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BLOWUP OF SOLUTIONS FOR SOME NONLINEAR EVOLUTION EQUATIONS

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ABSTRACT. We use the method of nonlinear capacity to give a sufficient condition for the blowup of solutions for a generalized Burgers equation and the Kuznetsov-Zabolotskaya-Khokhlov equation in a finite spatial domain.

1. Introduction

The Burgers equation

$$u_t - uu_x + u_{xx} = 0$$

where $u = u(t, x)$ has many applications. It arises in the study of fluid dynamics, nonlinear acoustics, and traffic flows [3, 4, 5]. It is obtained as a result of combining nonlinear motion with linear diffusion.

In this paper, we consider the generalized Burgers equation in a finite spatial interval

$$u_t - u^{2p-1}u_x + \sum_{k=1}^{q} \frac{\partial^{2k} u}{\partial x^{2k}} = 0. \tag{1.1}$$

where $t > 0, 0 \leq x \leq L$, p = 1 or 2, and q is a positive integer. We will show that the solution blows up in finite time for a certain class of the initial-boundary value problem.

We will also consider the Kuznetsov-Zabolotskaya-Khokhlov equation

$$(u_t - uu_x + u_{xx})_x + u_{yy} = 0, \tag{1.2}$$

which is a two-dimensional generalization of the Burgers equation with a weakly perturbation in the y direction, where $u = u(t, x, y), t > 0, 0 \leq x \leq L, 0 \leq y \leq M$, It was derived for the description of nonlinear acoustic beams [2,11,18]. It was also used in the study of finite-amplitude compressional waves in solids [16]. The method that we are going to use is the nonlinear capacity method which was first developed by Pokhozhaev [15]. It has been very successful in giving sufficient conditions for the nonexistence of global classical solutions of many nonlinear evolution equations for a certain class of initial-boundary value problems [1, 6-10, 12-14, 17, 19].

2. A generalized Burgers equation

Theorem 2.1. *Given the generalized Burgers equation*

$$u_t - u^{2p-1}u_x + \sum_{k=1}^{q} \frac{\partial^{2k} u}{\partial x^{2k}} = 0.$$

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where $u(0,x) = u_0(x), t > 0, 0 \le x \le L$, *with the boundary conditions* $\partial^r u/\partial x^r = 0$ *at* $x = 0$, *for r =0, 1,..., 2q-1, p = 1 or 2, and q is a positive integer. Assume* $J_0 = \int_0^L (L-x)^m u dx > 0$, *where* $m > 4q$ *is a positive integer. Then there is no global classical solution for the above initial-boundary value problem.*

Proof. Multiplying both sides of the equation (1.1) by a test function h(x) which is smooth and depends on x only, we get

$$(hu)_t - \frac{1}{2p} h(u^{2p})_x + h \sum_{k=1}^{q} \frac{\partial^{2k} u}{\partial x^{2k}} = 0. \tag{2.1}$$

Assume $h^{(s)} = 0$ at x = L, for s = 0, 1,...,2q. Integrating both sides of equation (2.1) in x from 0 to L, we get

$$\frac{dJ}{dt} + \frac{1}{2p} \int_0^L h' u^{(2p)} dx + \sum_{k=1}^{q} \int_0^L h^{(2k)} u dx = 0, \tag{2.2}$$

where J = $\int_0^L hudx$. Let h = $(L-x)^m$, where m is a positive number. Then we get from (2.2) the equation

$$\frac{dJ}{dt} - \frac{m}{2p} \int_0^L (L-x)^{m-1} u^{2p} dx + \sum_{k=1}^{q} c_k \int_0^L (L-x)^{m-2k} u dx = 0 \tag{2.3}$$

where $c_k = m(m-1)\dots(m-2k+1)$. Thus from equation (2.3), we get

$$\begin{aligned} \frac{dJ}{dt} &= \frac{m}{2p} \int_0^L (L-x)^{m-1} u^{2p} dx - \sum_{k=1}^{q} c_k \int_0^L (L-x)^{m-2k} u dx \\ &\ge \frac{m}{2pL} \int_0^L (L-x)^{m} u^{2p} dx - \sum_{k=1}^{q} c_k \int_0^L (L-x)^{m-2k} u dx. \end{aligned}$$

Now

$$\begin{aligned} \left|\int_0^L (L-x)^{m-2k} u dx\right| &\le \left(\int_0^L (L-x)^{m-\frac{4kp}{2p-1}} dx\right)^{\frac{2p-1}{2p}} \left(\int_0^L (L-x)^m u^{2p} dx\right)^{\frac{1}{2p}} \\ &\le \left(\frac{2p-1}{2p}\right) b_k^{\frac{1}{1-2p}} \left(\int_0^L (L-x)^{m-\frac{4kp}{2p-1}} dx\right) + \left(\frac{b_k}{2p}\right)\left(\int_0^L (L-x)^m u^{2p} dx\right) \end{aligned}$$

where $b_k = \frac{m}{2qc_kL}$. Therefore

$$\begin{aligned}\frac{dJ}{dt} &\geq \frac{m}{2pL}\int_0^L (L-x)^m u^{2p}dx - \sum_{k=1}^q \frac{c_kb_k}{2p}\int_0^L (L-x)^m u^{2p}dx) \\ &\quad -\sum_{k=1}^q c_k(\frac{2p-1}{2p})b_k^{\frac{1}{1-2p}}(\int_0^L (L-x)^{m-\frac{4kp}{2p-1}}dx) \\ &= (\frac{m}{2pL} - \sum_{k=1}^q \frac{c_kb_k}{2p})(\int_0^L (L-x)^m u^{2p}dx) \\ &\quad -\sum_{k=1}^q c_k(\frac{2p-1}{2p})b_k^{\frac{1}{1-2p}}(\int_0^L (L-x)^{m-\frac{4kp}{2p-1}}dx) \\ &= \frac{m}{4pL}\int_0^L (L-x)^m u^{2p}dx \\ &\quad -\sum_{k=1}^q c_k(\frac{2p-1}{2p})b_k^{\frac{1}{1-2p}}(\int_0^L (L-x)^{m-\frac{4kp}{2p-1}}dx) \\ &\geq \frac{m}{4pL}(\frac{m+1}{L^{m+1}})^{2p-1}J^{2p} \\ &\quad -\sum_{k=1}^q c_k(\frac{2p-1}{2p})b_k^{\frac{1}{1-2p}}(\int_0^L (L-x)^{m-\frac{4kp}{2p-1}}dx).\end{aligned}$$

Thus we have

$$\frac{dJ}{dt} \geq \frac{m}{4pL}(\frac{m+1}{L^{m+1}})^{2p-1}J^{2p} - \sum_{k=1}^q c_k(\frac{2p-1}{2p})b_k^{\frac{1}{1-2p}}(\int_0^L (L-x)^{m-\frac{4kp}{2p-1}}dx). \tag{2.4}$$

Case 1. Let p = 1. We then have, from (2.4),

$$\frac{dJ}{dt} \geq A^2J^2 - B^2. \tag{2.5}$$

where $A = \sqrt{\frac{m(m+1)}{4L^{m+2}}}$ and $B = \sqrt{\sum_{k=1}^q \frac{c_k}{2} b_k^{-1} \int_0^L (L-x)^{m-4k}dx}$.

Consider the equation

$$\frac{dI}{dt} = A^2I^2 - B^2. \tag{2.6}$$

with $I(0) = J(0) = J_0$. Since $J_0 > 0$, we can easily show that $I = \frac{B(1+D)}{A(1-D)}$, where $D = \frac{AJ_0-B}{AJ_0+B}e^{2ABt}$.

Therefore from (2.5) and (2.6), we have $J(t) \geq I(t) = \frac{B(1+D)}{A(1-D)}$, Thus J(t) blows up as t approaches T for some T $\leq \frac{1}{2AB}\ln\frac{AJ_0+B}{AJ_0-B}$.

Case 2. p = 2. We have, from (2.4),

$$\frac{dJ}{dt} \geq E^4J^4 - F^4, \tag{2.7}$$

where E $= (\frac{m}{8L}(\frac{m+1}{L^{m+1}})^3)^{\frac{1}{4}}$ and F $= (\sum_{k=1}^q (\frac{3c_k}{4})b_k^{\frac{-1}{3}}(\int_0^L (L-x)^{m-\frac{8k}{3}}dx))^{\frac{1}{4}}$.

Consider the equation

$$\frac{dI}{dt} = E^4 I^4 - F^4. \tag{2.8}$$

with $I(0) = J(0) = J_0$. Since $J_0 > 0$, we can easily show that $\ln(\frac{EI-F}{EI+F}) - 2tan^{-1}(\frac{EI}{F}) = \ln(\frac{EJ_0-F}{EJ_0+F}) - 2tan^{-1}(\frac{EJ_0}{F}) + 4EF^3t$. Therefore we get

$$\frac{EI-F}{EI+F} \geq (\frac{EJ_0-F}{EJ_0+F})R, \tag{2.9}$$

where R $= e^{-\pi - 2tan^{-1}(\frac{EJ_0}{F}) + 4EF^3 t}$. Hence, from (2.9),

$$I \geq \frac{F(1+G)}{E(1-G)} \tag{2.10}$$

where G $= (\frac{EJ_0-F}{EJ_0+F})R$. Therefore, from (2.7), (2.8), and (2.10), we get $J(t) \geq I(t) = \frac{F(1+G)}{E(1-G)}$, Thus J(t) blows up as t approaches T for some T $\leq \frac{1}{4EF^3}(\pi + 2tan^{-1}(\frac{EJ_0}{F}) + \ln\frac{EJ_0+F}{EJ_0-F})$. □

Remark 2.2. This method can be applied to a generalized Korteweg-de Vries equation

$$u_t - u^{2p-1}u_x - \sum_{k=1}^{q} \frac{\partial^{2k+1}u}{\partial x^{2k+1}} = 0,$$

where p = 1 or 2.

3. The Kuznetsov-Zabolotskaya-Khokhove equation

Theorem 3.1. *Given the Kuznetsov-Zabolotskaya-Khokhlov equation*

$$(u_t - uu_x + u_{xx})_x + u_{yy} = 0$$

where $u = u(t,x,y), u(0,x,y) = u_0(x,y), t > 0, 0 \leq x \leq L, 0 \leq y \leq M$, *with the following boundary conditions at* $x = 0$: $\partial u/\partial t = 0, u = 0, \partial u/\partial x = 0$, *and* $\partial^2 u/\partial x^2 = 0$, *and at* $y = 0, u = 0$ *and* $\partial u/\partial y = 0$.

Assume $J_0 = \int_0^L \int_0^M (L-x)^m (M-y)^n u_0 dxdy > 0$, where m > 3 and n >2 are positive integers. Then there is no global classical solution for the above initial-boundary value problem in the bounded domain $0 \leq x \leq L, 0 \leq y \leq M$.

Proof. Multiplying both sides of the equation (1.2) by the test function $h(x,y) = (L-x)^{m+1}(M-y)^n$ and integrating from 0 to L in x and from 0 to M in y with the given boundary conditions, we get

$$\begin{aligned}(m+1)\frac{dJ}{dt} &= \int_0^M \int_0^L (m+1)m(L-x)^{m-1}(M-y)^n u^2 dxdy \\ &- \int_0^M \int_0^L (m+1)m(m-1)(L-x)^{m-2}(M-y)^n u dxdy \\ &- \int_0^M \int_0^L n(n-1)(L-x)^{m+1}(M-y)^{n-2} u dxdy,\end{aligned}$$

where $J = \int_0^M \int_0^L (L-x)^m (M-y)^n u dxdy$. Using the inequalities

$$\int_0^M \int_0^L (m+1)m(m-1)(L-x)^{m-2}(M-y)^n u dxdy$$
$$\leq \frac{1}{4}\int_0^M \int_0^L (m+1)^2 m^2 (m-1)^2 (L-x)^{m-3}(M-y)^n dxdy$$
$$+\int_0^M \int_0^L (L-x)^{m-1}(M-y)^n u^2 dxdy,$$

$$\int_0^M \int_0^L n(n-1)(L-x)^{m+1}(M-y)^{n-2} u dxdy$$
$$\leq \frac{1}{4}\int_0^M \int_0^L n^2(n-1)^2 (L-x)^{m+3}(M-y)^n dxdy$$
$$+\int_0^M \int_0^L (L-x)^{m-1}(M-y)^n u^2 dxdy,$$

and

$$\int_0^M \int_0^L (L-x)^{m-1}(M-y)^n u^2 dxdy \geq \frac{(\int_0^M \int_0^L (L-x)^m (M-y)^n u dxdy)^2}{\int_0^M \int_0^L (L-x)^{m+1}(M-y)^n dxdy},$$

we get

$$\frac{dJ}{dt} \geq R^2 J^2 - S^2 \tag{3.1}$$

where $R = \sqrt{\frac{(m+1)m-2}{(m+1)\int_0^M \int_0^L (L-x)^{m+1}(M-y)^n dxdy}}$ and
$S = \frac{1}{2\sqrt{m+1}}$
$\sqrt{\int_0^M \int_0^L [(m+1)^2 m^2 (m-1)^2 + n^2(n-1)^2(L-x)^6](L-x)^{m-3}(M-y)^n dxdy}.$

Following the same argument in the previous section, we see that J(t) blows up as t approaches T for some T $\leq \frac{1}{2RS}\ln\frac{RJ_0+S}{RJ_0-S}$, where $J_0 = J(0)$. □

Remark 3.2. The method can also be extended to the generalized Kuznetsov-Zabolotskaya-Khokhlov equation

$$(u_t - u^{2p-1}u_x + \sum_{k=1}^{q} \frac{\partial^{2k} u}{\partial x^{2k}})_x + u_{yy} = 0,$$

where p = 1 or 2, and the generalized Kadomtsev-Petviashvili equation

$$(u_t - u^{2p-1}u_x - \sum_{k=1}^{q} \frac{\partial^{2k+1} u}{\partial x^{2k+1}})_x + u_{yy} = 0,$$

where p = 1 or 2.

4. Conclusion

The method of nonlinear capacity offers a very elegant way for providing the conditions that the solutions of some nonlinear evolution equations would blow up in a finite time. However since the method depends on the test function that we use, it remains as an interesting problem to find the exact time that the solutions of such a nonlinear evolution equation would blow up.

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