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SEPARATION THEOREM VIA SUPERLINEAR FUNCTIONS AND CHARACTERIZATION OF MAXIMAL ELEMENTS IN MULTIOBJECTIVE OPTIMIZATION

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Abstract. This paper presents a version of the conic scalarization method, for characterizing maximal elements of nonlinear multiobjective optimization problems without the convexity condition. The original conic scalarization method uses monotonically increasing sublinear functions and characterizes minimal points. In this paper, a special class of monotonically increasing superlinear functions is used to prove a nonlinear separation theorem. This theorem is then used to prove characterization theorems for maximal elements of multiobjective problems, where the superlinear separating functions are used as scalarizing functions.

1. Introduction

Scalarization of multiobjective optimization problems, generally means a formulation of a suitable scalar optimization problem, optimal solutions of which allows to characterize efficient solutions of the given multiobjective problem.

One of the mostly used ways to scalarize the given multiobjective problem, is to define scalarizing functions which may involve some parameters or additional constraints. Multiobjective optimization methods utilize different scalarizing functions in different ways.

Among the most popular scalarizing functions used in the literature, the Gerstewitz functionals [5, 6, 19], and the sublinear monotone functionals used by Kasimbeyli [3, 15, 16] can be mentioned. These functions have strong mathematical justifications for scalarization.

In this paper, we continue to study the conic scalarization method introduced by Kasimbeyli, which use a specifically defined sublinear functions [16]. These functions are used to prove many interesting theorems for existence and characterization the efficient solutions [2, 11–14, 17, 20] in vector optimization. The conic scalarization method has a numerious applications in the literature (see, for example [4, 8, 18, 21])

It is proved in [15, 16] that the conic scalarization method, is able to generate all proper efficient solutions corresponding to the preferences of decision maker, such as the weights and the reference points, and does not require the restrictive conditions such as convexity, boundedness and differentiability.

The conic scalarization method given in [15, 16], is based on the nonlinear separation theorem, which uses sublinear functions $g_{w,\alpha}(y) = \langle w, y\rangle + \alpha\|y\|$, where the pair of parameters (w, α) is taken from the augmented dual cone C^{a*}. These functions

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were used to separate the cone $-C$ and some cone K which contains the objective space A. By this way, the whole apparatus of the conic scalarization method is constructed on the characterization of minimal elements. Since the function $g_{w,\alpha}(y) = \langle w, y\rangle + \alpha\|y\|$, used for this purpose, is neither odd nor even, the simple change of y to $-y$, does not lead to obvious justification of the separation theorem for the ordering cone C itself and some cone K (which contains the objective space A). On the other hand, such a change requires to justify the corresponding class of monotonically increasing functions which will be used as scalarizing functions for maximization problems.

To overcome these problems, in this paper we use a special class of monotonically increasing superlinear functions and prove a separation theorem for the ordering cone C and some cone K. These tools then are used to formulate a scalar problem and obtain characterization results for maximal points of multiobjective problems, via scalarization.

The rest of the paper is organized as follows. Section 2 presents a nonlinear separation theorem. The separation theorem is used in section 3 to prove different characterization theorems for maximal, weakly maximal and properly maximal elements of nonconvex sets. Finally, section 4 draws some conclusions from the paper.

2. Separation via superlinear functions

We begin this section by recalling the separation property and the separation theorem for two cones $-C$ and K, where C is an ordering cone in R^n and K is a closed cone with $-C \cap K = \{0\}$, such that $-C$ and K satisfy the separation property. After these explanations we will introduce a special class of superlinear functions and present a separation theorem for two cones C and K, by using these functions.

Let $(\mathbb{Y}, \|\cdot\|)$ be a real normed space whose partial ordering is induced by a closed convex pointed cone $\mathbb{C}$. Recall that the dual cone $\mathbb{C}^*$ of $\mathbb{C}$ and its quasi-interior $\mathbb{C}^\#$ are defined by

$$\mathbb{C}^* = \{y^* \in \mathbb{Y}^* : \langle y^*, y\rangle \geq 0 \text{ for all } y \in \mathbb{C}\} \tag{2.1}$$

and

$$\mathbb{C}^\# = \{y^* \in \mathbb{Y}^* : \langle y^*, y\rangle > 0 \text{ for all } y \in \mathbb{C} \setminus \{0\}\}, \tag{2.2}$$

respectively.

The following three cones called augmented dual cones of $\mathbb{C}$ were introduced in [15].

$$\mathbb{C}^{a*} = \{(y^*, \alpha) \in \mathbb{C}^\# \times \mathbb{R}_+ : \langle y^*, y\rangle - \alpha\|y\| \geq 0 \text{ for all } y \in \mathbb{C}\}, \tag{2.3}$$

$$\mathbb{C}^{a\circ} = \{(y^*, \alpha) \in \mathbb{C}^\# \times \mathbb{R}_+ : \langle y^*, y\rangle - \alpha\|y\| > 0 \quad \forall y \in \operatorname{int}(\mathbb{C})\}, \tag{2.4}$$

and

$$\mathbb{C}^{a\#} = \{(y^*, \alpha) \in \mathbb{C}^\# \times \mathbb{R}_+ : \langle y^*, y\rangle - \alpha\|y\| > 0 \quad \forall y \in \mathbb{C} \setminus \{0\}\}, \tag{2.5}$$

where $\mathbb{C}$ is assumed to have a nonempty interior in the definition of $\mathbb{C}^{a\circ}$.

The following relationship between the three kinds of augmented dual cones $\mathbb{C}^{a*}$, $\mathbb{C}^{a\circ}$, and $\mathbb{C}^{a\#}$ is straightforward from their definitions:

$$\mathbb{C}^{a\#} \subset \mathbb{C}^{a\circ} \subset \mathbb{C}^{a*}. \tag{2.6}$$

Definition 2.1. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a given function.

(a) g is called monotonically increasing, if $g(y) \geq g(z)$ for all $y, z \in \mathbb{R}^n$ such that $y - z \in \mathbb{C}$,

(b) g is called strictly monotonically increasing, if $g(y) > g(z)$ for all $y, z \in \mathbb{R}^n$ such that $y - z \in \mathsf{int}(\mathbb{C})$,

(c) g is called strongly monotonically increasing, if $g(y) > g(z)$ for all $y, z \in \mathbb{R}^n$ such that $y - z \in \mathbb{C} \setminus \{0_{\mathbb{R}^n}\}$.

The following theorem is proved in [15].

Theorem 2.2. *Let $\mathbb{C} \subset \mathbb{R}^n$ be the ordering cone, $w \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}_+$, and let function $\xi_{(w,\alpha)} : \mathbb{R}^n \to \mathbb{R}$ be defined as*

$$\xi_{(w,\alpha)}(y) = \langle w, y \rangle + \alpha \|y\|, \tag{2.7}$$

where $\|\cdot\|$ is a given norm. Then, $\xi_{(w,\alpha)}$ is monotonically increasing, strictly monotonically increasing, and strongly monotonically increasing on $\mathbb{R}^n$ if and only if $(w, \alpha) \in \mathbb{C}^{a*}$, $(w, \alpha) \in \mathbb{C}^{a\circ}$, and $(w, \alpha) \in \mathbb{C}^{a\#}$, respectively.

Now we recall the separation property introduced in [15].

Definition 2.3. Let $\mathbb{C}$ and $\mathbb{K}$ be closed cones in $\mathbb{R}^n$ with norm-bases $\mathbb{C}_{\mathbb{U}} = \mathbb{C} \cap \mathbb{U}$ and $\mathbb{K}_{\mathbb{U}} = \mathbb{K} \cap \mathbb{U}$, respectively, where $\mathbb{U} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Let $\mathbb{K}_{\mathbb{U}}^{\partial} = \mathbb{K}_{\mathbb{U}} \cap \mathsf{bd}(\mathbb{K})$, and let $\widetilde{\mathbb{C}}$ and $\widetilde{\mathbb{K}}^{\partial}$ be the closures of the sets $\mathsf{co}(\mathbb{C}_{\mathbb{U}})$ and $\mathsf{co}(\mathbb{K}_{\mathbb{U}}^{\partial} \cup \{0_{\mathbb{X}}\})$, respectively, where co denotes the convex hull. The cones $\mathbb{C}$ and $\mathbb{K}$ are said to satisfy the separation property with respect to the norm $\|\cdot\|$ if

$$\widetilde{\mathbb{C}} \cap \widetilde{\mathbb{K}}^{\partial} = \emptyset. \tag{2.8}$$

The following theorem is a finite dimensional version of Theorem 4.3 proved in [15] for reflexive Banach spaces. It concerns the existence of a pair $(w, \alpha) \in \mathbb{C}^{a\#}$ for which the corresponding sublevel set of the strongly monotonically increasing sublinear function $\xi_{w,\alpha}(y) = \langle w, y \rangle + \alpha \|y\|$ separates the given two cones $-\mathbb{C}$ and $\mathbb{K}$.

Theorem 2.4. *Let $\mathbb{C}$ and $\mathbb{K}$ be nonempty closed cones in $\mathbb{R}^n$. Then, $-\mathbb{C}$ and $\mathbb{K}$ satisfy the separation property*

$$-\widetilde{\mathbb{C}} \cap \widetilde{\mathbb{K}}^{\partial} = \emptyset, \tag{2.9}$$

defined in Definition 2.3, if and only if $\mathbb{C}^{a\#} \neq \emptyset$ and there exists a pair $(w, \alpha) \in \mathbb{C}^{a\#}$ such that the corresponding sublevel set of function $\xi_{w,\alpha}(y) = \langle w, y \rangle + \alpha \|y\|$ separates the cones $-\mathbb{C}$ and $\mathsf{bd}(\mathbb{K})$ in the following sense:

$$\langle w, y \rangle + \alpha \|y\| < 0 \leq \langle w, z \rangle + \alpha \|z\| \tag{2.10}$$

for all $y \in -\mathbb{C} \setminus \{0_{\mathbb{R}^n}\}$, and $z \in \mathsf{bd}(\mathbb{K})$. In this case the cone $-\mathbb{C}$ is pointed.

The separation Theorem 2.4 given above, is constructed on the separation of the negative ordering cone $-C$ and some cone K. This construction allows to characterize minimal points of some set. It is not obvious, how to apply the above theorem to the case of cones C and K. The aim of this paper is to formulate and prove a separation theorem which answers this question, and apply it in characterization of maximal elements.

First we introduce a class of superlinear functions and then give a separation theorem using these functions.

Lemma 2.5. *Let $\mathbb{R}^n$ be partially ordered by a pointed closed convex cone C. Let $y^* \in \mathbb{R}^n \backslash \{0\}$ and $\alpha \in \mathbb{R}_+$, and let the superlinear function $\zeta_{(y^*,\alpha)} : \mathbb{R}^n \to \mathbb{R}$ be defined as*

$$\zeta_{(y^*,\alpha)}(y) = \langle y^*, y\rangle - \alpha\|y\|. \tag{2.11}$$

Then, the function $\zeta_{(y^*,\alpha)}$ is monotonically increasing, strictly monotonically increasing (if $\operatorname{int}(\mathbb{C}) \neq \emptyset$), and strongly monotonically increasing on $\mathbb{R}^n$ if and only if $(y^*, \alpha) \in \mathbb{C}^{a*}$, $(y^*, \alpha) \in \mathbb{C}^{a\circ}$, and $(y^*, \alpha) \in \mathbb{C}^{a\#}$, respectively.

Proof. Sufficiency. Let $y_1, y_2 \in \mathbb{R}^n$ and $y_1 - y_2 \in \mathbb{C}$. Then,

$$\begin{aligned}&\zeta_{(y^*,\alpha)}(y_1) - \zeta_{(y^*,\alpha)}(y_2)\\ &= \langle y^*, y_1\rangle - \alpha\|y_1\| - \langle y^*, y_2\rangle + \alpha\|y_2\|\\ &\geq \langle y^*, y_1 - y_2\rangle - \alpha\|y_1 - y_2\|.\end{aligned}$$

Hence the statement follows from the definitions of $\mathbb{C}^{a*}$, $\mathbb{C}^{a\circ}$, and $\mathbb{C}^{a\#}$, respectively.

Necessity. Assume that for some pair (y^*, α) with $y^* \in \mathbb{R}^n \backslash \{0\}$ and $\alpha \in \mathbb{R}_+$, the function $\zeta_{(y^*,\alpha)}(y) = \langle y^*, y\rangle - \alpha\|y\|$ is monotonically increasing. Then, for every $y \in \mathbb{C}$ we have

$$\zeta_{(y^*,\alpha)}(y) = \langle y^*, y\rangle - \alpha\|y\| \geq \zeta_{(y^*,\alpha)}(0) = 0,$$

which by definition of $\mathbb{C}^{a*}$ implies that $(y^*, \alpha) \in \mathbb{C}^{a*}$. The proofs of $(y^*, \alpha) \in \mathbb{C}^{a\circ}$ and $(y^*, \alpha) \in \mathbb{C}^{a\#}$ in the cases if the function $\zeta_{(y^*,\alpha)}$ is strictly and strongly monotonically increasing respectively, are similar to that of the case given above. ☐

The proof of the following theorem regarding the separation via superlinear functions, is similar (but not trivial) to that of Theorem 2.4, where sublinear separating functions were used. We present the separation theorem in general form, in infinite dimensional reflexive Banach spaces.

Theorem 2.6. *Let C and K be closed cones in reflexive Banach space $\mathbb{Y}$ which satisfy the separation property* (2.8) *defined in Definition 2.3:*

$$\tilde{C} \cap \tilde{K}^{\partial} = \emptyset.$$

Then, $C^{a\#} \neq \emptyset$, and there exists a pair $(y^*, \alpha) \in C^{a\#}$ such that the strongly monotonically increasing superlinear function $\zeta_{(y^*,\alpha)}(y) = \langle y^*, y\rangle - \alpha\|y\|$ separates the cones C and $bd(K)$ in the following sense:

$$\langle y^*, k\rangle - \alpha\|k\| < 0 < \langle y^*, c\rangle - \alpha\|c\| \tag{2.12}$$

for all $k \in bd(\mathbb{K}) \setminus \{0_{\mathbb{Y}}\}$ and $c \in C \setminus \{0_Y\}$. In this case the cone C is pointed. Conversely, if there exists a pair $(y^*, \alpha) \in C^{a\#}$ such that the strongly monotonically *increasing superlinear function* $\zeta_{y^*,\alpha}(y) = \langle y^*, y\rangle - \alpha\|y\|$ *separates the cones* C *and* K *in the sense of* (2.12)*, and if either the cone* C *is closed and convex or* $(\mathbb{Y}, \|\cdot\|)$ *is a finite dimensional space, then the cones* C *and* K *satisfy the separation property* (2.8)*.*

Proof. Sufficiency. Let $\widetilde{\mathbb{C}} \cap \widetilde{\mathbb{K}}^{\partial} = \emptyset$. It follows from the definitions of these sets that they are subsets of the unit ball $\mathbb{B}$ which is weakly compact by reflexivity of $\mathbb{Y}$. Moreover, since $\widetilde{\mathbb{C}}$ and $\widetilde{\mathbb{K}}^{\partial}$ are closed and convex sets, they are weakly closed (see [9, Theorem 3.24]) and hence weakly compact. Then, by the James's theorem [10], there is a nonzero continuous linear functional y^* over $\mathbb{Y}$ such that the following separation relation is satisfied:

$$\sup\{\langle y^*, k\rangle : k \in \widetilde{\mathbb{K}}^{\partial}\} < \inf\{\langle y^*, c\rangle : c \in \widetilde{\mathbb{C}}\}. \tag{2.13}$$

Let $\inf\{\langle y^*, c\rangle : y \in \widetilde{\mathbb{C}}\} = \gamma$. Then, since $0 \in \widetilde{\mathbb{K}}^{\partial}$, by (2.13) we have $\gamma > 0$, and there exists a positive number $\varepsilon < \gamma$ such that

$$\langle y^*, k\rangle + \varepsilon < \gamma \le \langle y^*, c\rangle \text{ for all } c \in \widetilde{\mathbb{C}} \text{ and } k \in \widetilde{\mathbb{K}}^{\partial}. \tag{2.14}$$

Since $\mathbb{C}_{\mathbb{U}} \subset \widetilde{\mathbb{C}}$ and $\mathbb{K}^{\partial}_{\mathbb{U}} \subset \widetilde{\mathbb{K}}^{\partial}$, the inequalities (2.14) imply

$$\langle y^*, k\rangle + \varepsilon < \gamma \le \langle y^*, c\rangle \text{ for all } c \in \mathbb{C}_{\mathbb{U}} \text{ and } k \in \mathbb{K}^{\partial}_{U}. \tag{2.15}$$

Every element c of $\mathbb{C} \setminus \{0_{\mathbb{Y}}\}$ can be represented as $c = \beta y$ for some $\beta > 0$ and $y \in \mathbb{C}_{\mathbb{U}}$. Then, from (2.15) and from the fact that $\gamma > 0$, we obtain

$$\langle y^*, c\rangle > 0 \text{ for all } c \in \mathbb{C} \setminus \{0_{\mathbb{Y}}\}, \tag{2.16}$$

which implies that $y^* \in \mathbb{C}^{\#}$.

Now, since $\|k\| = 1$ for each $k \in \mathbb{K}^{\partial}_{\mathbb{U}}$, it follows from (2.15) that

$$\langle y^*, k\rangle - (\gamma - \varepsilon)\|k\| < 0 \text{ for all } k \in \mathbb{K}^{\partial}_{U}$$

or

$$\langle y^*, k\rangle - (\gamma - \varepsilon)\|k\| < 0 \text{ for all } k \in bd(\mathbb{K}) \setminus \{0_{\mathbb{Y}}\}. \tag{2.17}$$

In a similar way, since $\|c\| = 1$ for each $c \in \mathbb{C}_{\mathbb{U}}$, it follows from (2.15) that

$$\langle y^*, c\rangle - (\gamma - \varepsilon)\|c\| \ge \gamma - (\gamma - \varepsilon) = \varepsilon > 0 \text{ for all } c \in \mathbb{C}_{\mathbb{U}},$$

which implies that

$$\langle y^*, c\rangle - (\gamma - \varepsilon)\|c\| > 0 \text{ for all } c \in \mathbb{C} \setminus \{0_{\mathbb{Y}}\}. \tag{2.18}$$

The last inequality means that $(y^*, \gamma - \varepsilon) \in \mathbb{C}^{a\#}$. Then, Corollary 3.3 in [15] implies that $\mathbb{C}$ is pointed. Thus, the sufficiency of the theorem is proved.

Necessity. (a) First suppose that $(\mathbb{Y}, \|\cdot\|)$ is a finite dimensional space and that, there exists a pair $(y^*, \alpha) \in \mathbb{C}^{a\#}$ such that

$$\langle y^*, k\rangle - \alpha\|k\| < 0 < \langle y^*, c\rangle - \alpha\|c\| \tag{2.19}$$

for all $c \in \mathbb{C}\setminus\{0_{\mathbb{Y}}\}$ and $k \in bd(\mathbb{K})\setminus\{0_{\mathbb{Y}}\}$. Then, since $\mathbb{C}_{\mathbb{U}} \subset \mathbb{C}$ and $\mathbb{K}^{\partial}_{\mathbb{U}} \subset bd(\mathbb{K})\setminus\{0_{\mathbb{Y}}\}$, the inequalities (2.19) are true also for all $c \in \mathbb{C}_{\mathbb{U}}$ and $k \in \mathbb{K}^{\partial}_{\mathbb{U}}$. We have $\|c\| = 1 = \|k\|$ for all $c \in \mathbb{C}_{\mathbb{U}}$ and $k \in \mathbb{K}^{\partial}_{\mathbb{U}}$, which together with (2.19) implies that

$$\langle y^*, k\rangle < \alpha < \langle y^*, c\rangle \tag{2.20}$$

for all $c \in \mathbb{C}_{\mathbb{U}}$ and $k \in \mathbb{K}^{\partial}_{\mathbb{U}}$. Due to the compactness of $\mathbb{C}_{\mathbb{U}}$ and the continuity of y^*, there exists a positive number ε such that the inequalities (2.20) can also be written in the form

$$\langle y^*, k\rangle < \alpha - \varepsilon < \alpha < \langle y^*, c\rangle \tag{2.21}$$

for all $c \in \mathbb{C}_{\mathbb{U}}$ and $k \in \mathbb{K}^{\partial}_{\mathbb{U}}$. These inequalities mean that the sets $\mathbb{K}^{\partial}_{\mathbb{U}} \cup \{0_{\mathbb{Y}}\}$ and $\mathbb{C}_{\mathbb{U}}$ are contained in the closed half-spaces $H^-(y^*, \alpha - \varepsilon) = \{y \in Y : \langle y^*, y\rangle \leq \alpha - \varepsilon\}$ and $H^+(y^*, \alpha) = \{z \in Y : \langle y^*, z\rangle \geq \alpha\}$, respectively. Since $\alpha - \varepsilon > 0$, and the half-spaces $\mathbb{H}^-(y^*, \alpha - \varepsilon)$ and $\mathbb{H}^+(y^*, \alpha)$ are closed convex sets, they also contain the closures $\widetilde{\mathbb{K}}^{\partial}$ and $\widetilde{\mathbb{C}}$ of the sets $co(\mathbb{K}^{\partial}_{\mathbb{U}} \cup \{0_{\mathbb{Y}}\})$ and $co(\mathbb{C}_{\mathbb{U}})$ respectively, and since the half-spaces $\mathbb{H}^-(y^*, \alpha - \varepsilon)$ and $\mathbb{H}^+(y^*, \alpha)$ are disjoint, consequently so are $\widetilde{\mathbb{K}}^{\partial}$ and $\widetilde{\mathbb{C}}$, which completes the proof.

Necessity. (b) Now suppose that $\mathbb{C}$ is a closed and convex cone of a reflexive Banach space $(\mathbb{Y}, \|\cdot\|)$. In this case the set $\widetilde{\mathbb{C}}$ is a subset of $\mathbb{C}$, and therefore (2.19) is also satisfied for all $y \in \widetilde{\mathbb{C}}$. Now, since $\widetilde{\mathbb{C}}$ is weakly compact and the function $\zeta(y) = \langle y^*, y\rangle - \alpha\|y\|$ is weakly lower semicontinuous, there exists some positive number ε with

$$\langle y^*, k\rangle - \alpha\|k\| \leq -\varepsilon < 0 < \langle y^*, c\rangle - \alpha\|c\| \tag{2.22}$$

for all $c \in \widetilde{\mathbb{C}}$ and $k \in bd(\mathbb{K}) \setminus \{0_{\mathbb{Y}}\}$. Then (2.21) follows from (2.22), and the remaining part of the proof is similar to the proof of part (a). The proof is completed. □

The class of monotonically increasing superlinear functions given in this section, are used in the following section, to formulate scalarization theorems for maximization problems in multiobjective optimization.

3. Scalarization

Let $\mathbb{X} \subset R^m$ be a nonempty set and let $f_i : \mathbb{X} \to \mathbb{R}, i = 1, ..., n$ are real-valued functions. Let $f(x) = (f_1(x), \ldots, f_n(x))$ for every $x \in \mathbb{X}$ and let $\mathbb{A} := f(\mathbb{X})$. We will assume in this paper that R^n is partially ordered by a closed convex cone $\mathbb{C} \subset R^n$.

Consider the following multi-objective optimization problem:

$$\max_{x\in\mathbb{X}} [f_1(x), ..., f_n(x)]. \tag{3.1}$$

Remark 3.1. Let $(y^*, \alpha) \in C^{a*}$ and let $r \in R^n$ be an arbitrary vector. Consider the function

$$\zeta_{(y^*,\alpha,r)}(y) = \langle y^*, y - r\rangle - \alpha\|y - r\|. \tag{3.2}$$

Then it is clear that (see Lemma 2.5), $\zeta_{(y^*,\alpha,r)}$ is monotonically increasing, strictly monotonically increasing and strongly monotonically increasing if and only if $(y^*, \alpha) \in C^{a*}$, $(y^*, \alpha) \in C^{a\circ}$ and $(y^*, \alpha) \in C^{a\#}$, respectively. Moreover, it follows from the definitions of augmented dual cones that,

$$\{r\} + C \subset \{y \in R^n : \langle y^*, y - r\rangle - \alpha\|y - r\| \geq 0\} \tag{3.3}$$

for every $(y^*, \alpha) \in C^{a*}$,

$$\{r\} + C \setminus \{0\} \subset \{y \in R^n : \langle y^*, y - r\rangle - \alpha\|y - r\| > 0\} \tag{3.4}$$

for every $(y^*, \alpha) \in C^{a\#}$ and

$$\{r\} + int(C) \subset \{y \in R^n : \langle y^*, y - r\rangle - \alpha\|y - r\| > 0\} \tag{3.5}$$

for every $(y^*, \alpha) \in C^{a\circ}$.

Lemma 3.2. *Let $\mathbb{C}$ be a nonempty cone of a real normed space $(\mathbb{Y}, \|\cdot\|)$, let $(y^*, \alpha) \in \mathbb{C}^{a*}$ and let*

$$C(y^*, \alpha) = \{y \in \mathbb{Y} : \langle y^*, y\rangle - \alpha\|y\| \geq 0\}. \tag{3.6}$$

Then, the following statements hold true.

(i) *For each $(y^*, \alpha) \in \mathbb{C}^{a*}$, the set $C(y^*, \alpha)$ is a closed convex cone containing $\mathbb{C}$. Moreover, it is pointed if $\alpha > 0$.*

(ii) *If $(y^*, \alpha) \in \mathbb{C}^{a\#}$, then*

$$\mathrm{int}(C(y^*, \alpha)) = \{y \in \mathbb{Y} : \langle y^*, y\rangle - \alpha\|y\| > 0\} \neq \emptyset \tag{3.7}$$

and $\mathbb{C} \setminus \{0\} \subset \mathrm{int}(C(y^*, \alpha))$.

Proof. The proof is omitted because it is similar to the proof of Lemma 3.6 in [15]. □

The following theorem gives characterization of maximal elements of multiobjective optimization problem (3.1), in terms of solutions of scalar optimization problem with the objective function ζ defined in (3.2).

Theorem 3.3. *Let $\mathbb{A} = f(X) \subset \mathbb{R}^n$ be a given nonempty set and let $r \in \mathbb{R}^n$ be a given vector. Denote $\mathbb{C} = \mathbb{R}^n_+$. Let*

$$\begin{aligned}(w, \alpha) \in \mathbb{C}^{a*} = \{&((w_1, \dots, w_n), \alpha) :\\ &0 \leq \alpha \leq w_i, w_i > 0, i = 1, \dots, n\},\end{aligned}$$

and let $Sol(SP(w, \alpha, r))$ be the set of optimal solutions of the scalar optimization problem

$$\max_{y\in\mathbb{A}}\{\langle w, y - r\rangle - \alpha\|y - r\|_1\}. \tag{3.8}$$

Suppose that $Sol(SP(w, \alpha, r)) \neq \emptyset$ for a given pair $(w, \alpha) \in \mathbb{C}^{a*}$. Then the following hold.

(i) *If*

$$\begin{aligned}(w, \alpha) \in \mathbb{C}^{a\circ} = \{&((w_1, \dots, w_n), \alpha) : 0 \leq \alpha \leq w_i,\\ &w_i > 0, i = 1, \dots, n \ \text{ and there exists}\\ &k \in \{1, \cdots, n\} \text{ such that } w_k > \alpha\},\end{aligned}$$

then every element of $Sol(SP(w, \alpha, r))$ is a weakly maximal element of $\mathbb{A}$.

(ii) *If $Sol(SP(w, \alpha, r))$ consists of a single element, then this element is a maximal element of $\mathbb{A}$.*

(iii) *If*

$$(w, \alpha) \in \mathbb{C}^{a\#} = \{((w_1, \dots, w_n), \alpha) : 0 \leq \alpha < w_i, i = 1, \dots, n\},$$

then then every element of $Sol(SP(w, \alpha, r))$ *is a properly maximal element of* $\mathbb{A}$ *(in the sense of both Henig and Benson [1]).*

Proof. (i) Let $(w, \alpha) \in \mathbb{C}^{a\circ}$ and $\overline{y} \in Sol(SP)$. Assume to the contrary that $\overline{y}$ is not a weakly maximal element of of $\mathbb{A}$. Then, by definition of the weak maximality, there exists an element $a\ in \mathbb{A}$ such that $a > \overline{y}$. Then since the function $\zeta_{(w,\alpha,r)}(y) = \langle w, y - r\rangle - \alpha\|y - r\|_1$ is strictly monotonically increasing (see Remark 3.1), this leads to $\zeta_{(w,\alpha,r)}(a) > \zeta_{(w,\alpha,r)}(\overline{y})$, which is a contradiction.

(ii) Let $(w, \alpha) \in \mathbb{C}^{a*}$ and let $Sol(SP(w, \alpha, r))$ consists of a single element $\overline{y}$. Assume that this element is not a maximal element of $\mathbb{A}$. Then by definition of the maximality, there exists an element $a\ in \mathbb{A}$ such that $a \geq \overline{y}$. Then since the function $\zeta_{(w,\alpha,r)}(y) = \langle w, y-r\rangle - \alpha\|y-r\|_1$ is monotonically increasing (see Remark 3.1), this leads to $\zeta_{(w,\alpha,r)}(a) \geq \zeta_{(w,\alpha,r)}(\overline{y})$. Since $\overline{y}$ is a single solution, this is a contradiction.

(iii) Let $(w, \alpha) \in \mathbb{C}^{a\#}$ and let $\overline{y} \in Sol(SP(w, \alpha, r))$. Then we have:

$$\langle w, y - r\rangle - \alpha\|y - r\| \leq \langle w, \overline{y} - r\rangle - \alpha\|\overline{y} - r\|$$

for every $y \in \mathbb{A}$. This implies

$$\langle w, y - \overline{y}\rangle - \alpha\|y - \overline{y}\| \leq 0 \tag{3.9}$$

for every $y \in \mathbb{A}$. On the other hand, since $\{\overline{y}\} + C(w, \alpha) = \{y : \langle w, y - \overline{y}\rangle - \alpha\|y - \overline{y}\| \geq 0\}$, we obtain by (3.9) that, $\overline{y} \in \mathbb{A} \cap \{\overline{y}\} + C(w, \alpha)$. If there was a point $\{a\} \in \mathbb{A} \cap (\{\overline{y}\} + C(w, \alpha))$ with $a \neq \overline{y}$, this would lead to $\zeta_{(w,\alpha,r)}(a) > \zeta_{(w,\alpha,r)}(\overline{y})$ since $(w, \alpha) \in \mathbb{C}^{a\#}$. Hence it is proved that $\overline{y}$ is a maximal element of A with respect to the pointed closed convex cone $C(w, \alpha)$. Now since by Lemma 3.2, $\mathbb{C} \setminus \{0\} \subset \mathrm{int}(C(y^*, \alpha))$, we obtain that $\overline{y}$ is a properly maximal element of A in the sense of Henig. Finally, from the equivalence of Benson and Henig proper efficiencies (see, e.g. [7, Theorem 2.1]), it follows that $\overline{y}$ is also a Benson properly minimal element. □

The following theorem presents a necessary and sufficient condition for the given point to be a Henig properly maximal element. This theorem actually demonstrates that every properly maximal element of a given set can be generated by the scalarization method suggested in this paper.

Theorem 3.4. *Let* $\mathbb{A} \subset \mathbb{R}^n$ *be a given nonempty set and let* $\mathbb{C} = \mathbb{R}^n_+$. *The point* $\overline{y} \in \mathbb{A}$ *is a Henig properly maximal element of* $\mathbb{A}$ *with respect to* $\mathbb{C}$ *if and only if there exists a pair*

$$(w, \alpha) \in \mathbb{C}^{a\#} = \{((w_1, \dots, w_n), \alpha) : 0 \leq \alpha < w_i, i = 1, \dots, n\}$$

such that the scalar optimization problem

$$\max_{y \in \mathbb{A}} \{\langle w, y - \overline{y}\rangle - \alpha\|y - \overline{y}\|_1\} \tag{3.10}$$

attains its maximum at $\overline{y}$ *that is, (3.9) holds.*

Proof. **Necessity.** Assume that $\overline{y}$ is a properly maximal element of $\mathbb{A}$ in the sense of Henig. Then by the definition, $\overline{y}$ is a maximal element of $\mathbb{A}$ with respect to some convex cone $\mathbb{K}$ with $\mathbb{R}^n_+ \setminus \{0_{\mathbb{R}^n}\} \subset \mathsf{int}(\mathbb{K})$. By Theorem [16, Theorem 3], the cones $\mathbb{R}^n_+$ and $\mathbb{K}$ satisfy the separation property given in Definition 2.3, with respect to the l_1 norm. Then, by Theorem 2.6, there exists a pair $(w, \alpha) \in \mathbb{C}^{a\#}$ such that

$$\langle w, k\rangle - \alpha\|k\|_1 < 0 < \langle w, c\rangle - \alpha\|c\|_1$$

for all $k \in \mathsf{bd}(\mathbb{K}) \setminus \{0_{\mathbb{R}^n}\}$ and $c \in \mathbb{R}^n_+ \setminus \{0_{\mathbb{R}^n}\}$. Then, since $\overline{y}$ is a maximal element of $\mathbb{A}$ with respect to $\mathbb{K}$, $\mathbb{R}^n_+ \setminus \{0_{\mathbb{R}^n}\} \subset \mathsf{int}(\mathbb{K})$ and $(w, \alpha) \in \mathbb{C}^{a\#}$, we obtain that

$$\langle w, y\rangle - \alpha\|y\|_1 \leq 0$$

for all $y \in (\mathbb{A} - \{\overline{y}\})$, or

$$\langle w, y - \overline{y}\rangle - \alpha\|y - \overline{y}\|_1 \leq 0$$

for all $y \in \mathbb{A}$. Thus, the necessity of the theorem is proved.

Sufficiency. The proof follows from Theorem 3.3(iii).

□

4. Conclusions

In this paper, a version of the conic scalarization method for maximizing a multi-objective optimization problems is presented. A new class of superlinear monotone functions is presented and a nonliner separation theorem with the help of the functions from this class, is proved. The presented class of functions and the separation theorem are used to characterize maximal elements of maximization problems without convexity and boundedness conditions.

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