Applied Analysis and Optimization
Volume 3, Number 3, 2019, 333–342

Yokohama Publishers
ISSN 2189-1664 Online Journal

BOCHNER-RIESZ OPERATORS BETWEEN MORREY SPACES

HUA WANG, JIE XIAO, AND SHAOZHEN XU

Abstract. This note concerns the boundedness of $\mathscr{I}_\delta$ (the Bochner-Riesz operator) mapping $L^{p,\kappa}$ (the (p,κ)-Morrey space) to $L^{q,\lambda}$ (the (q,λ)-Morrey space) or $L^{q,\lambda;\ln}$ (the $(q,\lambda;\ln)$-Morrey space), thereby showing

$$\|\mathscr{I}_\delta f\|_{L^{p,\lambda}} \lesssim \|f\|_{L^{p,\kappa}}\ \forall\ f\in L^{p,\kappa}\ \text{ under } \begin{cases} n\ge\kappa>\lambda>0;\\ 1\le p<\infty;\\ \delta\ge\frac{n-1}{2}+\frac{\lambda-\kappa}{p},\end{cases}$$

which may be regarded as the Morrey ($\kappa>\lambda$)-variant of the unsolved Bochner-Riesz conjecture (cf. [3] or [14, p.390]):

$$\|\mathscr{I}_\delta f\|_{L^p}\lesssim\|f\|_{L^p}\ \forall\ f\in L^p\ \text{ under }\ 2\ne p\in(1,\infty)\ \ \&\ \ \delta>n\left|\frac{1}{p}-\frac{1}{2}\right|-\frac{1}{2}.$$

1. Introduction

On the one hand, for a function f on $\mathbb{R}^n$ with the Fourier transform $\hat f$ and the inverse Fourier transform $\check f$:

$$\hat f(\xi)=\int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}\,dx\quad\&\quad \check f(\xi)=\hat f(-\xi)\quad\forall\quad\xi\in\mathbb{R}^n,$$

the Bochner-Riesz operator $\mathscr{I}_\delta f$ of f with order $\delta>-1$ is defined as:

$$\mathscr{I}_\delta f(x)=\int_{|\xi|\le1}(1-|\xi|^2)_+^\delta\hat f(\xi)e^{2\pi i x\cdot\xi}\,d\xi\quad\forall\quad x\in\mathbb{R}^n,$$

where

$$t_+=\begin{cases} t & \text{as}\quad t>0;\\ 0 & \text{otherwise.}\end{cases}$$

For $z\in\mathbb{C}$ and $k\in(-1/2,\infty)$ let (cf. [7, Appendix B])

$$\begin{cases} m_\delta(\xi)=(1-|\xi|^2)_+^\delta;\\ \Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt;\\ J_k(z)=\frac{(\frac{z}{2})^k}{\Gamma(k+2^{-1})\Gamma(2^{-1})}\int_{-1}^1 e^{izs}(1-s^2)^{k-\frac12}\,ds.\end{cases}$$

Then

$$\begin{cases} \check m_\delta(x)=\pi^{-\delta}\Gamma(1+\delta)|x|^{-\frac{n}{2}-\delta}J_{\frac{n}{2}+\delta}(2\pi|x|)\ \ \forall\ \ x\in\mathbb{R}^n;\\ \widehat{\mathscr{I}_\delta f}(\xi)=m_\delta(\xi)\hat f(\xi)\ \ \forall\ \ \xi\in\mathbb{R}^n.\end{cases}$$

2010 *Mathematics Subject Classification.* 42B15 42B35.

Key words and phrases. Bochner-Riesz operator, Fourier transform, Morrey space.

H. Wang was partially supported by NSERC of Canada (#20171864); J. Xiao was supported by NSERC of Canada (#20171864); S. Xu was supported partially by the National Natural Science Foundation of China (#11471309) and NSERC of Canada (#20171864).

On the other hand, given $(p,\kappa)\in[1,\infty)\times[0,n]$, the Morrey space $L^{p,\kappa}$ (as a useful tool to study the local behavior of solutions to second order elliptic partial differential equations; see also [6]) or its ln-analogue $L^{p,\kappa;\ln}$ is defined by

$$\left\{f\in L^p_{loc}(\mathbb{R}^n):\|f\|_{L^{p,\kappa}}=\sup_{r>0,x_0\in\mathbb{R}^n}\left(r^{\kappa-n}\int_{B(x_0,r)}|f(y)|^p\,dy\right)^{\frac{1}{p}}<\infty\right\},$$

or

$$\left\{f\in L^p_{loc}(\mathbb{R}^n):\|f\|_{L^{p,\kappa;\ln}}=\sup_{r>0,x_0\in\mathbb{R}^n}\left(\frac{r^{\kappa-n}}{\ln e(1+r)}\int_{B(x_0,r)}|f(y)|^p\,dy\right)^{\frac{1}{p}}<\infty\right\},$$

where $B(x_0,r)$ is the Euclidean ball with center $x_0\in\mathbb{R}^n$ and radius $r\in(0,\infty)$ as well as the Lebesgue measure $|B(x_0,r)|\approx r^n$. In particular, one has

$$L^{p,0}=L^\infty \ \ \& \ \ L^{p,n}=L^p.$$

In this note we are motivated by [8, 9, 10, 12] and the basic equivalence (cf. [11])

$$\lim_{r\to\infty}\|\mathscr{I}_{\delta,r}f-f\|_{L^p}=0\Leftrightarrow\|\mathscr{I}_\delta f\|_{L^p}\lesssim\|f\|_{L^p}\quad\forall\quad p\in[1,\infty),$$

where

$$\mathscr{I}_{\delta,r}f(x)=\int_{B(0,r)}\hat{f}(\xi)\big(1-r^{-2}|\xi|^2\big)e^{2\pi ix\cdot\xi}\,d\xi\quad\forall\quad r\in(0,\infty)$$

is the spherical mean of multiple Fourier integral of f (cf. [14, p.390]), to achieve

Theorem 1.1. *If*

$$\begin{cases}n\ge\kappa\ge\lambda>0;\\ \infty>p>1;\\ \frac{n-1}{2}>\delta>\frac{n-1}{2}-\frac{\kappa}{p};\\ q=\frac{p\lambda}{\kappa-\left(\frac{n-1}{2}-\delta\right)p},\end{cases}\tag{1.1}$$

then $\mathscr{I}_\delta:L^{p,\kappa}\to L^{q,\lambda}$ *is continuous, i.e.,*

$$\|\mathscr{I}_\delta f\|_{L^{q,\lambda}}\lesssim\|f\|_{L^{p,\kappa}}\quad\forall\quad f\in L^{p,\kappa}.\tag{1.2}$$

Remark 1.2. Interestingly, Theorem 1.1 has three by-products.

(i) If

$$\begin{cases}n\ge\kappa>\lambda>0;\\ p=q\in(1,\infty);\\ \frac{n-1}{2}>\delta=\frac{n-1}{2}+\frac{\lambda-\kappa}{p},\end{cases}$$

then

$$\|\mathscr{I}_\delta f\|_{L^{p,\lambda}}\lesssim\|f\|_{L^{p,\kappa}}\quad\forall\quad f\in L^{p,\kappa},$$

which validates the Morrey ($\kappa>\lambda$)-variant of the Bochner-Riesz conjecture (cf. [3] or [14, p.390]):

$$\|\mathscr{I}_\delta f\|_{L^p}\lesssim\|f\|_{L^p}\ \forall\ f\in L^p\Leftarrow 2\ne p\in(1,\infty)\ \ \&\ \ \delta>\left(\frac{n}{2}\right)\left(\left|\frac{2}{p}-1\right|-1\right)$$

whose truth for $n=2$ is given in [5].

(ii) If

$$0<\kappa=\lambda\le n\ \ \&\ \ 1<p=q<\infty\ \text{ in }\ (1.1),$$

then δ is forced to equal $\frac{n-1}{2}$ and hence $\mathscr{I}_{\frac{n-1}{2}}$ is bounded on the weighted Lebesgue space $L^p(w)$ under w being A_p-weight (cf. [13]) - accordingly - an application of $A_1\subset A_p$ and [1, Lemma 11] saying that

$$\|f\|_{L^{p,\kappa}}\approx\sup_{w\in A_1\ \&\ \int_{\mathbb{R}^n}w\,d\Lambda^{(\infty)}_{n-\kappa}\le1}\left(\int_{\mathbb{R}^n}|f(y)|^p\,w(y)\,dy\right)^{\frac{1}{p}}$$

holds with A_1-weight w obeying $(n-\kappa)$-dimensional Hausdorff integral

$$\int_{\mathbb{R}^n}w\,d\Lambda^{(\infty)}_{n-\kappa}\le1,$$

gives that $\mathscr{I}_{\frac{n-1}{2}}$ is bounded on $L^{p,\kappa}$ - of course - this boundedness is extendible to $\delta>\frac{n-1}{2}$ via the well-known fact that $\mathscr{I}_{\delta>\frac{n-1}{2}}$ is bounded on $L^p(w)$ with $w\in A_p$ (cf. [4]).

(iii) If

$$0<\kappa=\lambda\le n\ \ \&\ \ 1<p<q=\frac{p\kappa}{\kappa-\left(\frac{n-1}{2}-\delta\right)p},$$

then

$$\|\mathscr{I}_\delta f\|_{L^{q,\kappa}}\lesssim\|f\|_{L^{p,\kappa}}\quad\forall\quad f\in L^{p,\kappa},$$

thereby properly extending [2, Theorem 1.1] (from $p=q$ to $p<q$)- if

$$0<\kappa\le n\ \ \&\ \ p>1>\frac{2\delta}{n-1}>\max\left\{\left|\frac{2}{p}-1\right|,\left(\frac{n-\kappa}{n}\right)\left(\left|\frac{2}{p}-1\right|+\frac{2}{p}\right)\right\},$$

then

$$\|\mathscr{I}_\delta f\|_{L^{p,\kappa}}\lesssim\|f\|_{L^{p,\kappa}}\quad\forall\quad f\in L^{p,\kappa}.$$

Next, as a compensation for Theorem 1.1 we discover

Theorem 1.3. *If*

$$\begin{cases}n\ge\kappa>\lambda>0;\\ \infty>q,p\ge1;\\ \frac{n-\lambda}{q}+\frac{\kappa}{p}>n;\\ \delta\ge\frac{n-1}{2}-\frac{\kappa}{p}+\frac{\lambda}{q},\end{cases}\tag{1.3}$$

or

$$\begin{cases}n\ge\kappa\ge\lambda>0;\\ \infty>q,p\ge1;\\ \frac{n-\lambda}{q}+\frac{\kappa}{p}=n;\\ \delta\ge\frac{n-1}{2}-\frac{\kappa}{p}+\frac{\lambda}{q},\end{cases}\tag{1.4}$$

then $\mathscr{I}_\delta:L^{p,\kappa}\to L^{q,\lambda}$ *or* $L^{q,\lambda;\ln}$ *is bounded; i.e.,*

$$\|\mathscr{I}_\delta f\|_{L^{q,\lambda}}\ \text{or}\ \|\mathscr{I}_\delta f\|_{L^{q,\lambda;\ln}}\lesssim\|f\|_{L^{p,\kappa}}\quad\forall\quad f\in L^{p,\kappa}.\tag{1.5}$$

Remark 1.4. Even more interesting are the following two comments on Theorem 1.3:

(i) If

$$\begin{cases} n \geq \kappa > \lambda > 0; \\ 1 \leq p = q < \infty; \\ \delta \geq \frac{n-1}{2} - \frac{\kappa}{p} + \frac{\lambda}{p} > \frac{n-1}{2} - \frac{\kappa}{p}, \end{cases}$$

then

$$\|\mathscr{I}_\delta f\|_{L^{p,\lambda}} \lesssim \|f\|_{L^{p,\kappa}} \quad \forall \quad f \in L^{p,\kappa},$$

and hence Remark 1.2(i) can be extended to the endpoint $p = 1 = q$.

(ii) From

$$\kappa = \lambda = n \ \ \& \ \ p = q = 1 \ \ \& \ \ \delta \geq \frac{n-1}{2} \text{ in (1.4)}$$

it follows that

$$\|\mathscr{I}_\delta f\|_{L^{1,n;\ln}} \lesssim \|f\|_{L^1} \ \forall \ f \in L^1 \quad \text{under} \ \ \delta \geq \frac{n-1}{2},$$

holds - nevertheless - it is worth to point out a two-fold fact. On the one hand, this last estimation under $\delta > (n-1)/2$ is weaker than the easily-checked case $p \to 1$ of the above-quoted Bochner-Riesz conjecture:

$$\|\mathscr{I}_\delta f\|_{L^1} \lesssim \|f\|_{L^1} \ \forall \ f \in L^1 \quad \text{under} \ \ \delta > \frac{n-1}{2}$$

thanks to

$$\|\mathscr{I}_\delta f\|_{L^{1,n;\ln}} \lesssim \|\mathscr{I}_\delta f\|_{L^1}.$$

On the other hand, the special inequality

$$\|\mathscr{I}_{\frac{n-1}{2}} f\|_{L^{1,n;\ln}} \lesssim \|f\|_{L^1} \ \forall \ f \in L^1$$

is locally stronger than the well-known weak estimate

$$\|\mathscr{I}_{\frac{n-1}{2}} f\|_{L^{1,*}} = \sup_{t>0} \left(t \int_{\left\{x \in \mathbb{R}^n:\ \left|\mathscr{I}_{\frac{n-1}{2}} f(x)\right| > t\right\}} dx \right) \lesssim \|f\|_{L^1} \ \forall \ f \in L^1.$$

Notation. In the above and below, $U \lesssim V$ stands for $U \leq cV$ for a constant $c > 0$. Furthermore, $U \approx V$ means both $U \lesssim V$ and $V \lesssim U$.

2. Verification

For $\delta > -1$ we take into an account of the convolution operator

$$\mathscr{T}_\delta f(x) = f * \frac{J_{\frac{n}{2}+\delta}(2\pi|x|)}{|x|^{\frac{n}{2}+\delta}},$$

where the Bessel function $J_k(\cdot)$ with $k > -1/2$ satisfies

$$\begin{cases} J_k(r) \lesssim r^k & \forall \quad r \in (0,1); \\ J_k(r) \lesssim r^{-1/2} & \forall \quad r \in [1,\infty); \\ \frac{d}{dr}\left(r^{-k} J_k(r)\right) = -r^{-k} J_{k+1}(r) & \forall \quad r \in (0,\infty). \end{cases}$$

Accordingly, we have

$$\mathscr{J}_\delta f(x) = \pi^{-\delta}\Gamma(\delta+1) f * \frac{J_{\frac{n}{2}+\delta}(2\pi|x|)}{|x|^{\frac{n}{2}+\delta}} = \pi^{-\delta}\Gamma(\delta+1)\mathscr{T}_\delta f(x),$$

thereby finding

$$\begin{cases} \frac{J_{\frac{n}{2}+\delta}(2\pi|x|)}{|x|^{\frac{n}{2}+\delta}} \lesssim (1+|x|)^{-\frac{n+1}{2}-\delta}; \\ \check{d\sigma}(x) = \int_{\mathbb{S}^{n-1}} e^{2\pi i x\cdot y} d\sigma(y) = 2\pi|x|^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(2\pi|x|); \\ f * \check{d\sigma}(x) = 2\pi \mathscr{T}_{-1} f(x). \end{cases} \tag{2.1}$$

Proof of Theorem 1.1. From (2.1) it follows that

$$|\mathscr{T}_\delta f(x)| \lesssim I_\alpha |f|(x) = \int_{\mathbb{R}^n} |f(y)||x-y|^{\alpha-n}\, dy$$

where

$$n > \alpha = \frac{n-1}{2} - \delta > 0.$$

According to [8, Theorem 1.1] with the above α and $\beta = n$ (the Radon measure $d\mu$ is chosen as the n-dimensional Lebesgue measure), we have that if (1.2) is valid then $I_\alpha : L^{p,\kappa} \to L^{q,\lambda}$ is continuous, thereby getting

$$\|\mathscr{J}_\delta f\|_{L^{q,\lambda}} \lesssim \|\mathscr{T}_\delta f\|_{L^{q,\lambda}} \lesssim \big\|I_\alpha|f|\big\|_{L^{q,\lambda}} \lesssim \|f\|_{L^{p,\kappa}}.$$

□

Proof of Theorem 1.3. Thanks to the translation invariance of the convolution operator, in order to prove the implication (1.3) or (1.4)⇒(1.5), it suffices to verify that

$$\|f\|_{L^{p,\kappa}} \gtrsim \begin{cases} \left(r^{\lambda-n} \int_{B(o,r)} |\mathscr{T}_\delta f(x)|^q \, dx\right)^{\frac{1}{q}} & \text{under } (1.3); \\ \left(\frac{r^{\lambda-n}}{\ln e(1+r)} \int_{B(o,r)} |\mathscr{T}_\delta f(x)|^q \, dx\right)^{\frac{1}{q}} & \text{under } (1.4), \end{cases} \tag{2.2}$$

holds for any ball $B(o,r)$ with center o and radius r.

To do so, upon denoting by χ_E the characteristic function of $E \subset \mathbb{R}^n$ and writing

$$\begin{cases} f = f_1 + f_2 \in L^{p,\kappa}; \\ f_1 = f\chi_{B(o,2r)}; \\ f_2 = f\chi_{\mathbb{R}^n \setminus B(o,2r)}, \end{cases}$$

we have

$$\left(\int_{B(o,r)} |\mathscr{T}_\delta f(x)|^q \, dx\right)^{\frac{1}{q}} \lesssim \left(\int_{B(o,r)} |\mathscr{T}_\delta f_1(x)|^q \, dx\right)^{\frac{1}{q}} + \left(\int_{B(o,r)} |\mathscr{T}_\delta f_2(x)|^q \, dx\right)^{\frac{1}{q}},$$

whence considering two cases $r \in [1,\infty)$ and $r \in (0,1)$.

Case - $r \in [1,\infty)$. On the one hand, the Minkowski inequality is used to derive

$$\left(\int_{B(o,r)} |\mathscr{T}_\delta f_1(x)|^q \, dx\right)^{\frac{1}{q}} = \left(\int_{B(o,r)} \left| f_1 * \frac{J_{\frac{n}{2}+\delta}(2\pi|x|)}{|x|^{\frac{n}{2}+\delta}}\right|^q dx\right)^{\frac{1}{q}}$$

$$= \left(\int_{B(o,r)} \left| \int_{B(o,2r)} f(y) \frac{J_{\frac{n}{2}+\delta}(2\pi|x-y|)}{|x-y|^{\frac{n}{2}+\delta}} dy \right|^q dx \right)^{\frac{1}{q}}$$

$$\lesssim \int_{B(o,2r)} |f(y)| \left(\int_{B(o,r)} \left| \frac{J_{\frac{n}{2}+\delta}(2\pi|x-y|)}{|x-y|^{\frac{n}{2}+\delta}} \right|^q dx \right)^{\frac{1}{q}} dy.$$

Using

$$\begin{cases} z = x - y; \\ |z| \leq 3r; \\ \frac{n-1}{2} - \frac{\kappa}{p} + \frac{\lambda}{q} \leq \delta; \\ \frac{n-\lambda}{q} + \frac{\kappa}{p} > n, \end{cases}$$

the first inequality in (2.1) and the Hölder inequality we get

$$\left(\int_{B(o,r)} |\mathscr{T}_\delta f_1(x)|^q \, dx \right)^{\frac{1}{q}}$$

$$\lesssim \int_{B(o,2r)} |f(y)| \left(\int_{B(o,3r)} (1+|z|)^{-(\frac{n+1}{2}+\delta)q} dz \right)^{\frac{1}{q}} dy$$

$$\lesssim \int_{B(o,2r)} |f(y)| \left(\int_{B(o,3r)} (1+|z|)^{-(n-\frac{\kappa}{p}+\frac{\lambda}{q})q} \, dz \right)^{\frac{1}{q}} dy$$

$$\approx r^{-(n+\frac{\lambda-n}{q}-\frac{\kappa}{p})} \int_{B(o,2r)} |f(y)| \, dy$$

$$\lesssim r^{\frac{n-\lambda}{q}} \|f\|_{L^{p,\kappa}},$$

whence

$$\left(r^{\lambda-n} \int_{B(o,r)} |\mathscr{T}_\delta f_1(x)|^q \, dx \right)^{\frac{1}{q}} \lesssim \|f\|_{L^{p,\kappa}} \ \forall \ r \in [1, \infty). \tag{2.3}$$

At the same time, if

$$\begin{cases} z = x - y; \\ |z| \leq 3r; \\ \frac{n-1}{2} - \frac{\kappa}{p} + \frac{\lambda}{q} \leq \delta; \\ \frac{n-\lambda}{q} + \frac{\kappa}{p} = n, \end{cases}$$

then the previous analysis gives

$$\left(r^{\lambda-n} \int_{B(o,r)} |\mathscr{T}_\delta f_1(x)|^q \, dx \right)^{\frac{1}{q}} \lesssim (\ln(1+r))^{\frac{1}{q}} \|f\|_{L^{p,\kappa}} \tag{2.4}$$

On the other hand, an application of

$$\delta \geq \frac{\lambda}{q} + \frac{n-1}{2} - \frac{\kappa}{p} > \frac{n-1}{2} - \frac{\kappa}{p}$$

and the first estimate in (2.1) yields

$$
\begin{aligned}
&\left(\int_{B(o,r)}|\mathscr{T}_\delta f_2(x)|^q\,dx\right)^{\frac{1}{q}}\\
&=\left(\int_{B(o,r)}\left|\int_{\mathbb{R}^n\setminus B(o,2r)}f(y)\frac{J_{\frac{n}{2}+\delta}(2\pi|x-y|)}{|x-y|^{\frac{n}{2}+\delta}}dy\right|^q\,dx\right)^{\frac{1}{q}}\\
&\le\sum_{j=1}^{\infty}\left(\int_{B(o,r)}\left(\int_{B(o,2^{j+1}r)\setminus B(o,2^jr)}|f(y)|\left|\frac{J_{\frac{n}{2}+\delta}(2\pi|x-y|)}{|x-y|^{\frac{n}{2}+\delta}}\right|dy\right)^q\,dx\right)^{\frac{1}{q}}\\
&\lesssim\sum_{j=1}^{\infty}\left(\int_{B(o,r)}\left(\int_{B(o,2^{j+1}r)\setminus B(o,2^jr)}|f(y)|(1+|x-y|)^{-\frac{n}{2}-\delta-\frac{1}{2}}dy\right)^q\,dx\right)^{\frac{1}{q}}\\
&\lesssim\sum_{j=1}^{\infty}r^{\frac{n}{q}}(2^jr)^{-\frac{n}{2}-\delta-\frac{1}{2}}\int_{B(o,2^{j+1}r)}|f(y)|dy\\
&\le\sum_{j=1}^{\infty}r^{\frac{n}{q}}(2^jr)^{-\frac{n}{2}-\delta-\frac{1}{2}}(2^{j+1}r)^{n-\frac{\kappa}{p}}\|f\|_{L^{p,\kappa}}\\
&\lesssim r^{\frac{n-1}{2}-\delta+\frac{n}{q}-\frac{\kappa}{p}}\|f\|_{L^{p,\kappa}}\sum_{j=1}^{\infty}2^{(\frac{n-1}{2}-\delta-\frac{\kappa}{p})j}\\
&\lesssim r^{\frac{n-1}{2}-\delta+\frac{n}{q}-\frac{\kappa}{p}}\|f\|_{L^{p,\kappa}}.
\end{aligned}
$$

Consequently, from

$$\frac{\lambda}{q}+\frac{n-1}{2}-\delta-\frac{\kappa}{p}\le 0$$

it follows that

$$\left(r^{\lambda-n}\int_{B(o,r)}|\mathscr{T}_\delta f_2(x)|^q\,dx\right)^{\frac{1}{q}}\lesssim\frac{\|f\|_{L^{p,\kappa}}}{r^{-\frac{\lambda}{q}-\frac{n-1}{2}+\delta+\frac{\kappa}{p}}}\lesssim\|f\|_{L^{p,\kappa}}\quad\forall\quad r\in[1,\infty).\tag{2.5}$$

A combination of (2.3) or (2.4) and (2.7) gives that if $r\in[1,\infty)$ then

$$\left(r^{\lambda-n}\int_{B(o,r)}|\mathscr{T}_\delta f(x)|^q\,dx\right)^{\frac{1}{q}}\lesssim\begin{cases}\|f\|_{L^{p,\kappa}}\text{ under (1.3)};\\ \big(\ln e(1+r)\big)^{\frac{1}{q}}\|f\|_{L^{p,\kappa}}\text{ under (1.4)}.\end{cases}\tag{2.6}$$

Case - $r\in(0,1)$. Since there is an integer $j_0\ge 0$ such that

$$2^{-j_0-1}\le r<2^{-j_0},$$

we utilize the first estimate in (2) to get

$$\left(\int_{B(o,r)}|\mathscr{T}_\delta f_1(x)|^q\,dx\right)^{\frac{1}{q}}$$

$$\leq \int_{B(o,2r)} |f(y)| \left(\int_{B(o,r)} \left| \frac{J_{\frac{n}{2}+\delta}(2\pi|x-y|)}{|x-y|^{\frac{n}{2}+\delta}} \right|^q dx \right)^{\frac{1}{q}} dy$$
$$\lesssim \int_{B(o,2r)} |f(y)| \left| \int_{B(o,r)} 1dz \right|^{\frac{1}{q}} dy$$
$$\lesssim r^{n+\frac{n}{q}-\frac{\kappa}{p}} \|f\|_{L^{p,\kappa}},$$

whence

$$\left(r^{\lambda-n} \int_{B(o,r)} |\mathscr{T}_\delta f_1(x)|^q \, dx \right)^{\frac{1}{q}} \lesssim r^{\frac{\lambda}{q}+n-\frac{\kappa}{p}} \|f\|_{L^{p,\kappa}} \lesssim \|f\|_{L^{p,\kappa}} \quad \forall \;\; r \in (0,1). \tag{2.7}$$

Meanwhile, we utilize the first inequality in (2.1), the Hölder inequality and

$$\delta \geq \frac{\lambda}{q} + \frac{n-1}{2} - \frac{\kappa}{p} > \frac{n-1}{2} - \frac{\kappa}{p}$$

to calculate

$$\left(\int_{B(o,r)} |\mathscr{T}_\delta f_2(x)|^q \, dx \right)^{\frac{1}{q}}$$
$$\lesssim \left(\int_{B(o,r)} \left(\int_{\mathbb{R}^n \setminus B(o,2r)} |f(y)| \left| \frac{J_{\frac{n}{2}+\delta}(2\pi|x-y|)}{|x-y|^{\frac{n}{2}+\delta}} \right| dy \right)^q dx \right)^{\frac{1}{q}}$$
$$\lesssim \left(\int_{B(o,r)} \left(\sum_{j=1}^{\infty} \int_{B(o,2^{j+1}r) \setminus B(o,2^j r)} |f(y)| \left| \frac{J_{\frac{n}{2}+\delta}(2\pi|\xi-y|)}{|x-y|^{\frac{n}{2}+\delta}} \right| dy \right)^q dx \right)^{\frac{1}{q}}$$
$$\lesssim \left(\int_{B(o,r)} \left(\Big(\sum_{j=1}^{j_0+1} + \sum_{j=j_0+2}^{\infty} \Big) \int_{B(o,2^j r) \setminus B(o,2^{j-1}r)} |f(y)| \left| \frac{J_{\frac{n}{2}+\delta}(2\pi|x-y|)}{|x-y|^{\frac{n}{2}+\delta}} \right| dy \right)^q dx \right)^{\frac{1}{q}}$$
$$\lesssim \left(\int_{B(o,r)} \left(\sum_{j=1}^{j_0+1} \int_{B(x,2^{j+2}r)} |f(y)| \, dy \right)^q dx \right)^{\frac{1}{q}}$$
$$+ \left(\int_{B(o,r)} \left(\sum_{j=j_0+1}^{\infty} \int_{B(x,2^{j+2}r) \setminus B(x,2^{j-2}r)} \frac{|f(y)|}{(1+|x-y|)^{\frac{1+n}{2}+\delta}} \, dy \right)^q dx \right)^{\frac{1}{q}}$$
$$\lesssim \left(\int_{B(o,r)} \left(\sum_{j=1}^{j_0+1} \int_{B(x,2^{j+2}r)} |f(y)| \, dy \right)^q dx \right)^{\frac{1}{q}}$$
$$+ \left(\int_{B(o,r)} \left(\sum_{j=j_0+1}^{\infty} \int_{B(x,2^{j+2}r) \setminus B(x,2^{j-2}r)} \frac{|f(y)|}{(1+|x-y|)^{n-\frac{\kappa}{p}+\frac{\lambda}{q}}} \, dy \right)^q dx \right)^{\frac{1}{q}}$$

$$\lesssim \|f\|_{L^{p,\kappa}} \left(\sum_{j=1}^{j_0+1} 2^{j(n-\frac{\kappa}{p})} r^{n-\frac{\kappa}{p}+\frac{n}{q}} + \sum_{j=j_0+1}^{\infty} 2^{-\frac{j\lambda}{q}} r^{\frac{n-\lambda}{q}} \right)$$
$$\lesssim \|f\|_{L^{p,\kappa}} r^{\frac{n}{q}}.$$

Consequently, we find that if $r \in (0,1)$ then

$$\left(r^{\lambda-n} \int_{B(o,r)} |\mathscr{T}_\delta f_2(x)|^q \, dx \right)^{\frac{1}{q}} \lesssim \|f\|_{L^{p,\kappa}} r^{\frac{\lambda}{q}} \lesssim \|f\|_{L^{p,\kappa}}. \tag{2.8}$$

Clearly, putting together (2.7) and (2.8) implies that if $r \in (0,1)$ then

$$\left(r^{\lambda-n} \int_{B(o,r)} |\mathscr{T}_\delta f(x)|^q \, dx \right)^{\frac{1}{q}} \lesssim \begin{cases} \|f\|_{L^{p,\kappa}} & \text{under (1.3);} \\ \big(\ln e(1+r) \big)^{\frac{1}{q}} \|f\|_{L^{p,\kappa}} & \text{under (1.4).} \end{cases} \tag{2.9}$$

Now, a combination of (2.6) and (2.9) produces (2.2). □

References

[1] D. R. Adams and J. Xiao, *Morrey spaces in harmonic analysis*, Ark. Mat. **50** (2012), 201–230.

[2] D. R. Adams and J. Xiao, *Bochner-Riesz means of Morrey functions*, J. Fourier Anal. Appl. DOI: 10.1007/s00041-019-09712-x.

[3] C. Benea, F. Bernicot and T. Luque, *Sparse bilinear forms for Bochner Riesz multipliers and applications*, Trans. London Math. Soc. **4** (2017), 110–128.

[4] S. M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. **340** (1993), 253–272.

[5] L. Carleson and P. Sjölin, *Osillatory integrals and a multiplier problem for the disc*, Studia Math. **44** (1972), 287–299.

[6] M. Giaquinta, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Math. Studies, Princeton Univ. Press, 1983.

[7] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Inc., 2004.

[8] L. Liu and J. Xiao, *Restricting Riesz-Morrey-Hardy potentials*, J. Differential Equations **262** (2017), 5468–5496.

[9] L. Liu and J. Xiao, *Morrey potentials from Campanato classes*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **18** (2018), 1503–1517.

[10] L. Liu and J. Xiao, *A trace law for the Hardy-Morrey-Sobolev space*, J. Funct. Anal. **274** (2018), 80–120.

[11] S. Lu, *Conjectures and problems on Bochner-Riesz means*, Front. Math. China **8** (2013), 1237–1251.

[12] R. Moser, *An L^p regularity theory for harmonic maps*, Trans. Amer. Math. Soc. **367** (2015), 1–30.

[13] X. Shi and Q. Sun, *Weighted norm inequalities for Bochner-Riesz operators and singular integral operators*, Proc. Amer. Math. Soc. **116** (1992), 665–673.

[14] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

Manuscript received February 10 2019
revised May 16 2019

H. Wang
College of Mathematics and Econometrics, Hunan University, Changsha 410082 Hunan, China
E-mail address: wanghua@pku.edu.cn

J. Xiao
Department of Mathematics and Statistics, Memorial University, St. John's, NL A1C 5S7, Canada
E-mail address: jxiao@mun.ca

S. Xu
School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China
E-mail address: xushaozhen14b@mails.ucas.ac.cn