(X^{\wedge}\right)$ for some real $\beta$. Then, according to our scheme of notation, $n=n(H)$ is the dimension of the model cone $X^{\Delta}$, which is equal to $1+\operatorname{dim} X$, i.e., $(1+n(H)) / 2-\gamma=(2+\operatorname{dim} X) / 2-\gamma$.

Later on it will be necessary to employ a relationship between norms (1.33), i.e., spaces like $\mathcal{W}^{s}\left(\mathbb{R}^{1+q}, H\right)$ and $\mathcal{H}^{s, \gamma}\left(\mathbb{R}_{+} \times \mathbb{R}^{q}, H\right)$ with the norm (1.34). We employ here the transformation

$$
\begin{equation*}
\left(S_{\gamma} u\right)(\boldsymbol{t}):=e^{-(1 / 2-\gamma) \boldsymbol{t}} u\left(e^{-\boldsymbol{t}}\right) \tag{1.35}
\end{equation*}
$$

cf., also [37, formula (2.1.28)] and we have an isomorphism $S_{\gamma}: C_{0}^{\infty}\left(\mathbb{R}_{+}\right) \rightarrow C_{0}^{\infty}(\mathbb{R})$. For the one-dimensional Fourier transform $\boldsymbol{F}_{\boldsymbol{t} \rightarrow \boldsymbol{\tau}}$ we then have the relation

$$
\begin{equation*}
\left(M_{\gamma} u\right)(1 / 2-\gamma+i \boldsymbol{\tau})=\left(\boldsymbol{F} S_{\gamma} u\right)(\boldsymbol{\tau}) \tag{1.36}
\end{equation*}
$$

Comparing (1.34) and (1.33) we get an isomorphism

$$
\begin{equation*}
S_{\gamma-(1+n) / 2}: \mathcal{H}^{s, \gamma}\left(\mathbb{R}_{+} \times \mathbb{R}^{q}, H\right) \rightarrow \mathcal{W}^{s}\left(\mathbb{R}_{t, y}^{1+q}, H\right) \tag{1.37}
\end{equation*}
$$

Let us now give a definition of (1.15) for any $d \in \mathbb{N}$ which includes the case $d=0$. We start with $L_{\mathrm{cl}}^{\mu}\left(X ; \mathbb{R}^{1+q+d}\right)$, the class of classical parameter-dependent pseudodifferential operators on $X$ of order $\mu$, with the parameters $(\rho, \eta, \lambda)$ in its natural Fréchet topology, and we consider functions

$$
\begin{equation*}
p(r, y, \rho, \eta, \lambda):=\tilde{p}(r, y, r \rho, r \eta, r \lambda) \tag{1.38}
\end{equation*}
$$

for

$$
\begin{equation*}
\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}, \tilde{\lambda}) \in C^{\infty}\left(\overline{\mathbb{R}}_{+, r} \times \mathbb{R}_{y}^{q}, L_{\mathrm{cl}}^{\mu}\left(X ; \mathbb{R}_{\tilde{\rho}, \tilde{\eta}, \tilde{\lambda}}^{1+q+d}\right)\right) \tag{1.39}
\end{equation*}
$$

Here the variables $y \in \mathbb{R}^{q}$ play the role of local coordinates on the edge belonging to a coordinate neighborhood on $Y$. Families of operators $r^{-\mu} \mathrm{Op}_{r, y}(p)(\lambda)$ are pseudo-differential analogues of edge-degenerate differential operators (1.4), here with parameters $\lambda \in \mathbb{R}^{d}$. Later on the parameter will be employed for formulating corner-degenerate families of operators. The degenerate behavior of operators for $r$ close to zero suggests to consider also classes of holomorphic Mellin symbols. For the definition we first recall the meaning of

$$
\begin{equation*}
M_{\mathcal{O}_{w}}^{\mu}\left(X ; \mathbb{R}_{\tilde{\eta}, \tilde{\lambda}}^{q+d}\right) \tag{1.40}
\end{equation*}
$$

defined to be the set of all

$$
\tilde{h}(w, \tilde{\eta}, \tilde{\lambda}) \in \mathcal{A}\left(\mathbb{C}_{w}, L_{\mathrm{cl}}^{\mu}\left(X ; \mathbb{R}^{q+d}\right)\right)
$$

which induce elements

$$
\tilde{h}(\alpha+i \rho, \tilde{\eta}, \tilde{\lambda}) \in L_{\mathrm{cl}}^{\mu}\left(X ; \Gamma_{\alpha} \times \mathbb{R}^{q+d}\right)
$$

for every real $\alpha$, uniformly in compact $\alpha$-intervals, cf., formula (1.25). We then look at operator functions

$$
\begin{equation*}
h(r, y, w, \eta, \lambda):=\tilde{h}(w, r \eta, r \lambda) \tag{1.41}
\end{equation*}
$$

such that

$$
\begin{equation*}
\tilde{h}(r, y, w, \tilde{\eta}, \tilde{\lambda}) \in C^{\infty}\left(\overline{\mathbb{R}}_{+, r} \times \mathbb{R}_{y}^{q}, M_{\mathcal{O}_{w}}^{\mu}\left(X ; \mathbb{R}_{\tilde{\eta}, \tilde{\lambda}}^{q+d}\right)\right) . \tag{1.42}
\end{equation*}
$$

Let us now give the explcit description of the edge-operator spaces (1.15). For convenience we content ourselfes to the case of a compact manifold $B$ with edge $Y$ of dimension $q$. The case of non-compact $B$ is a simple generalization when we give a definition in terms of countable locally finite covering of $B$ by wedge-neighborhoods when those intersect $Y$, otherwise by corresponding neighborhoods on the smooth manifold $B \backslash Y$. Recall that a wedge neighborhood close to $Y$ has the form of a wedge $X^{\Delta} \times \Omega$ for an open set $\Omega \subseteq Y$ and closed $X$.

Definition 1.2. The space $L^{\mu}\left(B, \boldsymbol{g} ; \mathbb{R}_{\lambda}^{d}\right), d \in \mathbb{N}$, of parameter-dependent edge pseudo-differential operators of order $\mu \in \mathbb{R}$ on a compact manifold $B$ with edge $Y$ and associated with weight data $\boldsymbol{g}=(\beta, \beta-\mu)$ is defined to be the set of all families of operators
$A_{\text {edge }}(\lambda):=\omega_{\text {glob }} A_{\text {sing }}(\lambda) \omega_{\text {glob }}^{\prime}+\left(1-\omega_{\text {glob }}\right) A_{\text {int }}(\lambda)\left(1-\omega_{\text {glob }}^{\prime \prime}\right)+C(\lambda)+(M+G)(\lambda)$ for global cut-off functions

$$
\begin{equation*}
\omega_{\text {glob }}^{\prime \prime} \prec \omega_{\text {glob }} \prec \omega_{\text {glob }}^{\prime}, \tag{1.44}
\end{equation*}
$$

cf., notation in Section 4 below, $A_{\text {int }}(\lambda) \in L_{\mathrm{cl}}^{\mu}\left(B \backslash Y ; \mathbb{R}_{\lambda}^{d}\right)$ and

$$
\begin{equation*}
A_{\operatorname{sing}}(\lambda):=\sum_{j=0}^{N} A_{j, \operatorname{sing}}(\lambda) \tag{1.45}
\end{equation*}
$$

for

$$
\begin{equation*}
A_{j, \text { sing }}(\lambda):=\varphi_{j}\left(\chi_{j}^{-1}\right)_{*} \mathrm{Op}_{y}\left(\left[r^{-\mu} \omega \mathrm{Op}_{M}^{\beta-n / 2}\left(h_{j}\right) \omega^{\prime}\right](y, \eta, \lambda)\right) \varphi_{j}^{\prime}, \tag{1.46}
\end{equation*}
$$

and

$$
\left.h_{j}(r, y, w, \eta, \lambda) \in C^{\infty}\left(\overline{\mathbb{R}}_{+, r} \times \mathbb{R}_{y}^{q}, M_{\mathcal{O}_{w}}^{\mu}\left(X ; \mathbb{R}_{\tilde{\eta}, \tilde{\lambda}}^{q+d}\right)\right)\right|_{\tilde{\eta}=r \eta, \tilde{\lambda}=r \lambda} ;
$$

$\left\{U_{1}, \ldots, U_{N}\right\}$ is an open covering of $Y$ by coordinate neighborhoods, $\left\{\varphi_{1}, \ldots, \varphi_{N}\right\}$ a subordinate partition of unity, $\varphi_{j}^{\prime} \succ \varphi_{j}$ are other functions in $C_{0}^{\infty}\left(U_{j}\right)$, and $\chi_{j}$ : $U_{j} \rightarrow \mathbb{R}^{q}$ are charts. Moreover, $\omega, \omega^{\prime}$ are cut-off functions on the $r$ half-axis. The smoothing elements $C(\lambda)$ and $(M+G)(\lambda)$ in (1.43) will also be defined in Section 4 below.

In order to keep notation more concise we also write (hopefully without creating new confusion)

$$
\begin{equation*}
A_{\operatorname{sing}}(\lambda)=\operatorname{Op}_{y}\left(\left[r^{-\mu} \omega \operatorname{Op}_{M}^{\beta-n / 2}(h) \omega^{\prime}\right](y, \eta, \lambda)\right) \tag{1.47}
\end{equation*}
$$

where $h$ is given by (1.41), (1.42). Later on we employ similar abbreviations for hinger singularities.

## 2. Corner differential operators

Spaces $L \in \mathfrak{M}_{2}$ with

$$
\begin{equation*}
s(L)=\left(s_{0}(L), s_{1}(L), s_{2}(L)\right) \tag{2.1}
\end{equation*}
$$

and assume

$$
\operatorname{dim} s_{2}(L)=0
$$

Such an $L$ is called a singular manifold with corner $s_{2}(L)$. By definition of the category $\mathfrak{M}_{2}$ there is a compact element $B \in \mathfrak{M}_{1}$ such that a cone-neighborhood $V_{2}$ of $s_{2}(L)$ in $L$ is modelled on $B^{\Delta}$. Moreover, the space $L \backslash s_{2}(L) \in \mathfrak{M}_{1}$ is a manifold with edge $s_{1}(L)$ of dimension $>0$ and there is a compact $X \in \mathfrak{M}_{0}$ such that there is a wedge neighborhood $V_{1}$ of $s_{1}(L)$ in $L$ with the structure of a (locally trivial) $X^{\triangle}$-bundle over $s_{1}(L)$. Furthermore we have $s_{0}(L) \in \mathfrak{M}_{0}$.

A differential operator on $s_{0}(L)$ is called corner-degenerate if it has the form

$$
\begin{equation*}
A_{\text {corner }}=t^{-\mu} \sum_{j=0}^{\mu} b_{j}(t)\left(-t \frac{\partial}{\partial t}\right)^{j} \tag{2.2}
\end{equation*}
$$

for coefficients $b_{j} \in C^{\infty}\left(\overline{\mathbb{R}}_{+}, \operatorname{Diff}_{\operatorname{deg}}^{\mu-j}(B)\right)$, cf., notation (1.17). The principal symbol hierarchy in this case consists of three components

$$
\begin{equation*}
\sigma\left(A_{\text {corner }}\right)=\left(\sigma_{0}\left(A_{\text {corner }}\right), \sigma_{1}\left(A_{\text {corner }}\right), \sigma_{2}\left(A_{\text {corner }}\right)\right) \tag{2.3}
\end{equation*}
$$

where $\sigma_{0}\left(A_{\text {corner }}\right)$ is the homogeneous principal symbol of $A_{\text {corner }}$ interpreted as an element of $\operatorname{Diff}^{\mu}\left(s_{0}(L)\right)$ and $\sigma_{1}\left(A_{\text {corner }}\right)$ is the operator-valued principal edge symbol, according to the interpretation $A_{\text {corner }} \in \operatorname{Diff}{ }^{\mu}\left(L \backslash s_{2}(L)\right)$, cf., also the relations (1.11), (1.12). Moreover, we have the corner conormal symbol which is a family of continuous operators

$$
\begin{equation*}
\sigma_{2}\left(A_{\text {corner }}\right)(v):=\sum_{j=0}^{\mu} b_{j}(0) v^{j}: H^{s, \beta_{\varphi}}(B) \rightarrow H^{s-\mu, \beta_{\varphi}-\mu}(B) \tag{2.4}
\end{equation*}
$$

with $v$ varying over $\Gamma_{(1+\operatorname{dimB}) / 2-\gamma}$ for some corner weight $\gamma \in \mathbb{R}$.
A space $D \in \mathfrak{M}_{2}$ with

$$
\begin{equation*}
s(D)=\left(s_{0}(D), s_{1}(D), s_{2}(D)\right) \tag{2.5}
\end{equation*}
$$

and $Z:=s_{2}(D) \in \mathfrak{M}_{0}$ of dimension $l>0$ is a manifold with edge of second singularity order, also indicated by edge. $D \backslash s_{2}(D) \in \mathfrak{M}_{1}$ is a manifold with edge $s_{1}(D)$ of dimension $>0$. There is a compact $B \in \mathfrak{M}_{1}$ and a wedge neighborhood $V_{2}$
 compact $X \in \mathfrak{M}_{0}$ such that there is a wedge neighborhood $V_{1}$ of $s_{1}(D)$ in $D \backslash s_{2}(D)$ with the structure of a (locally trivial) $X^{\Delta}$-bundle over $s_{1}(D)$. Furthermore, we have $s_{0}(D) \in \mathfrak{M}_{0}$.

A differential operator $A_{\text {edge }}$ on $s_{0}(D)$ is called edge-degenerate if it induces in a wedge neighborhood $V_{2}$ of $s_{2}(D)$ in local splittings of variables $(t, b, z) \in B^{\wedge} \times \mathbb{R}^{l}$ of the form

$$
\begin{equation*}
A_{\text {edge }}=t^{-\mu} \sum_{j+|\alpha| \leq \mu} b_{j, \alpha}(t, z)\left(-t \frac{\partial}{\partial t}\right)^{j}\left(t D_{z}\right)^{\alpha} \tag{2.6}
\end{equation*}
$$

for coefficients $b_{j, \alpha} \in C^{\infty}\left(\overline{\mathbb{R}}_{+} \times \mathbb{R}^{l}\right.$, $\left.\operatorname{Diff}_{\text {deg }}^{\mu-(j+|\alpha|)}(B)\right)$. Moreover, $A_{\text {edge }}$ as an operator on $D \backslash s_{2}(D) \in \mathfrak{M}_{1}$ in a wedge neighborhood $V_{1}$ of $s_{1}(D)$ in $D \backslash s_{2}(D)$ in the splitting of variables $(r, x, y) \in X^{\wedge} \times \mathbb{R}^{q}$ is of the form (2.6) for coefficients $a_{j, \alpha} \in$ $C^{\infty}\left(\overline{\mathbb{R}}_{+} \times \mathbb{R}^{q}, \operatorname{Diff}^{\mu-(j+|\alpha|)}(X)\right)$.

Similarly to notation (1.17) by

$$
\begin{equation*}
\operatorname{Diff}_{\mathbf{d e g}}^{\mu}(D) \tag{2.7}
\end{equation*}
$$

we denote the set of all degenerate differential operators of the form (2.6). A similar notation makes sense for the set of all operators (2.2), but this case is included in (2.7) anyway, since the dimension of $Z$ may also be equal to 0 . Since $\mu=0$ is an admitted case we also have smooth functions on a singular manifold $M \in \mathfrak{M}_{2}$ and we set

$$
\begin{equation*}
C^{\infty}(M):=\operatorname{Diff}_{\operatorname{deg}}^{0}(M) \tag{2.8}
\end{equation*}
$$

It also makes sense to define $C_{0}^{\infty}(M)$ of compactly supported elements of (2.8). Clearly we can define spaces of degenerate differential operators like $\operatorname{Diff}^{\mu}{ }_{\text {deg }}(M)$ for $M \in \mathfrak{M}_{k}$ for every $k \in \mathbb{N}$ and we also have corresponding spaces (2.8). For $k=0$ those coincide with the well-known spaces on a smooth manifold.

## 3. Weighted corner spaces

One of the examples of (1.34) are spaces $H:=\mathcal{K}^{s, \beta}\left(X^{\wedge}\right)$ with the group action

$$
\begin{equation*}
\left(\kappa_{\delta} u\right)(r, x)=\delta^{\frac{(1+n)}{2}} u(\delta r, x) \tag{3.1}
\end{equation*}
$$

A combination of the construction of (1.30) with the idea of abstract edge spaces gives us spaces

$$
\begin{equation*}
\mathcal{W}_{\text {cone }}^{s}\left(Y^{\wedge}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right) \tag{3.2}
\end{equation*}
$$

where $Y^{\wedge}$ is understood in a similar manner as $X^{\wedge}=X_{r, x}^{\wedge}$, and $Y^{\wedge}$ equipped with the variables $\tilde{y}=(t, y)$ with $y \in \mathbb{R}^{q}$ being local coordinates on the edge $Y$. In other words we imitate the arguments around (1.26)-(1.30) by first looking at a coordinate neighborhood $V$ on the unit sphere in $\mathbb{R}^{1+q}$ and form

$$
\begin{equation*}
V^{\wedge}:=\left\{\tilde{y} \in \mathbb{R}^{1+q} \backslash\{0\}: \frac{\tilde{y}}{|\tilde{y}|} \in V\right\} \tag{3.3}
\end{equation*}
$$

Then we form

$$
\begin{align*}
\mathcal{W}_{\text {cone }}^{s}\left(V_{\tilde{y}}^{\wedge}, \mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)\right) & :=\left\{\left.u(\tilde{y}) \in \mathcal{W}_{\mathrm{loc}}^{s}\left(\mathbb{R}_{\tilde{y}}^{1+q}, \mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)\right)\right|_{\mathbb{R}_{+} \times V}\right.  \tag{3.4}\\
& \left.:\left.(1-\sigma(|\tilde{y}|)) u \in \mathcal{W}^{s}\left(\mathbb{R}_{\tilde{y}}^{1+q}, \mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)\right)\right|_{\mathbb{R}_{+} \times V}\right\}
\end{align*}
$$

A next step in the definition of (3.2) is that we choose a covering of $Y$ by coordinate neighborhoods $\left\{U_{j}\right\}_{j=1, \ldots, N}$ and a subordinate partition of unity $\left\{\varphi_{j}\right\}_{j=1, \ldots, N}$. Then, for diffeomorphisms

$$
\begin{equation*}
\chi_{j}: U_{j} \longrightarrow V \tag{3.5}
\end{equation*}
$$

we form

$$
\begin{equation*}
\chi_{j}^{\wedge}: U_{j ; t, y}^{\wedge} \longrightarrow V_{\tilde{y}}^{\wedge} \quad \text { by setting } \quad \chi_{j}^{\wedge}(t, y)=t \chi_{j}(y)=: \tilde{y} \tag{3.6}
\end{equation*}
$$

and we set

$$
\begin{align*}
\mathcal{W}_{\text {cone }}^{s} & \left(U_{j ; t, y}^{\wedge}, \mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)\right) \\
& \left.:=\left\{u(t, y):=\left(\left(\chi_{j}^{\wedge}\right)_{*}^{-1}\right) v\right)(t, y): v \in \mathcal{W}_{\text {cone }}^{s}\left(V^{\wedge}, \mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)\right)\right\} \tag{3.7}
\end{align*}
$$

with $\left(\chi_{j}^{\wedge}\right)_{*}^{-1}$ indicating the push forward of $\mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)$-valued distributions from $V^{\wedge}$ to $U_{j}^{\wedge}$. Then we get spaces

$$
\begin{align*}
& \mathcal{W}_{\text {cone }}^{s}\left(Y_{t, y}^{\wedge}, \mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)\right) \\
&:=\left\{\sum_{j=1}^{N} \varphi_{j} u_{j}(t, y): u_{j} \in \mathcal{W}_{\text {cone }}^{s}\left(U_{j}^{\wedge}, \mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)\right), j=1, \ldots, N\right\} . \tag{3.8}
\end{align*}
$$

Moreover, we express the interior contributions, namely,

$$
\begin{equation*}
H_{\mathrm{int}}^{s}\left((B \backslash Y)^{\wedge}\right):=\left.H_{\text {cone }}^{s}\left((2 \mathbb{B})^{\wedge}\right)\right|_{(B \backslash Y)^{\wedge}} \tag{3.9}
\end{equation*}
$$

where $H_{\text {cone }}^{s}\left((2 \mathbb{B})^{\wedge}\right)$ is known from (1.30) since $2 \mathbb{B}$ is closed, and we form.

$$
\begin{equation*}
H_{\text {cone }}^{s, \beta}\left(B^{\wedge}\right):=\omega_{\text {glob }_{1}} \mathcal{W}_{\text {cone }}^{s}\left(Y_{t, y}^{\wedge}, \mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)\right)+\left(1-\omega_{\text {glob }_{1}}\right) H_{\mathrm{int}}^{s}\left((B \backslash Y)^{\wedge}\right) \tag{3.10}
\end{equation*}
$$

for a cut-off function $\omega_{\text {glob }_{1}}(r)$ on $B$ in the local variable $r$ close to $Y$. Next we employ the spaces

$$
\begin{equation*}
\mathcal{H}^{s, \gamma}\left(\mathbb{R}_{+} \times \mathbb{R}^{q}, H\right), \quad(t, y) \in \mathbb{R}_{+} \times \mathbb{R}^{q} \tag{3.11}
\end{equation*}
$$

for a Hilbert space with group action, here for $H=\mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)$ referring to (3.1) and $X_{r, x}^{\wedge}$ in the variables $(r, x)$. For the Mellin transform

$$
\begin{equation*}
M f(v)=\int_{\mathbb{R}_{+}} t^{v} f(t) \frac{d t}{t} \tag{3.12}
\end{equation*}
$$

the space (3.11) is defined by (1.34) while $F$ is the Fourier transform. We now choose charts $\lambda_{l}: U_{l} \longrightarrow \mathbb{R}^{q}$, and set

$$
\begin{align*}
& \mathcal{H}_{\text {edge }}^{s, \gamma}\left(\mathbb{R}_{+} \times U_{l}, H\right):=\left\{\left(\operatorname{id}_{\mathbb{R}_{+}} \times \lambda_{l}\right)_{*}^{-1} u: u \in \mathcal{H}^{s, \gamma}\left(\mathbb{R}_{+} \times \mathbb{R}^{q}, H\right)\right\}  \tag{3.13}\\
& \quad \mathcal{H}_{\text {edge }}^{s, \gamma}\left(Y^{\wedge}, H\right):=\left\{\sum_{j=1}^{N} \varphi_{l} u_{l}: u_{l} \in \mathcal{H}_{\text {edge }}^{s, \gamma}\left(\mathbb{R}_{+} \times U_{l}, H\right)\right\} \tag{3.14}
\end{align*}
$$

In other words, we defined the spaces

$$
\begin{equation*}
\mathcal{H}_{\mathrm{edge}}^{s, \gamma}\left(Y_{t, y}^{\wedge}, \mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)\right) \tag{3.15}
\end{equation*}
$$

for

$$
\begin{equation*}
\mathcal{H}^{s, \beta, \gamma}\left(B^{\wedge}\right):=\omega_{\mathrm{glob}_{1}} \mathcal{H}_{\mathrm{edge}}^{s, \gamma}\left(Y_{t, y}^{\wedge}, \mathcal{K}^{s, \beta}\left(X_{r, x}^{\wedge}\right)\right)+\left(1-\omega_{\mathrm{glob}_{1}}\right) \mathcal{H}_{\mathrm{int}}^{s, \gamma}\left((B \backslash Y)^{\wedge}\right) . \tag{3.16}
\end{equation*}
$$

Finally, we have

$$
\begin{equation*}
\mathcal{H}_{\text {int }}^{s, \gamma}\left((B \backslash Y)^{\wedge}\right):=\left.\mathcal{H}^{s, \gamma}\left((2 \mathbb{B})^{\wedge}\right)\right|_{(B \backslash Y)^{\wedge}} \tag{3.17}
\end{equation*}
$$

with $\mathcal{H}^{s, \gamma}\left((2 \mathbb{B})^{\wedge}\right)$ being the weighted Mellin Sobolev space known from the constructions before. Considering the space $B^{\wedge}$, indicating a structure on $\mathbb{R}_{+} \times B$ treated as an edge manifold with conical exit for $t \rightarrow \infty$, we form spaces

$$
\begin{equation*}
\mathcal{K}^{s, \beta, \gamma ; e}\left(B^{\wedge}\right):=[t]^{-e} \mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right) \tag{3.18}
\end{equation*}
$$

for

$$
\begin{equation*}
\mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right):=\sigma \mathcal{H}^{s, \beta, \gamma}\left(B^{\wedge}\right)+(1-\sigma) H_{\text {cone }}^{s, \beta}\left(B^{\wedge}\right) \tag{3.19}
\end{equation*}
$$

for some cut-off function $\sigma(t)$ on the $t$ half-axis. From (3.16) and (3.10) we have

$$
\left.\left.\begin{array}{rl}
\mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right) & =\sigma\left\{\omega_{\text {glob }_{1}} \mathcal{H}_{\text {edge } \left._{s, \gamma}^{s}\left(Y^{\wedge}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right)+\left(1-\omega_{\text {glob }_{1}}\right) \mathcal{H}_{\text {int }}^{s, \gamma}\left((B \backslash Y)^{\wedge}\right)\right\}}\right.  \tag{3.20}\\
& +(1-\sigma)\left\{\omega_{\text {glob }_{1}} \mathcal{W}_{\text {cone }}^{s}\left(Y^{\wedge}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right)+\left(1-\omega_{\text {glob }}^{1}\right.\right.
\end{array}\right) H_{\text {int }}^{s}\left((B \backslash Y)^{\wedge}\right)\right\} . .
$$

Note that the spaces do not depend on the involved cut-off functions. On elements $u(t, b)$ in (3.20) we define the group action

$$
\begin{equation*}
\Lambda:=\left\{\Lambda_{\delta}\right\}_{\delta \in \mathbb{R}_{+}} \quad \text { by } \quad\left(\Lambda_{\delta} u\right)(t, b)=\delta^{(1+\operatorname{dim} B) / 2} u(\delta t, b), \tag{3.21}
\end{equation*}
$$

and we obtain associated edge spaces

$$
\begin{equation*}
\mathcal{W}^{s}\left(\mathbb{R}_{z}^{l}, \mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right)\right) \tag{3.22}
\end{equation*}
$$

Those are again the source for higher corner analogues of weighted spaces of the kind (3.20) and (3.22).

Let $D \in \mathfrak{M}_{2}$ be a compact manifold with edge $Z$ of dimension $l>0$, locally near $Z$ modelled on $B^{\Delta} \times \mathbb{R}^{l}$, for compact $B \in \mathfrak{M}_{1}$ having an edge $Y$ of dimension $q>0$. Then

$$
\begin{equation*}
D_{\text {int }}=D \backslash Z \quad \text { is locally near } Z \text { identified with } \mathbb{R}_{+} \times B \times \mathbb{R}^{l} \tag{3.23}
\end{equation*}
$$

For references below we set

$$
\begin{equation*}
H^{s, \beta}(B):=\omega_{\text {glob }_{1}} \mathcal{W}^{s}\left(Y, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right)+\left(1-\omega_{\text {glob }_{1}}\right) H_{\text {loc }^{s}}^{s}\left(B_{\text {int }}\right), \tag{3.24}
\end{equation*}
$$

for $H_{\text {loc }}^{s, \beta}\left(B_{\text {int }}\right):=\left.H^{s, \beta}(2 \mathbb{B})\right|_{B_{\text {int }}}, B_{\text {int }}=B \backslash s_{1}(B)$, where

$$
\begin{equation*}
\mathcal{W}^{s}\left(Y, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right):=\sum_{j=1}^{N} \tilde{\varphi}_{j}\left(\tilde{\chi}_{j}^{-1}\right)_{*} \mathcal{W}^{s}\left(\mathbb{R}^{q}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right) \tag{3.25}
\end{equation*}
$$

for an open covering $\left\{\tilde{U}_{1}, \ldots, \tilde{U}_{N}\right\}$ of $Y$ by coordinate neighborhoods, a subordinate partition of unity $\left\{\tilde{\varphi}_{1}, \ldots, \tilde{\varphi}_{N}\right\}$, and charts $\tilde{\chi}_{j}: \tilde{U}_{j} \rightarrow \mathbb{R}^{q}$. Moreover, $\omega_{\text {glob }}$ is a global cut-off function on $B$ which is an element in $C^{\infty}\left(B_{\text {int }}\right)$ supported in a small wedge-neighborhood $V$ of $s_{1}(B)$ with $\omega_{\text {glob }_{1}} \equiv 1$ in a smaller wedge-neighborhood
$V_{1}$ of $s_{1}(B)$ such that $\overline{V_{1}} \subset V$. For any smoothness $s \in \mathbb{R}$ and weights $\beta, \gamma \in \mathbb{R}$ we have weighted edge Sobolev spaces

$$
\begin{equation*}
H^{s, \beta, \gamma}(D):=\omega_{\operatorname{glob}_{2}} \mathcal{W}^{s}\left(Z, \mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right)\right)+\left(1-\omega_{\mathrm{glob}_{2}}\right) H_{\mathrm{loc}}^{s, \beta}\left(D_{\mathrm{int}}\right) \tag{3.26}
\end{equation*}
$$

Here $H_{\text {loc }}^{s, \beta}\left(D_{\mathrm{int}}\right):=\left.H^{s, \beta}(2 \mathbb{D})\right|_{D_{\mathrm{int}}}$, cf., formula (3.23) and

$$
\begin{equation*}
\mathcal{W}^{s}\left(Z, \mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right)\right):=\sum_{j=1}^{N} \varphi_{j}\left(\chi_{j}^{-1}\right)_{*} \mathcal{W}^{s}\left(\mathbb{R}^{l}, \mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right)\right) \tag{3.27}
\end{equation*}
$$

for an open covering $\left\{U_{1}, \ldots, U_{N}\right\}$ of $Z$ by coordinate neighborhoods, a subordinate partition of unity $\left\{\varphi_{1}, \ldots, \varphi_{N}\right\}$, and charts $\chi_{j}: U_{j} \rightarrow \mathbb{R}^{l}$. Moreover, $\omega_{\text {glob }_{2}}$ is a global cut-off function on $D$ which is an element in $C^{\infty}\left(D_{\mathrm{int}}\right)$ supported in a small wedgeneighborhood $V_{2}$ of $s_{2}(D)$ in $D$ with $\omega_{\text {glob }_{2}} \equiv 1$ in a smaller wedge-neighborhood $V_{2}^{1}$ of $s_{2}(D)$ such that $\overline{V_{2}^{1}} \subset V_{2}$. Note that the space (3.26) is independent of the cut-off function $\omega_{\text {glob }_{2}}$. The following assertion is an analogue of the known Proposition 4.2 below, namely,
Theorem 3.1. We have

$$
\begin{equation*}
H^{s, \beta, \gamma}(D) \subset H_{\mathrm{loc}}^{s, \beta}\left(D_{\mathrm{int}}\right) \quad \text { and } \quad \psi H^{s, \beta, \gamma}(D)=\psi H_{\mathrm{loc}}^{s, \beta}\left(D_{\mathrm{int}}\right) \tag{3.28}
\end{equation*}
$$

for every $\psi \in C_{0}^{\infty}\left(D_{\mathrm{int}}\right)$, cf., notation (2.8).
Proof. Because of (3.26) and (3.27) it suffices to show

$$
\begin{equation*}
\left(1-\omega_{\mathrm{glob}_{2}}\right) \mathcal{W}^{s}\left(Z, \mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right)\right) \subset H_{\mathrm{loc}}^{s, \beta}\left(D_{\mathrm{int}}\right) \tag{3.29}
\end{equation*}
$$

for any cut-off function $\omega_{\text {glob }_{2}}$ and

$$
\begin{equation*}
\varphi \mathcal{W}^{s}\left(Z, \mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right)\right)=\varphi H_{\mathrm{loc}}^{s, \beta}\left(D_{\mathrm{int}}\right) \tag{3.30}
\end{equation*}
$$

for every $\varphi \in C_{0}^{\infty}\left(D_{\text {int }}\right)$. For the latter conclusions we may content ourselfes with $\mathcal{W}^{s}\left(\mathbb{R}^{l}, \mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right)\right)$ rather than $\mathcal{W}^{s}\left(Z, \mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right)\right)$ and $\varphi \in C_{0}^{\infty}\left(\mathbb{R}_{+} \times B \times \mathbb{R}^{l}\right)$ instead of the former $\varphi$. In other words, we verify

$$
\begin{equation*}
\varphi \mathcal{W}^{s}\left(\mathbb{R}^{l}, \mathcal{K}^{s, \beta, \gamma}\left(B^{\wedge}\right)\right)=\varphi H_{\mathrm{loc}}^{s, \beta}\left(\mathbb{R}_{+} \times B \times \mathbb{R}^{l}\right) \tag{3.31}
\end{equation*}
$$

The support of $\varphi$ may assumed to be contained in a $t$-interval $\left[\varepsilon_{0}, \varepsilon_{1}\right]$ for $\varepsilon_{0}>\varepsilon_{1}$ and $\varepsilon_{1}$ sufficiently small. The space (3.31) can be locally identified with the sum of spaces

$$
\begin{equation*}
\varphi \mathcal{W}^{s}\left(\mathbb{R}_{z}^{l}, \omega_{\text {glob }_{1}} \mathcal{H}_{\text {edge }}^{s, \gamma}\left(\mathbb{R}_{+} \times \mathbb{R}^{q}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right)\right) \tag{3.32}
\end{equation*}
$$

cf., (3.15), and

$$
\begin{equation*}
\varphi \mathcal{W}^{s}\left(\mathbb{R}_{z}^{l},\left(1-\omega_{\text {glob }_{1}}\right) \mathcal{H}_{\mathrm{int}}^{s, \gamma}\left((B \backslash Y)^{\wedge}\right)\right. \tag{3.33}
\end{equation*}
$$

cf. (3.17). By virtue of (1.37) in (3.32) we may replace $\mathcal{H}_{\text {edge }}^{s, \gamma}\left(\mathbb{R}_{+} \times \mathbb{R}^{q}, H\right)$ by $\mathcal{W}^{s}\left(\mathbb{R}_{t, y}^{1+q}, H\right)$ for $H=\mathcal{K}^{s, \beta}\left(X^{\wedge}\right)$; subscript "edge" may be dropped at this moment, and the corner weight $\gamma$ can be left out because of the factor $\varphi$ which is vanishing close to $t=0$. Moreover, the cut-off function $\omega_{\text {glob }_{1}}(r)$ is absorbed by the space $H$ and gives rise to a subspace of $H$, and this modification may be ignored. In other words, what in principle remains from (3.32) is a subspace of

$$
\begin{equation*}
\varphi \mathcal{W}^{s}\left(\mathbb{R}_{z}^{l}, \mathcal{W}^{s}\left(\mathbb{R}_{t, y}^{1+q}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right)\right. \tag{3.34}
\end{equation*}
$$

Now we may apply relation (4.8) from Section 4 below, and we thus obtain the first identity on the left-hand side of (3.28). Concerning the nature of the space (3.33) the factor $\varphi$ which is localizing far from $t=0$ makes the weight $\gamma$ disappear. Moreover, since the support of $\omega_{\text {glob }_{1}}$ may assumed to be concentated very close to $r=0$ and because we may look at our distribution in an open neighborhood on $B$ far from $Y$, where the function $1-\omega_{\text {glob }_{1}}$ already vanishes, the characterization of the space $\mathcal{H}_{\text {int }}^{s, \gamma}\left((B \backslash Y)^{\wedge}\right)$, in local coordinates $(t, \tilde{x}) \in \mathbb{R}^{1+\operatorname{dim} B}$ is a "usual" Sobolev space $H^{s}\left(\mathbb{R}^{1+\operatorname{dim} B}\right)$. Thus the space $(3.33)$ may be identified with

$$
\begin{equation*}
\mathcal{W}^{s}\left(\mathbb{R}_{z}^{l}, H^{s}\left(\mathbb{R}^{1+\operatorname{dim} B}\right)\right)=H^{s}\left(\mathbb{R}^{l+1+\operatorname{dim} B}\right) \tag{3.35}
\end{equation*}
$$

where we apply the identity (4.7) from Section 4 below.
Corollary 3.2. Together with Proposition 4.2 below it follows that

$$
\varphi \psi H_{\mathrm{loc}}^{s, \beta}\left(D_{\mathrm{int}}\right)=\varphi H_{\mathrm{loc}}^{s}\left(B_{\mathrm{int}}\right)
$$

for $B$ in the meaning of $D_{\mathrm{int}}$.

## 4. Appendix

The singular analysis is formulated here for specific stratified spaces with noncomplete metrices. In order to keep the material self-contained in this respect, we briefly outline some notation around singular manifolds, although such spaces have been discussed in other papers before, see, e.g., $[7,8]$. This will be an occation to complete further necessary aspects for the present exposition.

By $\mathfrak{M}_{0}$ we understand a category of $C^{\infty}$-manifolds with differentiable maps as morphisms and isomorphisms as diffeomorphisms. Moreover, $\mathfrak{M}_{k}$ is a category of topological spaces $M$, such that $\Omega \times M, M \times \Omega \in \mathfrak{M}_{k}$ for any $\Omega \in \mathfrak{M}_{0}$ and for $k>1$ every $M$ contains a subspace $s_{k}(M) \in \mathfrak{M}_{0}$ such that $M \backslash s_{k}(M)$ belongs to $\mathfrak{M}_{k-1}$. Moreover, $s_{k}(M)$ has a wedge-neighborhood $V \subset M$ which means it has the structure of an $X_{k-1}^{\Delta}$-bundle over $s_{k}(M)$ for a compact $X_{k-1} \in \mathfrak{M}_{k-1}$, cf. Remark 4.1 below, where the transition maps between the fibres $X_{k-1}^{\Delta}$ are induced by isomorphisms

$$
\begin{equation*}
\mathbb{R} \times X_{k-1} \rightarrow \mathbb{R} \times X_{k-1} \tag{4.1}
\end{equation*}
$$

in $\mathfrak{M}_{k-1}$ such that $(t, x) \rightarrow(\tilde{t}, \tilde{x})$ restricts to continuous maps $\overline{\mathbb{R}}_{+} \times X_{k-1} \rightarrow \overline{\mathbb{R}}_{+} \times$ $X_{k-1}$ and isomorphisms $X_{k-1} \rightarrow X_{k-1}$ in $\mathfrak{M}_{k-1}$.

Remark 4.1. For $M \in \mathfrak{M}_{k}, k>1$ the above-mentioned bundle $V$ in (i) over $s_{k}(M)$ has local trivializations $U \times X_{k-1}^{\triangle}$ for open sets $U \subseteq s_{k}(M)$. In local coordinates $y \in \mathbb{R}^{q}, q:=\operatorname{dim} s_{k}(M)$ we interpret $X_{k-1}^{\Delta}$ as an infinite straight cone with base (or link) $X_{k-1}^{\triangle}$ where the above-mentioned transition maps $\chi(y):(t, x) \rightarrow(\tilde{t}, \tilde{x})$ are homogeneous in the sense

$$
\begin{equation*}
\chi(y)(\delta t, x)=(\delta \tilde{t}, \tilde{x}) \quad \text { for all } \quad \delta \in \mathbb{R}_{+} \tag{4.2}
\end{equation*}
$$

The situation with $V$ is similar to a manifold $M \in \mathfrak{M}_{1}$ with smooth boundary $\partial M$, where a collar neighborhood of the boundary can be identified with $[0,1) \times \partial M$, and this in turn is isomorphic to $\overline{\mathbb{R}}_{+} \times \partial M$.

For $M \in \mathfrak{M}_{k}$ in general we call $X_{k-1}^{\Delta}$ the model cone in connection with the picture that $M$ locally close to $s_{k}(M)$ is modelled on a wedge $X_{k-1}^{\Delta} \times \mathbb{R}^{q}$. This is what we mean by " $V$ has the structure of such a cone bundle over $s_{k}(M)$ ". From the construction we have a stretched set $\mathbb{V}$ of $V$ with the structure of an $\overline{\mathbb{R}}_{+} \times X_{k-1^{-}}$bundle. This contains an $X_{k-1}$-bundle $\mathbb{V}_{\mathbb{O}}$ over $s_{k}(M)$ and we obtain $\mathbb{V}$ by invariantly attaching $\mathbb{V}_{\mathbb{O}}$ to $V \backslash s_{k}(M)$. The same can be done for $M$ itself, i.e., by attatching $\mathbb{V}_{\mathbb{O}}$ to $M \backslash s_{k}(M)$ we obtain the stretchend space $\mathbb{M}$ associated with $M$. By taking a second copy $\mathbb{M}_{-}$of $\mathbb{M}=: \mathbb{M}_{+}$and identifying points in $\mathbb{V}_{\mathbb{O}}$ on the corresponding $\pm$ sides, indicated by $\sim$, we obtain the double space

$$
\begin{equation*}
2 \mathbb{M}:=\left(\mathbb{M}_{-} \cup \mathbb{M}_{+}\right) / \sim \text { in } \mathfrak{M}_{k-1} . \tag{4.3}
\end{equation*}
$$

Example 2. Let $X$ be closed and $M:=X^{\Delta} \times \mathbb{R}^{q}$ belonging to $\mathfrak{M}_{1}$ which is a manifold with edge. Then we have $s_{1}(M)=\mathbb{R}^{q}$ identified with $\{0\} \times \mathbb{R}^{q}$. Moreover, $V=[0,1) \times X \times \mathbb{R}^{q}, \mathbb{V}_{\mathbb{O}}=X \times \mathbb{R}^{q}, \mathbb{M}=\overline{\mathbb{R}}_{+} \times X \times \mathbb{R}^{q}$ and $2 \mathbb{M}=\mathbb{R} \times X \times \mathbb{R}^{q}$.

The constructions give rise to a representation $M=\bigcup_{j=0}^{k} s_{j}(M)$ as a disjoint union of strata $s_{j}(M) \in \mathfrak{M}_{0}$ indicated by

$$
\begin{equation*}
s(M):=\left(s_{0}(M), s_{1}(M), \ldots, s_{k}(M)\right) \tag{4.4}
\end{equation*}
$$

where
(4.5) $\quad \operatorname{dim} M:=\operatorname{dim} s_{0}(M)>\operatorname{dim} s_{1}(M)>\cdots>\operatorname{dim} s_{k-1}(M)>\operatorname{dim} s_{k}(M) \geq 0$
by successively applying the definition of $\mathfrak{M}_{k}$.
This system of notation has been systematically used for manifolds $B$ with edge $Y$, where the dimension $q$ of $Y$ has been often assumed to be $>0$. But also the case $\operatorname{dim} Y=0$ makes sense; then the respective space has conical singularities and then $\mathbb{B}$ is locally close to $Y$ a cylinder the bottom of which equals $X \in \mathfrak{M}_{0}$. In any case,
according to (4.3), the space
is smooth.
Often we tacitly assume that $X$ is connected, but we also may admit that there are finitely many connected components $X_{j}, j=0, \ldots, N$, and speak about different weights associated with $X_{j}$ for different $j$. Such a situation only causes straighforward modifications of the calculus. We also may admit that base spaces $X_{j}$ of the respective model cones of wedges are of different dimensions. Such a case has been considered in [9] which illustates the role of weight normalizations in weighted Mellin operators referring to $\operatorname{dim} X_{j}$.

Our considerations around corner spaces employ some specific properties of edge spaces $\mathcal{W}^{s}\left(\mathbb{R}^{q}, H\right)$, cf., the norm expression (1.33) for some separable Hilbert space $H$ with group action, cf. the notation around (1.32). In particular, we employed [37, Proposition 3.1.21] which is based on

$$
\begin{equation*}
H^{s}\left(\mathbb{R}^{1+n} \times \mathbb{R}^{q}\right)=\mathcal{W}^{s}\left(\mathbb{R}^{q}, H^{s}\left(\mathbb{R}^{1+n}\right)\right) \tag{4.7}
\end{equation*}
$$

with $H^{s}\left(\mathbb{R}^{1+n}\right)$ being endowed with the group action $\left(\kappa_{\delta} f\right)(\tilde{x})=\delta^{(1+n) / 2} f(\delta \tilde{x})$, see, [37, Example 1.3.23]. A more general relation of this kind is

$$
\begin{equation*}
\mathcal{W}^{s}\left(\mathbb{R}^{p+q}, H\right)=\mathcal{W}^{s}\left(\mathbb{R}^{p}, \mathcal{W}^{s}\left(\mathbb{R}^{q}, H\right)\right) \tag{4.8}
\end{equation*}
$$

where $H$ is equipped with $\kappa=\left\{\kappa_{\delta}\right\}_{\delta \in \mathbb{R}_{+}}$and $\left.\mathcal{W}^{s}\left(\mathbb{R}^{q}, H\right)\right)$ with $\chi=\left\{\chi_{\lambda}\right\}_{\lambda \in \mathbb{R}_{+}}$for

$$
\begin{equation*}
\left(\chi_{\lambda} u\right)(y)=\kappa_{\lambda} \lambda^{q / 2} u(\lambda y) \quad \text { for } \quad u \in \mathcal{W}^{s}\left(\mathbb{R}^{q}, H\right) \tag{4.9}
\end{equation*}
$$

A proof is given in [37, Remark 1.3.43, Proposition 1.3.44].
Let $B$ be a compact manifold with edge $Y$ of dimension $q>0$; in particular, $B$ is locally close to $Y$ modelled on $X^{\Delta} \times \mathbb{R}^{q}$. For any smoothness $s \in \mathbb{R}$ and weight $\beta \in \mathbb{R}$ we have weighted edge Sobolev spaces $H^{s, \beta}(B)$, cf., (3.24).

Proposition 4.2. The space $(3.24)$ is independent of the cut-off function $\omega_{\text {glob }_{1}}$. In particular, we have

$$
\begin{equation*}
H^{s, \beta}(B) \subset H_{\mathrm{loc}}^{s}\left(B_{\mathrm{int}}\right) \quad \text { and } \quad \varphi H^{s, \beta}(B)=\varphi H_{\mathrm{loc}}^{s}\left(B_{\mathrm{int}}\right) \tag{4.10}
\end{equation*}
$$

for every $\varphi \in C_{0}^{\infty}\left(B_{\mathrm{int}}\right)$.
Since we did employ a similar proposition in more general context we briefly give the proof.

Proof. By virtue of $(3.24)$, (3.25), it suffices to show
(4.11) $(1-\omega) \mathcal{W}^{s}\left(\mathbb{R}^{q}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right) \subset H_{\mathrm{loc}}^{s}\left(X^{\wedge} \times \mathbb{R}^{q}\right) \quad$ for any cut-off function $\omega(r)$
and
(4.12) $\quad \varphi \mathcal{W}^{s}\left(\mathbb{R}^{q}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right)=\varphi H_{\mathrm{loc}}^{s}\left(X^{\wedge} \times \mathbb{R}^{q}\right) \quad$ for every $\quad \varphi \in C_{0}^{\infty}\left(X^{\wedge} \times \mathbb{R}^{q}\right)$.

We use the identity

$$
\begin{equation*}
\|u\|_{\mathcal{W}^{s}\left(\mathbb{R}^{q}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right)}^{2}=\int[\eta]^{2 s}\left\|[\eta]^{-(n+1) / 2} \hat{u}\left([\eta]^{-1} r, x, \eta\right)\right\|_{\mathcal{K}^{s, \beta}\left(X^{\wedge}\right)}^{2} d \eta . \tag{4.13}
\end{equation*}
$$

Without loss of generality we may assume $[\eta] \geq 1$ for all $\eta$. Consider the space

$$
\begin{equation*}
D_{\varepsilon}:=\left\{u \in \mathcal{W}^{s}\left(\mathbb{R}^{q}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right): \operatorname{supp} u \subset\{(\tilde{x}, y):|\tilde{x}| \geq \varepsilon\}\right\} \tag{4.14}
\end{equation*}
$$

Then $u \in D_{\varepsilon}$ implies that $\hat{u}(\tilde{x}, \eta)$ satisfies $\operatorname{supp} \hat{u} \subset\{(\tilde{x}, \eta):|\tilde{x}| \geq \varepsilon\}$ and then the same property holds for $\hat{v}(\tilde{x}, \eta):=\hat{u}\left([\eta]^{-1} r, x, \eta\right)$. In fact, $\hat{v}$ vanishes for $[\eta]^{-1}|\tilde{x}| \leq \varepsilon$ and hence, for $|\tilde{x}| \leq \varepsilon[\eta]$, i.e., it is supported by $|\tilde{x}| \geq \varepsilon$. Now the $\mathcal{K}^{s, \beta}\left(X^{\wedge}\right)$ )-norm on such functions is equivalent to their $H^{s}\left(\mathbb{R}^{1+n}\right)$-norm, i.e., we have an inequality

$$
\begin{equation*}
c_{1}(\varepsilon)\|\hat{v}(\cdot, \eta)\|_{H^{s}\left(\mathbb{R}^{1+n}\right)}^{2} \leq\|\hat{v}(\cdot, \eta)\|_{\left.\mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right)}^{2} \leq c_{2}(\varepsilon)\|\hat{v}(\cdot, \eta)\|_{H^{s}\left(\mathbb{R}^{1+n}\right)}^{2} \tag{4.15}
\end{equation*}
$$

for some constants $c_{i}(\varepsilon)>0$ which remains valid when we replace $\hat{v}(\cdot, \eta)$ by $[\eta]^{\varphi s-(1+n) / 2} \hat{v}(\cdot, \eta)$. Because of (4.13) and by virtue of (4.7) it follows that

$$
\begin{equation*}
c_{1}(\varepsilon)\|u\|_{H^{s}\left(\mathbb{R}^{1+n+q}\right)}^{2} \leq\|u\|_{\mathcal{W}^{s}\left(\mathbb{R}^{q}, \mathcal{K}^{s, \beta}\left(X^{\wedge}\right)\right)}^{2} \leq c_{2}(\varepsilon)\|u\|_{H^{s}\left(\mathbb{R}^{1+n+q}\right)}^{2} \tag{4.16}
\end{equation*}
$$

Let us finally add the explanation of $L_{\mathrm{cl}}^{\mu}\left(X ; \mathbb{R}_{\lambda}^{d}\right)$, which has been used in different constructions around pseudo-differential operators on a smooth manifold $X$ of dimension $n$. Modulo some standard globalization, in terms of a locally finite open covering of $X$ by coordinate neighborhoods and charts mapping to $\mathbb{R}_{x}^{n}$, taken as local coordinates, we assume operators to be of the form

$$
\begin{equation*}
A(\lambda):=\mathrm{Op}_{x}(a)(\lambda)+C(\lambda) \tag{4.17}
\end{equation*}
$$

for a classical symbol

$$
a\left(x, x^{\prime}, \xi, \lambda\right) \in S_{\mathrm{cl}}^{\mu}\left(\mathbb{R}_{x, x^{\prime}}^{2 n} \times \mathbb{R}_{\xi, \lambda}^{n+d}\right)
$$

Here, $S_{\mathrm{cl}}^{\mu}(\ldots)$ indicates classical symbols in Hörmander's sense (for $(\rho, \delta)=(1,0)$ ), "classical" means asymptotic expansions of symbols which are positively homogeneous of order $\mu-j, j \in \mathbb{N}$ for large $|\xi, \lambda|$, i.e., $\lambda \in \mathbb{R}^{d}$ is formally treated as a component of the covariable. Also the corresponding spaces of pseudo-differential operators with parameters, including $d=0$, will be equipped with subscript "cl". The larger classes without that subscript mean that the respective symbol estimates are required without asking the existence of homogeneous conponents of degree $\mu-j$. The operator $C(\lambda)$ is smoothing; for $d=0$ it is an integral operator with kernel in $C^{\infty}\left(\mathbb{R}_{x}^{n} \times \mathbb{R}_{x^{\prime}}^{n}\right)$; the corresponding global operator space $L^{-\infty}(X)$ is Fréchet, and for $d>0$ we ask $C(\lambda) \in \mathcal{S}\left(\mathbb{R}^{d}, L^{-\infty}(X)\right)$. More background may be found in articles or standard textbooks on pseudo-differential operators, cf., [17, 24, 30, 31].

A similar terminology is applied for pseudo-differential operators on a manifold with singularities with spaces of operator-valued symbols

$$
S^{\mu}\left(\mathbb{R}^{2 q} \times \mathbb{R}^{q+d} ; H, \tilde{H}\right)
$$

for separable Hilbert spaces $H$ and $\tilde{H}$ with group actions $\kappa=\left\{\kappa_{\delta}\right\}_{\delta \in \mathbb{R}_{+}}$and $\tilde{\kappa}=$ $\left\{\tilde{\kappa}_{\delta}\right\}_{\delta \in \mathbb{R}_{+}}$, respectively. Homogeneity (also called twisted homogeneity) of a function $f_{(\nu)} \in C^{\infty}\left(\Omega \times \mathbb{R}_{\eta, \lambda}^{q+d}, \mathcal{L}(H, \tilde{H})\right)$ of order $\nu \in \mathbb{R}$ for large $|\eta, \lambda|$ and open $\Omega \subseteq \mathbb{R}_{y}^{p}$ for some $p$ means in this case that

$$
f_{(\nu)}(y, \delta \eta, \delta \lambda)=\delta^{\nu} \kappa_{\delta} f_{(\nu)}(y, \eta, \lambda) \tilde{\kappa}_{\delta}^{-1}
$$

for $\delta \geq 1$ and $|\eta, \lambda| \geq c$ for some constant $c>0$. Concerning more material in this context, see [14, 35-37].

## Acknowledgment

This research project is partially supported by an NSF grant DMS-1408839 and a McDevitt Endowment Fund at Georgetown University. The paper is based on lectures which were given by the authors at China Medical University, Taichung, Taiwan They would like to express their profound gratitude to Professor Jen-Chih Yao for his invitation and for the warm hospitality extended to them during their stay in Taiwan.

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[^0]:    1991 Mathematics Subject Classification. Primary 35S35; Secondary 35J70.
    Key words and phrases. Boutet de Monvel's calculus, pseudo-differential operators, singular cones, Mellin symbols with values in the edge calculus, parametrices of elliptic operators, Kegel space.

