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# THE LEIBNIZ AND CHAIN RULES FOR FRACTIONAL DERIVATIVES

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Abstract. Recently, mathematical modeling of most of the dynamical processes and dynamical systems involves fractional derivatives (FDs) and fractional differential equations (FDEs). In this context, the Leibniz and Chain rules are frequently used in order to determine the fractional derivatives of the products or compositions of functions. This article investigates some currently popular formulas on the Leibniz and Chain rules for fractional derivatives and analyzes their consistency with the help of some counter-examples. For the purpose of this note, the fractional derivative is considered mainly in the Riemann-Liouville sense.

## 1. Introduction

The theory of Fractional Calculus (FC) generalizes the notion of derivatives and integrations of a function in the classical sense and it usually deals with the derivatives of arbitrary (real or complex) orders. The FC theory was initially developed mainly as a pure theoretical field of mathematics, but due to its major applications in nonlinear dynamical systems, in recent years the idea is extensively used in various applied fields such as quantum mechanics, electricity, ecological systems, and many other models for real-life problems.

The theory of FC is dynamic in nature, so the differential or difference equations involved are quite sufficiently capable of capturing and describing the dynamic and uncertain characteristics and behaviors of the physical world and the nature itself. Recently, fractional-order differential equations (FDEs) were increasingly used to model the problems in control of dynamical systems, visco-elasticity, electrochemistry, diffusion processes, physical and biological processes, and many other situations. Besides, the theory of FDEs has been frequently used in fractional-order models involving non-differentiable problems in fractal engineering such as non-differentiable solution of LC circuits, the heat conduction equation on Cantor sets, the damped wave equation in fractal strings, the perturbation solution of the oscillator of free damped vibrations, and many more (see, for details, [1–13]).

Certain formulas involving fractional derivatives and fractional integrals have various forms which are not compatible with each other. Consequently, the mathematical results involved have no clear visualization as those in the classical theory of calculus. This leads us to come across several violations and dynamics in the theory of FC. For instance, the formulas based on the Leibniz rule, the Chain rule and the Exponent rules for fractional derivatives are found to be inconsistent and

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with obvious violations. The primary objective of this article is to discuss some of these inconsistent and violated results.

A majority of violations and dynamic natures in fractional calculus are due to the fact that the fractional derivative of a constant function need not be zero. For example, in the Riemann-Liouville sense, the fractional derivative of a constant function is never zero, whereas it is exactly zero in other senses as the Liouville-Caputo sense and the Grünwald-Letnikov sense. Some of the deviations or violating behaviors based upon the Leibniz rules and the Chain rules are also found in [20–22, 24–27]. This violating and nonuniform behavior of fractional derivatives provides enough ingredients for the entire dynamic structure of FC.

## 2. Definitions and preliminaries

The most popular definitions of fractional derivatives and fractional integrals are those that are popularly called the Riemann-Liouville, the Liouville-Caputo and the Grünwald-Letnikov fractional derivatives and fractional integrals (see, for example, [16–19]), which are recalled below:

(i) The left and the right *Riemann-Liouville* fractional derivatives of order $\alpha$ $(n-1<\alpha<n)$, are given by

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\,\frac{d^n}{dt^n}\left(\int_a^t (t-x)^{n-\alpha-1}\, f(x)\,\mathrm{d}x\right) \tag{2.1} $$

$$ (t>a;\ n\in\mathbb{N}=\{1,2,3,\cdots\}) $$

and

$$ {}_tD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\,\frac{d^n}{dt^n}\left(\int_t^b (x-t)^{n-\alpha-1}\, f(x)\,\mathrm{d}x\right) \qquad (t<b;\ n\in\mathbb{N}), \tag{2.2} $$

respectively.

(ii) The left and the right *Riemann-Liouville* fractional integrals of order $\alpha$ $(n-1<\alpha<n)$ are given uby

$$ {}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}\, f(x)\,\mathrm{d}x \qquad (t>a) \tag{2.3} $$

and

$$ {}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}\, f(x)\,\mathrm{d}x \qquad (t<b), \tag{2.4} $$

respectively.

(iii) The left and the right *Liouville-Caputo* fractional integral of order $\alpha$ $(n-1<\alpha<n)$ are given by

$$ {}^{\mathrm{LC}}_{\;\;a}D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-x)^{n-\alpha-1}\, f^{(n)}(x)\,\mathrm{d}x \qquad (t>a;\ n\in\mathbb{N}) \tag{2.5} $$

and

$$
{}^{\mathrm{LC}}_{\ t}D^{\alpha}_{b} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b (x-t)^{n-\alpha-1}\, f^{(n)}(x)\,\mathrm{d}x \qquad (t < b;\ n \in \mathbb{N}), \tag{2.6}
$$

respectively.

(iv) The *Grünwald-Letnikov* fractional derivative of order $\alpha$ is given by

$$
{}^{\mathrm{GL}}D^{\alpha}_{t} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{\left[\frac{t-\alpha}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh), \tag{2.7}
$$

where $\Gamma(z) \quad (z \in \mathbb{C} \setminus \mathbb{Z}_0^- = \{0,-1,-2,-3,\cdots\})$ denotes the familiar (Euler's) Gamma function satisfying the following recurrence relation:

$$
\Gamma(z+1) = z\Gamma(z)
$$

and $[\lambda]$ denotes the integer part of $\lambda \in \mathbb{R}$.

Recently, Jumarie [1–3] modified the formulas for fractional derivatives of the Riemann-Liouville type as follows:

$$
\big(f(t)\big)^{(\alpha)}_{\mathrm{J}} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left( \int_0^t (t-x)^{-\alpha} [f(x) - f(0)]\,\mathrm{d}x \right), \tag{2.8}
$$

where, for convenience, J stands for Jumarie.

The primary formulas for the Leibniz rule and the Chain rule for fractional derivatives were given by Liouville as follows:

$$
{}_0D^{\alpha}_{t}[f(t)g(t)] = \sum_{k=0}^{\infty} \binom{\alpha}{k}\, {}_0D^{\alpha-k}_{t}[f(t)]\, {}_0D^{k}_{t}[g(t)], \tag{2.9}
$$

where ${}_0D^{\alpha}_{t}$ denotes the fractional derivative in the Riemann-Liouville sense, that is,

$$
{}_0D^{\alpha}_{t} f(t) = {}_aD^{\alpha}_{t} f(t) \bigg|_{a=0},
$$

as defined in Eq. (2.1).

Using the definition given in (2.8), Jumarie [1–3] provided two basic formulas regarding the Leibniz and Chain rules for the fractional derivatives involved. Let $f(t)$ and $g(t)$ be two functions satisfying the convergence and existence requirements for the definition (2.8). Then, for their product $f(t)g(t)$ and for their composition denoted by $f\big(g(t)\big)$, the formulas stated by Jumarie are recalled below (see [1–3]):

$$
\big(f(t)g(t)\big)^{(\alpha)}_{\mathrm{J}} = \big(f(t)\big)^{(\alpha)}_{\mathrm{J}} g(t) + f(t)\big(g(t)\big)^{(\alpha)}_{\mathrm{J}} \tag{2.10}
$$

and

$$
\Big(f\big(g(t)\big)\Big)^{(\alpha)}_{\mathrm{J}} = f'_g\big(g(t)\big)^{(\alpha)}_{\mathrm{J}}. \tag{2.11}
$$

Subsequently, however, Liu [13] showed that the formulas (2.10) and (2.11) are incorrect and gave the following modified versions of the corresponding formulas:

$$\big(f(t)g(t)\big)_J^{(\alpha)} = \sum_{j=0}^{\infty} \binom{\alpha}{j} f^j(t) g_{\mathrm{RL}}^{\alpha-j}(t) - \frac{f(0)g(0)}{t^\alpha \Gamma(1-\alpha)} \tag{2.12}$$

and

$$\Big(f\big(g(t)\big)\Big)_{\mathrm{J}}^{(\alpha)} = \sum_{j=1}^{\infty} \binom{\alpha}{j} \frac{t^{j-\alpha} j!}{\Gamma(j-\alpha+1)} \sum_{m=1}^{j} f^{(m)}\big(g(t)\big)$$
$$\cdot \sum_{P_1,\cdots,P_n \geqq 0} \prod_{k=1}^{j} \frac{1}{P_k!} \left(\frac{g^{(k)}(t)}{k!}\right)^{P_k} + \frac{f\big(g(t)\big) - f\big(g(0)\big)}{t^\alpha \Gamma(1-\alpha)}, \tag{2.13}$$

wherein RL stands for Riemann-Liouville and (as above) J stands for Jumarie, and the last sum in (2.13) extends over all combinations of nonnegative integer values of $P_1, \cdots, P_n$ such that (see [13, Eqs. (20) and (21)])

$$\sum_{k=1}^{n} kP_k = n \qquad \text{and} \qquad \sum_{k=1}^{n} P_k = m.$$

## 3. A set of main results

In this section, we determine the formulas for approximations of fractional derivatives and fractional integrals of finitely or infinitely differentiable functions and also verify the consistency and validity of the results (2.9) to (2.13) by using some counter-examples. We begin by presenting a known result as Lemma 3.1 below.

**Lemma 3.1.** *Let* $\alpha \in \mathbb{R}$ *and* $\beta \in \mathbb{R} \setminus \mathbb{Z}^-$ $(\mathbb{Z}^- := \mathbb{Z}_0^- \setminus \{0\})$. *Then*

$${}_aD_t^\alpha[(t-a)^\beta] = (t-a)^{\beta-\alpha} \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}. \tag{3.1}$$

*In particular, if* $0 < t < \infty$ $(t \neq a)$ *and* $\beta = 0$, *then*

$${}_aD_t^\alpha(1) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}.$$

**Remark 3.2.** By using the formulas in (iii) and (iv) of Section 2 as well as Eq. (2.8), it is directly verified that the fractional derivatives of a constant function is exactly zero.

**Proposition 3.3.** (see also [23, pp. 280–282, Theorem 15.1 and Section 15.2]) *Let* $f(t)$ *and* $g(t)$ *be two differentiable functions and* $0 < \alpha < 1$. *Then the formulas given in* (2.9), (2.10) *and* (2.12) *are inconsistent, that is, the formulas* (2.9), (2.10) *and* (2.12) *do not hold true in general.*

*Proof.* First of all, let us suppose that $f(t)$ and $g(t)$ are any given differentiable functions and that the formula (2.9) is consistent. Then, by putting $f(t) = 1$ in

(2.9), we have

$$\begin{aligned} D_t^\alpha[1\cdot g(t)] = D_t^\alpha g(t) &= \sum_{k=0}^{\infty}\binom{\alpha}{k} D_t^{\alpha-k}(1) D_t^k g(t) \\ &= \sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{t^{k-\alpha}}{\Gamma(1-\alpha+k)}\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\big(g(t)\big). \end{aligned}$$

Thus, upon setting $g(t) = t^{\frac{1}{2}}$ and using Lemma 3.1, we find that

$$D_t^\alpha\big(t^{\frac{1}{2}}\big) = \frac{\sqrt{\pi}\,t^{\frac{1}{2}-\alpha}}{2\Gamma\left(\frac{3}{2}-\alpha\right)}. \tag{3.2}$$

We also have

$$\begin{aligned} \sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{t^{k-\alpha}}{\Gamma(1-\alpha+k)}\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\big(t^{\frac{1}{2}}\big) &= \sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{t^{k-\alpha}}{\Gamma(1-\alpha+k)}\cdot\frac{\sqrt{\pi}\,t^{\frac{1}{2}-k}}{2\Gamma\left(\frac{3}{2}-k\right)} \\ &= \frac{\sqrt{\pi}\,t^{\frac{1}{2}-\alpha}}{2}\sum_{k=0}^{\infty}\binom{\alpha}{k}\frac{1}{\Gamma(1-\alpha+k)\Gamma\left(\frac{3}{2}-k\right)}. \end{aligned} \tag{3.3}$$

By using Eqs. (3.2) and (3.3) for all $0 < \alpha < 1$, we thus obtain a contradiction, that is,

$$\frac{1}{\Gamma\left(\frac{3}{2}-\alpha\right)} = \sum_{k=0}^{\infty}\frac{\binom{\alpha}{k}}{\Gamma(1-\alpha+k)\Gamma\left(\frac{3}{2}-k\right)}.$$

This proves that the formula given in (2.9) is inconsistent.

Secondly, let us assume the consistency of the formula (2.10). Then, by putting $f(t) = 1$, we have

$$\begin{aligned} \big(1\cdot g(t)\big)_{\mathrm{J}}^{(\alpha)} = \big(g(t)\big)_{\mathrm{J}}^{(\alpha)} &= (1)_{\mathrm{J}}^{(\alpha)} g(t) + 1\cdot\big(g(t)\big)_{\mathrm{J}}^{(\alpha)} \\ &= \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,g(t) + \big(g(t)\big)_{\mathrm{J}}^{(\alpha)}, \end{aligned}$$

which implies that, for any $0 < \alpha < 1$ and $g(t) \neq 0$, we get

$$\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\,g(t) = 0.$$

This leads to a contradiction and, therefore, concludes the inconsistency of the formula (2.10).

Finally, we assume that the formula (2.12) is valid for any arbitrary differentiable functions $f(t)$ and $g(t)$. Then, by setting $f(t) = 1$, we obtain

$$\begin{aligned} \big(f(t)g(t)\big)_{\mathrm{J}}^{(\alpha)} = \big(1\cdot g(t)\big)_{\mathrm{J}}^{(\alpha)} &= \sum_{j=0}^{\infty}\binom{\alpha}{j} 1^j\, g_{\mathrm{RL}}^{\alpha-j}(t) - \frac{g(0)}{t^\alpha\Gamma(1-\alpha)} \\ &= g_{\mathrm{RL}}^{\alpha}(t) - \frac{g(0)}{t^\alpha\Gamma(1-\alpha)}. \end{aligned} \tag{3.4}$$

It is noticed that Eq.(3.4) does not always hold true for any arbitrary functions. For instance, if

$$\alpha = \frac{1}{2} \qquad \text{and} \qquad g(t) = (t-a)^{\frac{1}{2}},$$

then we have

$$\big(g(t)\big)_{\mathrm{J}}^{(\alpha)} = -\frac{\sqrt{-a}}{t^{\alpha}\Gamma(1-\alpha)},$$

whereas, by using Lemma 3.1, we obtain

$$D_t^{\alpha}\big(g(t)\big) = 0.$$

This leads to a contradiction and, therefore, concludes the proof of the assertion in the above Proposition. □

**Theorem 3.4.** (see also [23, pp. 278–279, Lemma 15.3 and Section 15.1]) *Let $f(t)$ be an $(m+1)$-time differentiable and real-valued function in $[a,b]$ $(b > a)$ for $m \in \mathbb{N}$. Then, for any positive real number $\alpha$ such that $n-1 < \alpha < n$ $(n \in \mathbb{N})$,*

$$D_{a+}^{\alpha} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)}{\Gamma(1-\alpha+k)}\,(t-a)^{k-\alpha} + R_m \qquad (t > a),$$

*where*

$$R_m = \frac{1}{\Gamma(n-\alpha+m+1)}\,\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_a^t f^{(m+1)}(x)(t-x)^{n-\alpha+m}\,\mathrm{d}x\right)$$

*and, in terms of the Gamma function $\Gamma(z)$, the widely-used Pochhammer symbol $(\mu)_\nu$ $(\mu,\nu \in \mathbb{C})$ (or the shufted factorial) is defined, in general, by*

$$(3.5)\quad (\mu)_\nu := \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)} = \begin{cases} 1 & (\nu = 0;\ \mu \in \mathbb{C}\setminus\{0\}) \\[2ex] \mu(\mu+1)\cdots(\mu+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}$$

*it being understood conventionally that $(0)_0 := 1$ and assumed tacitly that the $\Gamma$-quotient in (3.5) exists.*

*Proof.* Here, in our demonstration of Theorem 3.4, we make use of Lemma 3.1 as well as termwise fractional differentiation. Indeed, if we suppose that $f(t)$ is an

$(m+1)$-time differentiable function in $[a,b]$ $(b>a)$ for $m \in \mathbb{N}$, then we may write

$$\begin{aligned}
D_{a+}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left( \int_a^t f(x)(t-x)^{n-\alpha-1} \,\mathrm{d}x \right) \\
&= \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left( f(a) \frac{(t-a)^{n-\alpha}}{n-\alpha} + f'(a) \frac{(t-a)^{n-\alpha+1}}{(n-\alpha)(n-\alpha+1)} + \cdots \right. \\
&\quad + f^{(m)}(a) \frac{(t-a)^{n-\alpha+m}}{(n-\alpha)_{m+1}} \\
&\quad \left. + \frac{1}{(n-\alpha)_{m+1}} \int_a^t f^{(m+1)}(x)(t-x)^{n-\alpha+m} \,\mathrm{d}x \right) \\
&= \frac{1}{\Gamma(n-\alpha)} \left[ f(a) \frac{(-\alpha+1)_n (t-a)^{-\alpha}}{n-\alpha} + f'(a) \frac{(-\alpha+2)_n (t-a)^{-\alpha+1}}{(n-\alpha)(n-\alpha+1)} + \cdots \right. \\
&\quad + f^{(m)}(a) \frac{(-\alpha+m)_n (t-a)^{-\alpha+m}}{(n-\alpha)_{m+1}} \\
&\quad \left. + \frac{1}{(n-\alpha)_{m+1}} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left( \int_a^t f^{(m+1)}(x)(t-x)^{n-\alpha+m} \,\mathrm{d}x \right) \right] \\
&= f(a) \frac{(-\alpha+1)_n (t-a)^{-\alpha}}{\Gamma(n-\alpha+1)} + f'(a) \frac{(-\alpha+2)_n (t-a)^{-\alpha+1}}{\Gamma(n-\alpha+2)} + \cdots \\
&\quad + f^{(m)}(a) \frac{(-\alpha+m)_n (t-a)^{-\alpha+m}}{\Gamma(n-\alpha+m)} \\
&\quad + \frac{1}{\Gamma(n-\alpha+m)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left( \int_a^t f^{(m+1)}(x)(t-x)^{n-\alpha+m} \,\mathrm{d}x \right) \\
&= \sum_{k=0}^{m} \frac{f^{(k)}(a)(1-\alpha+k)_n}{\Gamma(n-\alpha+k+1)} (t-a)^{k-\alpha} + R_m \\
&= \sum_{k=0}^{m} \frac{f^{(k)}(a)}{\Gamma(1-\alpha+k)} (t-a)^{k-\alpha} + R_m,
\end{aligned}$$

where, just as in the hypothesis of Theorem 3.4,

$$R_m = \frac{1}{\Gamma(n-\alpha+m+1)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left( \int_a^t f^{(m+1)}(x)(t-x)^{n-\alpha+m} \,\mathrm{d}x \right).$$

Moreover, if the function $f(t)$ is analytic in some neighborhood of the point $t=a$, then we may easily write

$$f(t) = \sum_{k=0}^{\infty} c_k \, (t-a)^k \qquad \left( c_k := \frac{f^{(k)}(a)}{k!} \right),$$

and hence

$$D_{a+}^{\alpha} f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{\Gamma(1-\alpha+k)} (t-a)^{k-\alpha}.$$

This completes the proof of Theorem 3.4. □

**Corollary 3.5.** *If $\alpha$ is any positive real number such that $0 < \alpha < 1$, then the following assertion hold true*:

$$D_{0+}^{\alpha} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(0)}{\Gamma(1-\alpha+k)} \, t^{-\alpha+k} + \frac{1}{\Gamma(2-\alpha+m)} \, \frac{d}{dt}\Big( \big(f^{(m+1)} * t^{1-\alpha+m}\big)(t) \Big)$$

*where $(f*g)(t)$ denotes the convolution of the functions $f(t)$ and $g(t)$ and is defined by*

$$(f*g)(t) = \int_0^t f(x)g(t-x) \, \mathrm{d}x = \int_0^t f(t-x)g(x) \, \mathrm{d}x.$$

*Proof.* Putting $a = 0$ and $n = 1$ in Theorem 3.4, we have the result asserted by Corollary 3.5. □

**Theorem 3.6.** *Let $f(t)$ be an $(m+1)$-time differentiable and real-valued function in $[a,b]$. Then, for any positive real $\alpha$ such that $n-1 < \alpha < n$ $(n \in \mathbb{N})$,*

$$D_{b-}^{\alpha} f(t) = \sum_{k=0}^{m} (-1)^k \, \frac{f^{(k)}(b)}{\Gamma(1-\alpha+k)} \, (b-t)^{k-\alpha} + R_m^* \qquad (t < b),$$

*where*

$$R_m^* = \frac{(-1)^{n+m+1}}{\Gamma(n-\alpha+m+1)} \, \frac{\mathrm{d}^n}{\mathrm{d}t^n} \left( \int_t^b f^{(m+1)}(x)(x-t)^{n-\alpha+m} \, \mathrm{d}x \right).$$

*Proof.* The proof follows similar arguments as those used in Theorem 3.4. □

**Corollary 3.7.** *If $\alpha$ is any positive real number such that $0 < \alpha < 1$, then*

$$\begin{aligned} D_{0-}^{\alpha} f(t) = {} & \sum_{k=0}^{m} (-1)^k \, \frac{f^{(k)}(0)}{\Gamma(1-\alpha+k)} \, (-t)^{k-\alpha} \\ & + \frac{(-1)^{m+1}}{\Gamma(2-\alpha+m)} \, \frac{\mathrm{d}}{\mathrm{d}t} \left( f^{(m+1)} * (-t)^{1-\alpha+m} \right)(t) \qquad (t < 0). \end{aligned}$$

*Proof.* Corollary 3.7 is a direct consequence of Theorem 3.6 when $b = 0$ and $n = 1$. □

**Theorem 3.8.** *Let $f(t)$ be a continuous, $(m+1)$-time differentiable and real-valued function in $[a,b]$. Then, for any positive real $\alpha$ such that $n-1 < \alpha < n$ $(n \in \mathbb{N})$,*

$$I_{a+}^{\alpha} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(a)}{\Gamma(\alpha+k+1)} \, (t-a)^{\alpha+k} + \mathbf{R}_m,$$

*where*

$$\mathbf{R}_m = \frac{1}{\Gamma(\alpha+m+1)} \, \left( \int_a^t f^{(m+1)}(x)(t-x)^{\alpha+m} \, \mathrm{d}x \right).$$

*Proof.* It is observed that

$$\begin{aligned}
I^{\alpha}_{a+}f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t f(x)(t-x)^{\alpha-1}\,\mathrm{d}x \\
&= \frac{1}{\Gamma(\alpha)} \left[ f(a)\,\frac{(t-a)^{\alpha}}{\alpha} + f'(a)\,\frac{(t-a)^{\alpha+1}}{\alpha(\alpha+1)} + \cdots + f^{(m)}(a)\frac{(t-a)^{\alpha+m}}{(\alpha)_{m+1}} \right. \\
&\qquad \left. + \frac{1}{(\alpha)_{m+1}} \int_a^t f^{(m+1)}(x)(t-x)^{\alpha+m}\,\mathrm{d}x \right] \\
&= f(a)\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} + f'(a)\,\frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+2)} + \cdots + f^{(m)}(a)\,\frac{(t-a)^{\alpha+m}}{\Gamma(\alpha+m+1)} \\
&\qquad \left. + \frac{1}{\Gamma(\alpha+m+1)} \int_a^t f^{(m+1)}(x)(t-x)^{\alpha+m}\,\mathrm{d}x \right] \\
&= \sum_{k=0}^{m} \frac{f^{(k)}(a)}{\Gamma(\alpha+k+1)}\,(t-a)^{\alpha+k} + \mathbf{R}_m,
\end{aligned}$$

where

$$\mathbf{R}_m = \frac{1}{\Gamma(\alpha+m+1)} \left( \int_a^t f^{(m+1)}(x)(t-x)^{\alpha+m}\,\mathrm{d}x \right).$$

This concludes the proof of Theorem 3.8. □

**Corollary 3.9.** *For any positive real number* $\alpha$ *such that* $0 < \alpha < 1$, *it is asserted that*

$$I^{\alpha}_{0+}f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(0)}{\Gamma(\alpha+k+1)}\,t^{\alpha+k} + \frac{1}{\Gamma(\alpha+m+1)}\,\left(f^{(m+1)} * t^{\alpha+m}\right)(t).$$

*Proof.* Corollary 3.9 is a direct consequence of Theorem 3.8. □

**Theorem 3.10.** *Let* $f(t)$ *be an* $(m+1)$*-time differentiable and real-valued function in* $[a,b]$. *Then, for any positive real* $\alpha$ *such that* $n-1 < \alpha < n$ $(n \in \mathbb{N})$,

$$I^{\alpha}_{b-}f(t) = \sum_{k=0}^{m} (-1)^k \frac{f^{(k)}(b)}{\Gamma(\alpha+1+k)}\,(b-t)^{\alpha+k} + R^*_m \qquad (t < b),$$

*where*

$$R^*_m = \frac{(-1)^{m+1}}{\Gamma(\alpha+1+m)} \int_t^b f^{(m+1)}(x)(x-t)^{\alpha+m}\,\mathrm{d}x.$$

*Proof.* The proof of Theorem 3.10 requires the same techniques as those used in the previous theorems. □

**Corollary 3.11.** *If* $\alpha$ *is a positive real number such that* $0 < \alpha < 1$, *then*

$$I^{\alpha}_{0-}f(t) = \sum_{k=0}^{m} (-1)^k \frac{f^{(k)}(0)}{\Gamma(\alpha+1+k)}\,(-t)^{\alpha+k} + \frac{(-1)^m}{\Gamma(\alpha+1+m)}\,\left(f^{(m+1)} * (-t)^{\alpha+m}\right)(t) \qquad (t < 0).$$

*Proof.* Corollary 3.11 is a direct consequence of Theorem 3.10 when $b = 0$ and $n = 1$. □

**Corollary 3.12.** *If $\alpha$ is any positive real number such that $0<\alpha<1$, then*

(3.6)
$$\big(f(t)\big)^{(\alpha)}=\sum_{k=1}^{m}\frac{f^{(k)}(0)}{\Gamma(1-\alpha+k)}\,t^{-\alpha+k}+\frac{1}{\Gamma(2-\alpha+m)}\,\frac{d}{dt}\big(f^{(m+1)}*t^{1-\alpha+m}\big)(t).$$

*Proof.* The proof of Corollary 3.12 follows similar lines as those used in Theorem 3.5. □

**Remark 3.13.** From Corollary 3.5 and the assertion (3.6) of Corollary 3.12, it follows that, if $f(0)\neq 0$, then

$$D^{\alpha}_{0+}f(t)=\frac{f(0)}{\Gamma(1-\alpha)}\,t^{-\alpha}+\big(f(t)\big)^{(\alpha)}.$$

We observe that the expression in (3.6) for a function $f(t)$ is well-posed for all values of $t$ away from the origin and for all values $\alpha$ near 1. For more clarifications, we have the following example.

**Example 3.14.** Let us consider a function $f(t)$ given by

$$f(t)=\cos\big(\sqrt{t}\big)\qquad(t\in\mathbb{R}).$$

Then, clearly, we get

$$f(0)=1\neq 0.$$

Now, using Corollary 3.5, we can readily deduce that

$$\begin{aligned}D^{\alpha}_{0+}\big(\cos(\sqrt{t})\big)&=\sum_{k=0}^{m}\frac{\dfrac{\mathrm{d}^k}{\mathrm{d}t^k}\big(\cos(\sqrt{t})\big)\Big|_{t=0}}{\Gamma(1-\alpha+k)}\,t^{-\alpha+k}\\&\quad+\frac{1}{\Gamma(2-\alpha+m)}\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t\cos(\sqrt{x})(t-x)^{1-\alpha+m}\,\mathrm{d}x\right)\\&\simeq\sum_{k=0}^{\infty}\frac{\dfrac{\mathrm{d}^k}{\mathrm{d}t^k}\big(\cos(\sqrt{t})\big)\Big|_{t=0}}{\Gamma(1-\alpha+k)}\,t^{-\alpha+k}\\&=\sum_{k=0}^{\infty}\frac{(-1)^k\,k!}{(2k)!\,\Gamma(1-\alpha+k)}\,t^{-\alpha+k}\\&=t^{-\alpha}E_{1,1-\alpha}\left(-\frac{t}{2}\right),\end{aligned}$$

where $E_{\alpha,\beta}(z)$ is the well-known Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}\qquad(\alpha>0;\ \beta\in\mathbb{C}).$$

Simultaneously, the above derivative can be approximated by using Corollary 3.12 as follows:

$$\big(\cos(\sqrt{t})\big)^{(\alpha)}\simeq\sum_{k=1}^{\infty}\frac{(-1)^k\,k!}{(2k)!\,\Gamma(1-\alpha+k)}\,t^{-\alpha+k}.$$

In fact, the difference between these two approximations based upon Corollary 3.5 and Corollary 3.12 can be found to be as given below:

$$\lambda_\alpha = \left| \frac{1}{\Gamma(1-\alpha)}\, t^{-\alpha} \right|.$$

In order to get the approximations up to a higher-order accuracy, the term $\lambda_\alpha$ should not be neglected. It significantly affects the accuracy and stability of approximation results (see Corollary 3.12) derived from the formula provided by Jumarie [1,2]. For instance, if $t >> 0$ (away from 0) and $\alpha = 0.9$, then it is noticed that

$$\lambda_{0.9} = \left| \frac{t^{-0.9}}{\Gamma(0.1)} \right| < \epsilon \to 0,$$

whereas, if $\alpha = 0.1$, the quantity $\lambda_{0.1}$ can not be neglected, that is,

$$\lambda_{0.1} = \left| \frac{t^{-0.1}}{\Gamma(0.9)} \right| \nrightarrow 0.$$

Secondly, for $t \to 0$ (nearer to 0), both of the terms $\lambda_{0.9}$ and $\lambda_{0.1}$ are unbounded, and hence can never be neglected. For details, one may find it to be convenient to refer to Figures 1, 2 and 3 below, in each of which the depicted approximations are taken up to the first 500 terms.

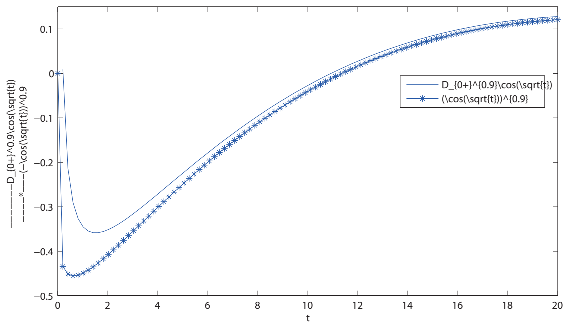


**Figure 1. Fractional derivatives of order 0.9 for the function $f(t) = \cos(\sqrt{t})$ for the Riemann-Liouville type (——) and the Jumarie type (*******)**

Therefore, in general, the fractional derivative formula due to Jumarie [1, 2] is inconsistent (see Figures 1 and 2 above). In fact, it is valid only for some specific functions.

Now, using some counter-examples, we state a number of observations on the Leibniz rule for fractional derivatives.

**Theorem 3.15.** *The well-known formula for the Leibniz rule defined in* (2.10) *is consistent only for* $\alpha = 1$ *and may not hold true for other fractional values of* $\alpha$ $(0 < \alpha < 1)$.

*Proof.* The proof of Theorem 3.15 follows from [26]. However, the idea is slightly violating nearer to $\alpha = 1$, but the degree of deviation is increased as we move $\alpha$

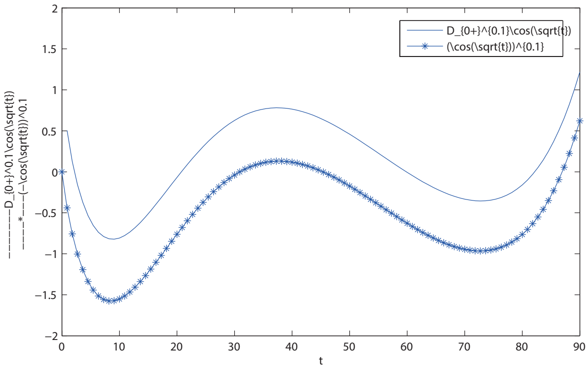


**Figure 2. Fractional derivatives of order $0.1$ for the function $f(t) = \cos(\sqrt{t})$ of the Riemann-Liouville type (——) and the Jumarie type (******)**

from 1 to 0. In fact, if $\alpha \to 0$, then it is observed that

$$\left\| \big(f(t)g(t)\big)_{\mathrm{J}}^{(\alpha)} - \Big(\big(f(t)\big)^{(\alpha)} g(t) + f(t)\big(g(t)\big)_{\mathrm{J}}^{(\alpha)}\Big) \right\|_{L_\infty} \backsimeq \|fg\|_{L_\infty},$$

where $L_\infty$ is the set of all essentially bounded measurable functions. $\square$

**Theorem 3.16.** *For any arbitrary functions $f(t)$ and $g(t)$, the general Leibniz rule based upon integer-order derivatives is commutative, whereas it does not commute in the case of fractional-order derivatives, that is, for a given fraction $\alpha$ $(0 < \alpha < 1)$,*

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} D_t^{\alpha-k} f(t) \cdot D_t^k g(t) \neq \sum_{k=0}^{\infty} \binom{\alpha}{k} D_t^{\alpha-k} g(t) \cdot D_t^k f(t) \tag{3.7}$$

*in general.*

*Proof.* The proof for the integer-order derivatives is trivial, so we choose to omit the details involved. We demonstrate the result for fractional-order derivatives by by means of the following illustrative example. $\square$

**Example 3.17.** Let us consider the following two functions:

$$f(t) = t^{-\frac{1}{2}} \qquad \text{and} \qquad g(t) = 25.$$

Then, by taking $\alpha = \frac{1}{2}$ in Eq. (2.9), we have

$$\begin{aligned} D_t^{\alpha}[f(t)g(t)] &= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} D_t^{\frac{1}{2}-k} \big(t^{-\frac{1}{2}}\big) D_t^k(25) \\ &= \binom{\frac{1}{2}}{0} D_t^{\frac{1}{2}} \big(t^{-\frac{1}{2}}\big) D_t^0(25) = 0. \end{aligned} \tag{3.8}$$

On the other hand, by interchanging the positions of $f(t)$ and $g(t)$, we have

$$\begin{aligned} D_t^{\alpha}[g(t)f(t)] &= \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} D_t^{\frac{1}{2}-k}\left[25 D_t^k\left(t^{-\frac{1}{2}}\right)\right] \\ &= 25 \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \frac{t^{-\frac{1}{2}+k}}{\Gamma\left(\frac{1}{2}+k\right)} \cdot \frac{\sqrt{\pi} t^{-\frac{1}{2}-k}}{\Gamma\left(\frac{1}{2}-k\right)} \\ &= \frac{25\sqrt{\pi}}{t} \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \frac{1}{\Gamma\left(\frac{1}{2}+k\right)\Gamma\left(\frac{1}{2}-k\right)} \\ &= K_{\frac{1}{2}}(t) \neq 0, \end{aligned} \tag{3.9}$$

where

$$K_{\frac{1}{2}}(t) = 25\sqrt{\frac{\pi}{t}} \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k} \frac{1}{\Gamma\left(\frac{1}{2}+k\right)\Gamma\left(\frac{1}{2}-k\right)}.$$

For the justification of the following equation:

$$K_{\frac{1}{2}}(t) \neq 0,$$

we estimate the approximated values of $K_{\frac{1}{2}}(t)$ by using some efficient in-built functions in MATLAB. By increasing the number of terms in its approximations, it is observed that the entire graph relevant to the function $K_{\frac{1}{2}}(t)$ never overlaps with the $t$-axis. However, we provide the plot of the function $K_{\frac{1}{2}}(t)$ versus $t$ (see Figure 3) by considering the approximation taken up to the first 500 terms. Now, by combining Eqs. (3.8) and (3.9), we complete the demonstration.

Finally, by using the result recorded by Samko *et al.* [23], Remark 3.2 and Theorem 3.16, we are led to Remark 3.18 below.

**Remark 3.18.** For any non-analytic or non-differentiable functions, the general Leibniz rule for fractional derivatives does not hold true and it is also noncommutative.

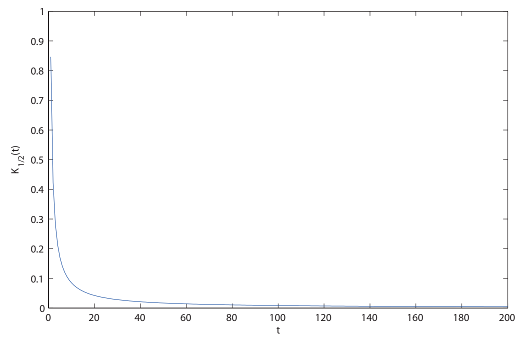


**Figure 3. Plot of the function $K_{\frac{1}{2}}(t)$ versus $t$**

## 4. Conclusion

In recent years, it has been observed that, in most of the nonlinear fractional modeling problems, the Leibniz rule for fractional derivatives is frequently used. With this prospect in view, the existing formulas for the Leibniz rule have been modified and restated by several researchers in respect of various types of fractional derivatives. The relevant idea for fractional derivatives involving non-differentiable and discrete functions have also been extended.

The present article shows that, in general, the Leibniz rule does not hold true for fractional derivatives of non-integer orders. Therefore, many attempts have been made in order to get a unified or generalized formula for the Leibniz rule, which can easily apply in both classical and fractional-order cases. However, owing to the nonlinearity of the fractional derivative, all attempts in this direction have not got the desired success so far and we still have this open problem in the theory of fractional calculus.

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