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QUANTITATIVE STABILITY FOR EQUILIBRIUM PROBLEMS UNDER RELAXED UPPER-SIGN PROPERTIES: APPLICATION TO QUASICONVEX PROGRAMMING

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Abstract. We establish Hölder and Lipschitz estimates for solutions to equilibrium problems under relaxed upper-sign properties introduced in M. Castellani and M. Guili [19] [*Refinements of existence results for relaxed quasimonotone equilibrium problems.* J Glob Optim. 57 (2013) 1213-1227]. This type of weak regularity will play a crucial role in our analysis by relaxing the strong monotonicity condition considered in [1] M. Ait Mansour and H. Riahi [*Sensitivity analysis for abstract equilibrium problems.* J Math Analysis Appl. 306 (2005) 684–691]. The alternative is the strong quasiconvexity assumption that enables us, via weak Minty solutions, to extend to equilibrium problems the quantitative stability of star or strict solutions to Stampacchia variational inequalities presented in M. Ait Mansour and D. Aussel [3] [*Quasimonotone variational inequalities and quasiconvex programming: Quantitative stability. Pacific* J. Optim. **3** (2006), 611–626]. The obtained results are thereafter applied to parametric quasiconvex programming where we show that the passage via variational formulations is not needed, by providing a quantitative stability under the perturbation of the objective function, completing henceforth the study presented in [3] (see above).

1. Introduction

One of the central topics related to optimization and equilibrium problems is the study of stability and sensitivity analysis of their solutions. It has implication in both existence theory and algorithmic procedures. Indeed, such a study enables us to evaluate the impact of external perturbations on equilibrium points, which is interesting for numerical treatments, optimality characterizations and approximation theory especially for infinite dimensional problems. The interested reader to this theme may consult [1–4, 9, 10] and the bibliography therein.

Let X, M and Λ be normed vector spaces and consider, for $p \in M$ and $\lambda \in \Lambda$, the parametric equilibrium problem: find $x \in K_\lambda$ such that

$$EP(f_p, K_\lambda) \quad f_p(x, y) \geq 0, \ \forall y \in K_\lambda, \tag{1.1}$$

where $\{K_\lambda\}_{\lambda\in\Lambda}$ is a family of subsets of X and $\{f_p\}_{p\in M}$ is a family of bifunctions i.e., for each $p \in M$, $f_p(.,.)$ is a real-valued function of two variables x and y. In this problem, M and Λ represent the spaces of parameters p and λ respectively. The

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formulation $EP(f_p, K_\lambda)$ corresponds to a parametric perturbation of an original equilibrium problem of the type: find $x \in K$ such that

$$EP(f,K) \qquad f(x,y) \geq 0, \ \forall y \in K, \tag{1.2}$$

where $f : K \times K \to \mathbb{R}$ and $K \subset X$.

In this paper we establish quantitative stability for the problem $EP(f_p, K_\lambda)$. Roughly speaking, we evaluate the influence of the parametric perturbation on the solutions set $S_{p,\lambda}$. Here, the evaluation in question is expected to be an estimate of a Lipschitz or Hölder type by computing the distance between two solutions sets $S_{p,\lambda}$ and $S_{p',\lambda'}$ in terms of the norm perturbations $\|p-p'\|$ and $\|\lambda-\lambda'\|$. This means that we have to find as far as possible the sharpest error bounds, say $e_1, e_2 > 0$, such that for some q_1, q_2 from $(0,1]$, for any p, p' and λ, λ' around initial values $\overline{p}$ and $\overline{\lambda}$ respectively,

$$haus(S_{p,\lambda}, S_{p',\lambda'}) \leqslant e_1\|p-p'\|^{q_1} + e_2\|\lambda-\lambda'\|^{q_2}. \tag{1.3}$$

Here, $haus(A,B)$ stands for the Hausdorff distance between two bounded subsets A and B. We would like to emphasize that it is not an easy task to obtain such a stability result by considering at the same time perturbations on both the bifunction and the constraint set because of technical difficulties besides the very weak format of the continuity and monotonicity assumptions, namely, weak quasimonotonicity and relaxed upper sign property. Thus, inspired by [1,3], we divide the treatment in two steps where we firstly deal with one parameter inherent in the objective bifunction level and secondly with the other one arising in the constraint set. As observed in the two lastly quoted references, the case of global perturbation in both of the constraints and the objective bifunction comes easily from these two cases by the triangle inequality.

The main primary observation of the present research is that estimates of the type (1.3), when the constraints set is fixed (i.e., $\lambda = \lambda'$ and $e_2 = 0$) could be derived from a similar estimate of Minty type solutions (see Theorem 4.6 below) under a very weak continuity assumption only supposed on the objective function namely relaxed upper sign continuity. Instead, we consider the condition of strong quasiconvexity -with respect to the second argument- that plays the role of strong generalized monotonicity usually required to obtain quantitative stability in variational problems. But we observed that under this weaker format of assumptions, a restriction on the required stability is in force. More precisely, Theorem 4.6 (below) ensures the Hölder estimate only by fixing p' at the initial value $\overline{p}$. Thus, with a similar technique as in Theorem 4.6, we prove quantitative stability of standard equilibrium points of weakly pseudomonotone perturbed bifunctions, see Theorem 4.11.

A particular but very important example of application is the parametric quasiconvex programming. In this context, one can expect a result of the following form: given a family of parametric quasiconvex real-valued functions, say $g_p : X \to \mathbb{R}$, by keeping K_λ as before in the abstract formulation, we are able to provide conditions through which the following can be envisaged:

$$haus(\underset{K_\lambda}{\mathrm{A}rgmin}\, g_p, \underset{K_{\lambda'}}{\mathrm{A}rgmin}\, g_{p'}) \leqslant e_1\|p-p'\|^{q_1} + e_2\|\lambda-\lambda'\|^{q_2}. \tag{1.4}$$

Moreover, our method leads to a new conclusion regarding what we call global *strong minimizers* which meet the so-called strict minimizers of order 2 also known as 2-order sharp minimizers with a certain modulus following [30]. In particular, we stress the role of strong quasiconvexity to obtain this sharpness with different modulus of the form $\tau\alpha$, where τ is any real number in $(0, \frac{1}{4})$ and α is the strong quasi-convexity constant of the function subject to parametric minimization.

Let us emphasise that the stability of star and strict solutions considered in [3] for variational inequalities is not evidently generalizable to the general formulation of the problem $EP(f, K)$, and in addition, this kind of solutions always requires the non-emptiness of the interior of the corresponding constraints. We are thus motivated in this work to consider rather weak Minty solutions which, thanks to relaxed upper-sign property, become standard solutions as underlined in [19]. Moreover, with an intimate link to strong quasiconvexity, we introduce the notion of *strong solutions* which are nothing else but a type of strong approximate solutions and may meet the class of star and strict solutions when we regard equilibrium points as solutions to Stampacchia variational inequalities, and will prove to have stability properties under mild assumptions.

The paper is organized as follows. In Section 2, we recall the material and fix the notations we need. In Section 3, we introduce and discuss solutions concepts as well as a fundamental penalization result. In Section 4, we present our main results on quantitative stability and situate them in the close recent literature. We begin with the case of perturbation on the objective bifunction and end by the one of perturbation at the level of the feasibility set K. Finally, in Section 5, we discuss applications to quasiconvex programming for which we establish two results.

2. Preliminaries

Throughout the paper, unless otherwise is specified, X is a normed vector space whose norm is denoted by $\|.\|$, X^* will denote its topological dual and $\langle ., . \rangle$ the corresponding duality pairing. The closed unit ball of X (resp. X^*) will be denoted by $\bar{B}_X$ (resp. $\bar{B}_{X^*}$) and we write $\bar{B}(x, r)$ for the closed ball of radius $r > 0$ and center $x \in X$. The notations Λ and M will stand for two other normed vector spaces of parameters, whose norms are also denoted by $\|.\|$. We further consider a family of arbitrary convex subsets $\{K_\lambda\}_{\lambda\in\Lambda}$ of X and a family of bifunctions $\{f_p := f_p(., .)\}_{p\in M}$ defined on $K \times K$, where K is a closed and convex subset of X. Given a pair $(p, \lambda) \in M \times \Lambda$, we consider the parametric equilibrium problem defined above in (1.1) and fix $\bar{p} \in M$ and $\bar{\lambda} \in \Lambda$.

2.1. **Weak Minty solutions and notations.** Let K be a closed and convex subset of X. For any $\mu \geqslant 0$, the *weak μ-Minty equilibrium problem* corresponding to (1.2) is as follows: find $x \in K$ such that

$$\mu - MEP(f,K) \quad f(y,x) \leqslant \mu\|y-x\|^2,\ \forall y \in K.$$

For $\mu = 0$, $\mu - MEP(f,K)$ collapses into the standard Minty equilibrium formulation $MEP(f,K)$: find $x \in K$ such that $f(y,x) \leq 0$ for all $y \in K$.

Definition 2.1. ([19]) Let $\mu \geq 0$ be fixed. A point $x \in K$ is said to be a *local weak μ-Minty* equilibrium for f if there exists a neighborhood $\mathcal{V}_x$ of x such that

$$f(y,x) \leqslant \mu\|y-x\|^2, \quad \forall y \in K \cap \mathcal{V}_x.$$

<u>**Notations**</u>:

- ▶ For any $\mu \geqslant 0$, $M^\mu(f,K)$ will stand for the set of solutions to $\mu - MEP(f,K)$ and $M_L^\mu(f,K)$ for the local solutions of this problem.
- ▶ $S(f,K)$ denotes the set of (standard) solutions to the equilibrium problem $EP(f,K)$.
- ▶ For any $p \in M$, S_p denotes the set of solutions to $EP(f_p,K)$ i.e., $S_p := S(f_p,K)$.
- ▶ For any $\lambda \in \Lambda$, S_λ denotes the set of solutions to $EP(f,K_\lambda)$ i.e., $S_\lambda := S(f,K_\lambda)$.
- ▶ For any $p \in M$, M_p^μ will stand for the solutions to $\mu - MEP(f_p,K)$ i.e., $M_p^\mu := M^\mu(f_p,K)$.
- ▶ For any $p \in M$, $L - M_p^\mu$ will stand for the local solutions to $\mu - MEP(f_p,K)$ i.e., $L - M_p^\mu := M_L^\mu(f_p,K)$.

2.2. **The $\mu-$upper sign property.** Let K be a closed and convex subset of X. The following concept of regularity will play an important role in our stability analysis.

Definition 2.2. ([19]) Let $\mu \geq 0$. A bifunction f is said to have the *μ-upper sign property* at $x \in K$ (with respect to the first variable) if there exists a convex neighborhood $\mathcal{V}_x$ of x such that for all $y \in \mathcal{V}_x \cap K$,

$$f(z_t,x) \leqslant \mu\|z_t - x\|^2, \forall t \in]0,1[\Longrightarrow f(x,y) \geqslant 0, \tag{2.1}$$

where $z_t = (1-t)x + ty$. If $\mu = 0$, the equilibrium bifunction f is said to have the upper sign property.

Remark 2.3.

- For the case $\mu = 0$, if f is upper hemicontinuous in x and semistrictly quasiconvex (in particular if it is strongly convex) in the second argument then it has the μ-upper sign property in x, see [19, Lemma 3 and Remark 1 pp 1221].
- The μ-upper sign property in x is not a kind of weak continuity, but if for every $x \in K$ there exists $r > 0$ such that for every $y \in K \cap \overline{B}(x,r)$ the following implication is satisfied:

$$(1-t)f(z_t,x) + tf(z_t,y) \geq 0,\ \forall t \in]0,1[, \tag{2.2}$$

  then the upper hemicontinuity of f in x implies the μ-upper sign property of f at x for every $\mu \geq 0$, see [19, Lemma 4, pp 1223] for more details. The condition (2.2) is satisfied if f is locally convex in the second argument, uniformly with respect to x, that is, there exists $r > 0$, independent of x, such that the function $f(x, .)$ restricted to the open ball $B(x, r)$ is convex.

For further discussion on this new concept of regularity in the context of variational inequalities, the interested reader is referred to [19, Section 3.2, pp 1224]. Here, we only recall its interest in our case in the following result.

Proposition 2.4. *Let K be a convex subset of X and $f : K \times K \longrightarrow \mathbb{R}$ be a bifunction. If f has the μ-upper sign property in x and for all $x, y \in K$,*

$$f(x,y) < 0 \Longrightarrow f(x,(1-t)x+ty) < 0, \ \ \forall t \in]0,1[, \tag{2.3}$$

then $M^\mu(f,K) \subset M_L^\mu(f,K) \subset S(f,K)$.

Proof. The inclusion $M^\mu(f,K) \subset M_L^\mu(f,K)$ is always true while the second one $M_L^\mu(f,K) \subset S(f,K)$ is exactly the result in [19, Theorem 1].

□

2.3. **Continuity concepts for set-valued maps and Stampacchia variational inequalities.** For any nonempty subset A of X and any point $x \in X$, $d(x, A) = \inf\{\|x-y\|| : \ y \in A\}$ will stand for the distance from x to A, and if B is another subset of X, $e(A, B)$ denotes the excess of A on B given by $e(A, B) = \sup\{d(a, B) : a \in A\}$. Finally, the Hausdorff distance between two bounded subsets A and B of X is given by

$$haus(A,B) = \max\{e(A,B), e(B,A)\}.$$

To make sense of the use of Hausdorff distances in our quantitative stability, throughout the remaining of the paper, the solutions sets to the considered equilibrium problems are supposed to be nonempty and **bounded**. As observed in [3], the case of unboundedness can obtained from the bounded one with the use of the so-called ρ-Hausdorff distance also known as Attouch-Wets distance.

Let us now recall the concept on continuity of set-valued maps needed for our purpose.

Definition 2.5. Let $(M, \|.\|)$ and $(X, \|.\|)$ be two normed vector spaces. A set valued mapping $T : M \rightrightarrows X$ is said to be Lipschitz continuous relative to a (nonempty) set D of the domain of T (i.e., $D \subset domT$) if there exists $\kappa \geq 0$ such that

$$haus(T(y'), T(y)) \leqslant \kappa\|y-y'\|, \ \ \forall y, y' \in D, \tag{2.4}$$

or equivalently, there exists $\kappa \geq 0$ such that

$$T(y') \subset T(y) + \kappa\|y-y'\| \, \overline{B}_X, \ \ \forall y, y' \in D, \tag{2.5}$$

where $\overline{B}_X = \{x \in X : \ \ \|x\| \leq 1\}$ denotes the closed unit ball in X.

Let us introduce some notations which will be useful throughout this paper and recall the classical definition of *Stampacchia variational inequality.*

Definition 2.6. ([3]) Let K be a closed and convex subset of X and let $T : X \rightrightarrows X^*$ be a set valued map. The *Stampacchia variational inequality* problem related to T and K is to find $x \in K$ and $x^* \in T(x)$ such that

$$SVI(T,K) \qquad \langle x^*, y - x\rangle \geq 0, \ \ \forall y \in K.$$

We denote by $S_T(K)$ the set of solutions to $SVI(T,K)$.
If the (dual)-element x^* in the inequality $SVI(T,K)$ is such that $x^* \in T(x)\backslash\{0\}$, then x is called a *star solution* to $SVI(T,K)$. We denote by $S_T^*(K)$ the set of star solutions of $SVI(T,K)$.
If the (dual)-element x^* in the inequality $SVI(T,K)$ is such that $\langle x^*, y - x\rangle > 0, \ \ \forall y \in K\backslash\{x\}$, then x is called a *strict solution* to $SVI(T,K)$. We denote by $S_T^>(K)$ the set of strict solutions of $SVI(T,K)$.

Remark 2.7. Strict and star solutions when they exist should belong to the boundary of K, see [4, Subsection 4.2] and the comment before relation (1) in [8]. Moreover, as considered in [3, 4, 10], this kind of solutions exist if K has a nonempty relative interior.

2.4. **Generalized convexity and generalized monotonicity.** Let K be a nonempty convex subset of X. A function $g : K \longrightarrow \mathbb{R}$ is said to be:

- quasiconvex if for all $x, y \in K$,
$$g(tx + (1-t)y) \leqslant \max\{g(x), g(y)\}, \ \forall t \in]0,1];$$
- strictly quasiconvex if for all $x, y \in K$ with $x \neq y$,
$$g(tx + (1-t)y) < \max\{g(x), g(y)\}, \ \forall t \in]0,1[;$$
- semistrictly quasiconvex if g is quasiconvex and for all $x, y \in K$,
$$g(x) < g(y) \Longrightarrow g(tx + (1-t)y) < g(y), \ \ \forall t \in]0,1[; \tag{2.6}$$
- α-strongly quasiconvex, for some $\alpha \geq 0$, if for any $x, y \in K$ and any $t \in [0,1]$,
$$g(tx + (1-t)y) \leqslant \max\Big(g(x), g(y)\Big) - \alpha t(1-t)\|x - y\|^2.$$

Definition 2.8. ([19]) Let $\mu \geqslant 0$ be fixed and f be a real-valued bifunction defined on $K \times K$, where K is a convex subset of X. The bifunction f is said to be

- μ-weakly monotone if, and only if for all $x, y \in K$,
$$f(x,y) + f(y,x) \leqslant \mu\|y - x\|^2; \tag{2.7}$$
- μ-weakly pseudomonotone if, and only if for all $x, y \in K$,
$$f(x,y) \geqslant 0 \Longrightarrow f(y,x) \leqslant \mu\|y - x\|^2; \tag{2.8}$$
- μ-weakly quasimonotone if, and only if for all $x, y \in K$,
$$f(x,y) > 0 \Longrightarrow f(y,x) \leqslant \mu\|y - x\|^2; \tag{2.9}$$
- strongly μ-quasimonotone if, and only if for all $x, y \in K$,
$$f(x,y) > 0 \Longrightarrow f(y,x) \leqslant -\mu\|y - x\|^2; \tag{2.10}$$

- properly μ-weakly quasimonotone if, and only if for all $n \in \mathbb{N}$, $x_1, \dots, x_n \in K$ and $\overline{x} \in Conv\{x_1, \dots, x_n\}$ there exists $i \in \{1, \dots, n\}$ such that

$$f(x_i, \overline{x}) \leqslant \mu \|x_i - \overline{x}\|^2.$$

Remark 2.9. Observe that

(1) The concept of μ-weakly pseudomonotonicity is closely tied with the set of weak μ-Minty equilibrium points of a bifunction f_p. Actually, if (2.8) is satisfied then standard equilibria are also of a μ-Minty type i.e., $S_p \subset M_p^\mu$.

(2) If, in addition to conditions of the previous point, f_p is α-strongly quasiconvex then M_p^μ coincides with S_p and reduces at most to a point for every μ such that f_p has the μ-upper sign property with $\mu < \frac{\alpha}{4}$.

3. Penalization and uniqueness

In this section, we introduce and discuss a new concept of equilibrium points, what we call μ-strong equilibrium point. Let K be a closed and convex subset of X and $f : K \times K \to \mathbb{R}$ a bifunction. Let us agree to introduce the following:

Definition 3.1. Let $\mu \geq 0$. A point $x \in K$ will be said to be *μ-strong solution* (or *μ-strong equilibrium point*) to $EP(f, K)$ if

$$\mu - EP(f, K) \qquad f(x, y) \geqslant \mu \|y - x\|^2, \ \ \forall y \in K. \tag{3.1}$$

We denote by $S^\mu(f, K)$ the set of solutions to $\mu - EP(f, K)$.

Remark 3.2.

- μ-strong solutions to $EP(f, K)$ are nothing else but standard solutions to the following (auxiliary) equilibrium problem $EP(g, K)$ with $g(x, y) = f(x, y) - \mu\|x - y\|^2$, $x, y \in K$. They are of course of strong type since $\mu \geq 0$. Weak solutions correspond to $\mu < 0$. A further motivation to qualify this type of solutions as strong ones comes from the strong quasiconvexity, see Proposition 3.4.
- It is straightforward to see that $S^\mu(f, K) \subset S(f, K)$ for all $\mu \geq 0$, the converse will be the object of Proposition 3.4 below.

The concept of μ- strong solutions in the case of Stampacchia variational inequalities is given in the following definition

Definition 3.3. Let K be a closed and convex subset of X and $T : K \rightrightarrows X^*$ be a set-valued map. For any $\mu \geq 0$, a point $x \in K$ will be said to be *μ-strong solution* to $SVI(T, K)$ if there exists a dual element $x^* \in T(x)$ such that

$$\mu - SVI(T, K) \qquad \langle x^*, y - x\rangle \geqslant \mu \|y - x\|^2, \ \ \forall y \in K. \tag{3.2}$$

We denote by $S_T^\mu(K)$ the set of μ-strong solutions to $SVI(T, K)$.

For the link between equilibrium problems and variational inequalities, we refer to the recent study by Aussel , Dutta and Pandit [7]. Next, we show that μ-strong equilibrium points and standard ones coincide under strong quasiconvexity condition.

Proposition 3.4. *Let K be a convex subset of X and $f : K \times K \longrightarrow \mathbb{R}$ be a bifunction. Assume that the following conditions hold:*

i) $f(x,x) = 0$ *for all* $x \in K$,
ii) f *is* μ*-strongly quasiconvex in the second argument for some* $\mu \geq 0$.

Then, for all $\tau \in [0, \frac{1}{4}]$, *the problems* $EP(f,K)$ *and* $\tau\mu$-$EP(f,K)$ *are equivalent i.e.,* $S(f,K) = S^{\tau\mu}(f,K)$.

Proof. The inclusion $S^{\tau\mu}(f,K) \subset S(f,K)$ is trivial. We have to prove the converse inclusion. Let $\overline{x}$ be a solution to $EP(f,K)$ i.e., $\overline{x} \in S(f,K)$. Then,

$$f(\overline{x}, x) \geqslant 0, \ \ \forall x \in K. \tag{3.3}$$

Let $t \in [0,1]$ and $x \in K$, put $x_t = t\overline{x} + (1-t)x$. By (3.3) and assumption ii) we have

$$0 \leqslant f(\overline{x}, x_t) \leqslant \max\{f(\overline{x},\overline{x}), f(\overline{x},x)\} - t(1-t)\mu\|\overline{x} - x\|^2, \ \ \forall x \in K. \tag{3.4}$$

Hence, by assumption i) and (3.3), we obtain

$$0 \leqslant f(\overline{x}, x_t) \leqslant f(\overline{x}, x) - t(1-t)\mu\|\overline{x} - x\|^2, \ \ \forall x \in K. \tag{3.5}$$

Therefore,

$$t(1-t)\mu\|\overline{x} - x\|^2 \leqslant f(\overline{x}, x), \ \ \forall x \in K, \ \forall t \in [0,1]. \tag{3.6}$$

Thus, given that the function $\psi(t) = t(1-t)$ is continuous with $\frac{1}{4}$ as least upper bound, it follows that

$$\tau\mu\|\overline{x} - x\|^2 \leqslant f(\overline{x}, x), \ \ \forall x \in K, \ \forall \tau \in [0, \frac{1}{4}]. \tag{3.7}$$

This means that $\overline{x} \in S^{\tau\mu}(f,K)$ for all $\tau \in [0, \frac{1}{4}]$, ending the proof. □

Remark 3.5. The condition i) in Proposition 3.4 is sufficient but not necessary as shows the following counter-example.

Example 3.6. Take $f(x,y) = y^2 - x$ with $K = [0,1]$. Clearly, f is α-strongly convex (and hence α-strongly quasiconvex) for all $\alpha \in]0,1]$ but $f(x,x) = x^2 - x < 0$ for all $x \in]0,1[$. However, 0 is a $\tau\alpha$-strong equilibrium (unique in this case) for f over $[0,1]$ for all $\tau \in (0, \frac{1}{4})$ since $f(0,y) = y^2 \geqslant \alpha|y-0|^2 \geqslant \tau\alpha|y-0|^2$ for all $y \in K$. If $K = [1, +\infty[$ then 1 is a $\tau\alpha$-strong equilibrium (unique in this case) of f over K since $f(1,y) \geqslant \alpha|y-1|^2 \geqslant \tau\alpha|y-1|^2$ for all $y \in K$ and all $\tau \in (0, \frac{1}{4})$. Remark also that $g(x,y) = y^2 - x^2$ satisfies the conditions i) and ii) (with $\alpha \leqslant 1$) of Proposition 3.4 and it has the same unique equilibrium point 0 (resp. 1) over $[0,1]$ (resp. $[1,+\infty[$) which is a $\tau\alpha$-strong equilibrium for f over $[0,1]$ (resp. $[1,+\infty[$) for any $\tau \in (0, \frac{1}{4})$.

Proposition 3.7. *Let* $\mu > 0$, K *be a convex subset of* X *and* $f : K \times K \longrightarrow \mathbb{R}$ *be a bifunction. Assume that the following conditions hold:*
i) $f(x,x) = 0$ *for all* $x \in K$;
ii) f *is* $\tau\mu$*-weakly quasimonotone for some* $\tau \in]0, \frac{1}{4}[$;
iii) f *is* μ*-strongly quasiconvex in the second argument.*

Then, the problem $EP(f,K)$ *admits at most one solution.*

Proof. Suppose for contradiction that $EP(f,K)$ admits two solutions x_1 and x_2 in K ($x_1 \neq x_2$). Since f is μ-strongly quasiconvex in y, from Proposition 3.4, for all $\tau' \in]0, \frac{1}{4}[$, it follows that

$$\tau'\mu\|x_1 - x_2\|^2 \leqslant f(x_1, x_2).$$

Clearly, $f(x_1, x_2) > 0$, then by the $\tau\mu$-weak quasimonotonicity of f we deduce that

$$f(x_2, x_1) \leqslant \tau\mu\|x_1 - x_2\|^2. \tag{3.8}$$

But x_2 is also a solution to $EP(f,K)$, then we have

$$\tau'\mu\|x_1 - x_2\|^2 \leqslant f(x_2, x_1),\ \forall \tau' \in]0, \frac{1}{4}[. \tag{3.9}$$

Hence, for any $\tau' > \tau$ ($\tau' \in]0, \frac{1}{4}[$), (3.8) and (3.9) are in conflict. □

Remark 3.8.

- The assumption ii) is essential in Proposition 3.7. To see this, let $g : \mathbb{R} \to \mathbb{R}$ be an α-strongly quasiconvex function with nonnegative real values, $\alpha > 0$, such that $g(0) = 0$ and

$$\lim_{x \to +\infty} g(x) = +\infty. \tag{3.10}$$

  Consider the bifunction $f : K \times K \to \mathbb{R}$ defined, for $x, y \in K$, by $f(x,y) = g(y-x)$. Clearly, it is a simple matter to check that f is α-strongly quasiconvex in the second argument. Observe that in view of (3.10), for a fixed $x \in \mathbb{R}$,

$$\lim_{y \to +\infty} f(x,y) = +\infty. \tag{3.11}$$

  This implies that f is not μ-weakly quasimonotone for any $\mu > 0$. Moreover, for all $x \in \mathbb{R}$, $f(x,x) = g(0) = 0$. However, $\mathbb{R}$ provides a set of equilibrium points of f over $\mathbb{R}$.
- The conclusion of Proposition 3.7 remains true if f is μ-weakly quasimonotone and $\frac{\mu}{\tau}$-strongly quasiconvex in the second argument for some $\tau \in]0, \frac{1}{4}[$.
- Under assumptions of Proposition 3.7 and according to proposition 3.4, also the problem $\mu - EP(f,K)$ admits at most one solution.

4. Quantitative stability

4.1. **Perturbation with respect to the objective bivariate function f.** We begin our treatment by considering the perturbation at the level of the bifunction, the convex constraints set K being fixed. In this respect, for every $p \in M$, let S_p be the set of solutions to the following perturbed problem: find $x_p \in K$ such that

$$EP(f_p, K) \quad f_p(x_p, y) \geqslant 0,\ \forall y \in K.$$

For a given $\mu \geq 0$ and $p \in M$, the weak-μ Minty problem (whose set of solutions is already fixed as M_p^μ) corresponding to $EP(f_p, K)$ is as follows: find $x_p^\mu \in K$ such that

$$\mu - MEP(f_p, K) \quad f_p(y, x_p^\mu) \leqslant \mu\|y - x_p^\mu\|^2, \ \forall y \in K.$$

For a fixed initial value $\bar{p} \in M$ of the parameter, the following assumption will be needed:

($\mathcal{A}_1$) There exist $\theta > 0$, $2 > \delta \geq 0$, and there exists γ with $\min\{1, 2-\delta\} \geqslant \gamma \geq 0$ and a neighborhood N of $\bar{p}$ such that for all $x, y \in K$ and all $p, p' \in N$ we have

$$|f_p(x,y) - f_{p'}(x,y)| \leqslant \theta \left\|p - p'\right\|^\gamma \|y - x\|^\delta .$$

Example 4.1. Let $g_p : K \longrightarrow \mathbb{R}$ be a family of Lipschitz functions for $p \in M$. Assume that the Clarke subdifferential of g_p satisfies the following Lipschitz property with respect to p: There exist $\theta > 0$ and a neighborhood N of $\bar{p}$ such that for all $p, p' \in N$,

$$\partial^C g_p(x) \subset \partial^C g_{p'}(x) + \theta\|p - p'\|\bar{B}_{X^*}, \quad \forall x \in K. \tag{4.1}$$

Then, the family of bifunctions $(f_p)_{p\in N}$ such that $f_p(x,y) := g_p^0(x, y-x)$ satisfies the assumption ($\mathcal{A}_1$) with $\delta = \gamma = 1$. Indeed, from (4.1) we easily obtain

$$\begin{aligned} g_p^0(x, y-x) &= \sup_{z^* \in \partial^C g_p(x)} \langle z^*, y - x\rangle \\ &\leq \sup_{w^* \in \partial^C g_{p'}(x), u^* \in B_{X^*}} \langle w^* + \theta\|p - p'\|u^*, y - x\rangle \\ &\leq \sup_{w^* \in \partial^C g_{p'}(x)} \langle w^*, y - x\rangle + \theta\|p - p'\|\|x - y\| \\ &= g_{p'}^0(x, y - x) + \theta\|p - p'\|\|x - y\|. \end{aligned}$$

Consequently,

$$g_p^0(x, y-x) - g_{p'}^0(x, y-x) \leq \theta\|p - p'\|\|x - y\| \quad \forall x, y \in K \quad \textit{and} \quad \forall p, p' \in N. \tag{4.2}$$

Since the role of p and p' is symmetric, the previous inequality is also true with p (resp. p') at the place of p' (resp. p). This shows that ($\mathcal{A}_1$) is satisfied for this example.

Remark 4.2. Taking into account that the subsets $\partial^C g_p(x)$ are weak* closed in X^*, the condition (4.1) is nothing else but the Lipschitz continuity relative to N of the map $p \rightrightarrows \partial^C g_p$.

Example 4.3. Let K be a compact subset of $\mathbb{R}^n$, $n \geq 1$. For a given $\alpha > 0$, for any $p > 1$, consider the bifunction f_p defined on $K \times K$ by $f_p(x,y) = \frac{p\alpha}{2}\|y - x\|^2$. Put $R = \sup\{\|y - x\|, \ x, y \in K\}$ and see easily that ($\mathcal{A}_1$) is satisfied with $\theta = \frac{R\alpha}{2}$, $\delta = 1$ and $\gamma = 1$. Moreover, remark that f_p is $p\alpha$-strongly quasiconvex and also α-strongly quasiconvex since $p > 1$.

The following result, which will be useful for the proof of Theorem 4.6 below, obtains the same conclusion of Proposition 2.4 in the case of strong quasiconvexity by removing the technical assumption (2.3).

Lemma 4.4. *Let $f_p : K \times K \longrightarrow \mathbb{R}$, $p \in M$, be a perturbed equilibrium bifunction such that*

i) $f_p(x,x) = 0$ *for all* $x \in K$;
ii) f_p *has the* μ*-upper sign property at* x *for some* $\mu \geq 0$;
iii) f_p *is* α*-strongly quasiconvex in the second argument for some* $\alpha \geq 0$.

Then, the set $L - M_p^\mu$ *of local solutions to the weak* μ*-Minty equilibrium problem is included in the set* S_p *of standard equilibrium points.*

Proof. Assume that $x \in L - M_p^\mu$ and take $y \in K \setminus \{x\}$. Then there exists $r_1 > 0$ such that

$$|f_p(z,x)| \leq \mu\|z-x\|^2, \; \forall z \in K \cap \overline{B}(x,r_1). \tag{4.3}$$

From the definition of μ-upper sign property in the first argument there exists $r_2 > 0$ such that

$$|f_p(z_t,x)| \leq \mu\|z_t - x\|^2 \Longrightarrow f_p(x,z) \geq 0, \; \forall z \in K \cap \overline{B}(x,r_2), \tag{4.4}$$

where $z_t = (1-t)x + tz$ for $t \in]0,1[$. Take $r = \frac{\inf(r_1,r_2)}{2}$ and combine between (4.3) and (4.4) to see that

$$f_p(x,z) \geq 0, \quad \forall z \in \overline{B}(x,r) \cap K. \tag{4.5}$$

Since $K \cap \overline{B}(x,r) \cap]x,y[\neq \emptyset$, we can choose $t \in]0,1[$ such that $y_t := (1-t)x + ty$ and $y_t \in \overline{B}(x,r) \cap K$. Consequently, y_t satisfies the relation (4.5), i.e., $f_p(x,y_t) \geq 0$. But f_p is α-strongly quasiconvex in the second argument, then

$$\begin{aligned} 0 \leq f_p(x,y_t) &\leq \max\big(f_p(x,x), f_p(x,y)\big) - \alpha t(1-t)\|x-y\|^2 \\ &< \max\big(0, f_p(x,y)\big). \end{aligned}$$

Hence, necessarily $\max\big(0, f_p(x,y)\big) = f_p(x,y)$ and $f_p(x,y) > 0$ for all $y \in K \setminus \{x\}$. Taking into account the condition $i)$ of this lemma we deduce that $f_p(x,y) \geq 0$ for all $y \in K$, this achieves the proof.

□

Lemma 4.5. *Let K be a closed and convex subset of X. Let f be a real-valued bifunction defined on $K \times K$. Let $\mu \geq 0$. Assume that the following conditions hold:*

i) f *is* α*-strongly quasiconvex for some* $\alpha > 0$;
ii) f *has the* μ*-upper sign property;*
iii) $\frac{\alpha}{4} > \mu$.

Then, $M^\mu(f,K)$ *reduces at most to a singleton.*

Proof. Assume that $M^\mu(f,K)$ is nonempty and suppose for a contradiction that there exists $x_1, x_2 \in M^\mu(f,K)$ such that $x_1 \neq x_2$. By definition of M^μ, $f(x_2,x_1) \leq \mu\|x_1 - x_2\|^2$. At the meantime, thanks to Lemma 4.4, $x_2 \in S(f,K)$. Then, Proposition 3.4 ensures that $S(f,K) = S^{\frac{\alpha}{4}}(f,K)$. Therefore,

$$\frac{\alpha}{4}\|x_1 - x_2\|^2 \leq f(x_2,x_1).$$

But $f(x_2, x_1) \leq \mu\|x_1 - x_2\|^2 < \frac{\alpha}{4}\|x_1 - x_2\|^2$, a contradiction.

□

Let us now state our main result on quantitative stability for μ-Minty solutions around the initial value $\overline{p}$ of the parameter p.

Theorem 4.6. *Let $f_p : K \times K \longrightarrow \mathbb{R}$ for $p \in M$ be a perturbed equilibrium bifunction. Assume that $(\mathcal{A}_1)$ holds and the following conditions are satisfied:*

i) *$f_{\overline{p}}$ is $\overline{\alpha}$-strongly quasiconvex in the second argument for some $\overline{\alpha} > 0$ and $f_{\overline{p}}(x, x) = 0$ for all $x \in K$;*

ii) *$f_{\overline{p}}$ has the $\overline{\mu}$-upper sign property in x and $M_{\overline{p}}^{\overline{\mu}}$ is nonempty and bounded for some $\overline{\mu} \geq 0$;*

iii) *For some $\mu \geq 0$ such that $\frac{1}{4}\overline{\alpha} > \mu$, M_p^{μ} is nonempty and bounded for all $p \in N \setminus \{\overline{p}\}$.*

Then, the corresponding sets of μ-Minty solutions satisfy the following Hölder estimate:

$$haus(M_p^{\mu}, M_{\overline{p}}^{\overline{\mu}}) \leqslant \kappa\|p - \overline{p}\|^{\tilde{\gamma}}, \ \ \forall p \in N. \tag{4.6}$$

Here, $\kappa = \left(\frac{\theta}{\frac{1}{4}\overline{\alpha} - \mu}\right)^{\frac{1}{2-\delta}}$ *and* $\tilde{\gamma} = \frac{\gamma}{2-\delta}$.

Proof. Let $p \in N$, $x \in M_p^{\mu}$ and $\overline{x} \in M_{\overline{p}}^{\overline{\mu}}$ such that $\overline{x} \neq x$. Then thanks to Lemma 4.4, it results that $\overline{x} \in M_{\overline{p}}^{\overline{\mu}} \subset L - M_{\overline{p}}^{\overline{\mu}} \subset S_{\overline{p}}$. Thus, using again *i*), from Proposition 3.4 it follows that

$$f_{\overline{p}}(\overline{x}, x) \geqslant \frac{1}{4}\overline{\alpha}\|x - \overline{x}\|^2. \tag{4.7}$$

Now, since $x \in M_p^{\mu}$, by the definition of μ-Minty solutions, we have

$$f_p(y, x) \leqslant \mu\|x - y\|^2, \ \ \forall y \in K. \tag{4.8}$$

In particular with $y = \overline{x}$ in (4.8) we get

$$f_p(\overline{x}, x) \leqslant \mu\|x - \overline{x}\|^2 \ \text{ or else } \ - f_p(\overline{x}, x) \geqslant -\mu\|x - \overline{x}\|^2. \tag{4.9}$$

Therefore, the sum of (4.7) and (4.9) leads to

$$f_{\overline{p}}(\overline{x}, x) - f_p(\overline{x}, x) \geqslant (\frac{1}{4}\overline{\alpha} - \mu)\|x - \overline{x}\|^2, \tag{4.10}$$

which, thanks to the assumption $(\mathcal{A}_1)$, yields

$$\Big\|x - \overline{x}\Big\| \leqslant \Big(\frac{\theta}{\frac{1}{4}\overline{\alpha} - \mu}\Big)^{\frac{1}{2-\delta}} \Big\|p - \overline{p}\Big\|^{\frac{\gamma}{2-\delta}}. \tag{4.11}$$

Since x and $\overline{x}$ are arbitrarily taken in M_p^{μ} and $M_{\overline{p}}^{\overline{\mu}}$ respectively, it results that

$$e(M_{\overline{p}}^{\overline{\mu}}, M_p^{\mu}) = \sup_{\overline{x}\in M_{\overline{p}}^{\overline{\mu}}} d(\overline{x}, M_p^{\mu}) \leqslant \Big(\frac{\theta}{\frac{1}{4}\overline{\alpha}-\mu}\Big)^{\frac{1}{2-\delta}} \Big\|p-\overline{p}\Big\|^{\frac{\gamma}{2-\delta}}. \tag{4.12}$$

Similarly, we also have

$$e(M_p^{\mu}, M_{\overline{p}}^{\overline{\mu}}) = \sup_{x\in M_p^{\mu}} d(x, M_{\overline{p}}^{\overline{\mu}}) \leqslant \Big(\frac{\theta}{\frac{1}{4}\overline{\alpha}-\mu}\Big)^{\frac{1}{2-\delta}} \Big\|p-\overline{p}\Big\|^{\frac{\gamma}{2-\delta}}. \tag{4.13}$$

From (4.12) and (4.13) we immediately obtain

$$haus(M_p^{\mu}, M_{\overline{p}}^{\overline{\mu}}) = \max\Big(e(M_{\overline{p}}^{\overline{\mu}}, M_p^{\mu}), e(M_p^{\mu}, M_{\overline{p}}^{\overline{\mu}})\Big) \leqslant \Big(\frac{\theta}{\frac{1}{4}\overline{\alpha}-\mu}\Big)^{\frac{1}{2-\delta}} \Big\|p-\overline{p}\Big\|^{\frac{\gamma}{2-\delta}}.$$

□

Remark 4.7.

- Theorem 4.6 extends, to weak μ-Minty solution, the result obtained in [5, Theorem 6.2] for strong μ- Minty solution in the sense of [5, (v) of Definition 4.4].
- A strong (remarkable) point in Theorem 4.6 is that hypotheses $i)$ and $ii)$ are done only on the initial bivariate function $f_{\overline{p}}$.

If the condition $i)$ and $ii)$ of Theorem 4.6 are satisfied for all $p \in N$, we can deduce that the set M_p^{μ} contains at most one element. Precisely, thanks to Lemma 4.5 we state the following:

Corollary 4.8. *Let $f_p : K \times K \longrightarrow \mathbb{R}$ for $p \in M$ be a perturbed equilibrium bifunction. Assume that $(\mathcal{A}_1)$ holds and the following conditions are satisfied for all $p \in N$:*

i) *f_p is α-strongly quasiconvex in the second argument for some $\alpha > 0$ and $f_p(x, x) = 0$ for all $x \in K$;*
ii) *f_p has the μ-upper sign property in x and M_p^{μ} is nonempty for some $\mu \geq 0$;*
iii) $\frac{1}{4}\alpha > \mu$.

Then, the corresponding sets of μ-Minty solutions M_p^{μ} are reduced to a singleton $\{x_p\}$ and we have the following estimate:

$$\|x_p - x_{p'}\| \leqslant \kappa\|p - p'\|^{\tilde{\gamma}}, \ \forall p, p' \in N. \tag{4.14}$$

Here, $\kappa = \Big(\frac{\theta}{\frac{1}{4}\overline{\alpha}-\mu}\Big)^{\frac{1}{2-\delta}}$ *and* $\tilde{\gamma} = \frac{\gamma}{2-\delta}$.

Example 4.9. Take $K := [-1, 1]$, $\overline{p} := 0$, $N := [-1, 1]$, and $(f_p)_{p\in\mathbb{R}}$ defined as follows:

$$f_p(x, y) := \begin{cases} \frac{-(y-x)^2}{|p|\exp(\frac{1}{p^2})}, & \text{if} \quad p \neq 0 \\ (y - x)^2, & \text{if} \quad p = 0. \end{cases}$$

It is straightforward to see that $M_p^0 = M_{\overline{p}}^{\frac{1}{p}} = [-1, 1]$. We verify that the functions f_p satisfies all assumptions of Theorem 4.6.

- Let $x, y \in K$ and $p, p' \in N$.
  First case: If x, y is in K such that $p \neq 0$. In this case we have

$$\begin{aligned}
|f_p(x,y) - f_{p'}(x,y)| &= \left|\frac{(y-x)^2}{|p|\exp(\frac{1}{p^2})} - \frac{(y-x)^2}{|p'|\exp(\frac{1}{p'^2})}\right| \\
&= |y-x||x+y|\left|\frac{1}{|p'|\exp(\frac{1}{p'^2})} - \frac{1}{|p|\exp(\frac{1}{p^2})}\right| \\
&\leq 2|y-x|\left(\frac{1}{|p'|\exp(\frac{1}{p'^2})} + \frac{1}{|p|\exp(\frac{1}{p^2})}\right).
\end{aligned}$$

  Since $\lim\limits_{|p|\to 0} \frac{1}{|p|\exp(\frac{1}{p^2})} = 0$, there exists $\theta > 0$ such that $\frac{1}{|p|\exp(\frac{1}{p^2})} \leq \frac{1}{4}\theta$ for $p \in N$. Therefore,

$$|f_p(x,y) - f_{p'}(x,y)| \leq \theta|y-x|.$$

  Second case: If $p = 0$ we have

$$|f_p(x,y) - f_{p'}(x,y)| = 0 \leq \theta|x-y|.$$

  Then, for all $x, y \in K$ and all $p, p' \in N$, we have $|f_p(x,y) - f_{p'}(x,y)| \leq \theta|x-y|$, hence the bifunctions f_p satisfy the assumption $(\mathcal{A}_1)$ with θ positive, $\gamma = 0$ and $\delta = 1$.
- Clearly, the bifunction $f_0(x,y) = (y-x)^2$ is null on the diagonal, moreover it has the $\overline{\mu}$-upper sign property for $\overline{\mu} = 1$ and it is in addition $\overline{\alpha}$-strongly quasiconvex with $\overline{\alpha} = 1$.
- For all $p \in N \setminus \{0\}$ and all $x, y \in [-1, 1]$ we have $f_p(y,x) \leq 0 \times (y-x)^2$.

As a conclusion, the bifunctions $(f_p)_{p\in\mathbb{R}}$ satisfy all conditions of Theorem 4.6 with $\mu = 0$, $\overline{\mu} = 1$ and $p \in [-1, 1]$.

Example 4.10. Take $K := [-1, 1]$, $N := [-1, 1]$, $\mu := 0$ and $(f_p)_{p\in\mathbb{R}}$ defined by

$$f_p(x,y) := (y-p)(y-x).$$

Remark that for all $p \in N$, the function f_p is null on the diagonal, α-strongly convex in the second argument (hence strongly quasiconvex) with $\alpha := 1$, moreover it has the μ-upper sign property for all $\mu \geq 0$. On the other hand, for all $p, p' \in N$ and all $x, y \in K$, $\mid f_p(x,y) - f_{p'}(x,y) \mid = \mid x - y \mid\mid p - p' \mid$, then the condition $(\mathcal{A}_1)$ is fulfilled with $\theta = \delta = \gamma = 1$. Consequently, the conditions of Corollary 4.8 are satisfied, and hence the estimate (4.14) holds. In this case, the corresponding set of 0-Minty solutions M_p^0 is reduced to a singleton $\{x_p\}$. In this case, (4.14) takes the following format:

$$\mid x_p - x_{p'} \mid \leq 4 \mid p - p' \mid, \quad \forall p, p' \in N.$$

Observe that in this example, for every $p \in N$,

$$x_p \in M_p^0 \Longleftrightarrow (x_p - p)(x_p - y) \leq 0, \ \forall y \in K$$
$$\Longleftrightarrow x_p = p.$$

Notation: In the sequel, we denote by D the domain of the map M^μ for some $\mu \geq 0$ defined by $p \rightrightarrows M^\mu(p) := M_p^\mu$, so under the assumptions of Theorem 4.6, N is a subset of D.

Next, we present a similar quantitative stability for two standard solutions sets S_p and $S_{p'}$ for arbitrary parameters p and p'.

Theorem 4.11. *Let $f_p : K \times K \longrightarrow \mathbb{R}$ for $p \in M$ be a perturbed equilibrium bifunction. Assume that $(\mathcal{A}_1)$ holds and the following conditions are satisfied:*

i) *$f_{\overline{p}}$ is $\overline{\alpha}$-strongly quasiconvex in the second argument for some $\overline{\alpha} > 0$ and $f_{\overline{p}}(x,x) = 0$ for all $x \in K$;*
ii) *$S_{\overline{p}}$ is nonempty and bounded;*
iii) *For all $p \in N \setminus \{\overline{p}\}$, S_p is nonempty and bounded;*
iv) *For some $\mu \geq 0$ such that $\frac{1}{4}\overline{\alpha} > \mu$, f_p is μ-weakly pseudomonotone and S_p is nonempty for all $p \in N \setminus \{\overline{p}\}$.*

Then, the corresponding sets of standard solutions satisfy the following Hölder estimate:

$$haus(S_p, S_{\overline{p}}) \leqslant \kappa \|p - \overline{p}\|^{\tilde{\gamma}}, \ \forall p \in N. \tag{4.15}$$

Here, $\kappa = \left(\frac{\theta}{\frac{1}{4}\overline{\alpha} - \mu}\right)^{\frac{1}{2-\delta}}$ and $\tilde{\gamma} = \frac{\gamma}{2-\delta}$.

Proof. Let $p \in N \setminus \{\overline{p}\}$, $x \in S_p$ and $\overline{x} \in S_{\overline{p}}$. Using $i)$ and Proposition 3.4 we obtain that

$$f_{\overline{p}}(\overline{x}, x) \geqslant \frac{1}{4}\overline{\alpha}\|x - \overline{x}\|^2. \tag{4.16}$$

Now, since $x \in S_p$, the μ-weak pseudomonotonicity of f_p ensures that $x \in M_p^\mu$. Thus, by definition of μ-Minty solutions, we have

$$f_p(y, x) \leqslant \mu\|x - y\|^2, \ \forall y \in K. \tag{4.17}$$

In particular with $y = \overline{x}$ in (4.17) it follows that

$$f_p(\overline{x}, x) \leqslant \mu\|x - \overline{x}\|^2 \text{ or else } -f_p(\overline{x}, x) \geqslant -\mu\|x - \overline{x}\|^2. \tag{4.18}$$

Therefore, the sum of (4.16) and (4.18) implies

$$f_{\overline{p}}(\overline{x}, x) - f_p(\overline{x}, x) \geqslant (\frac{1}{4}\overline{\alpha} - \mu)\|x - \overline{x}\|^2. \tag{4.19}$$

Hence, by involving the assumption $(\mathcal{A}_1)$, it results that

$$\Big\|x - \overline{x}\Big\| \leqslant \Big(\frac{\theta}{\frac{1}{4}\overline{\alpha} - \mu}\Big)^{\frac{1}{2-\delta}} \Big\|p - \overline{p}\Big\|^{\frac{\gamma}{2-\delta}}. \tag{4.20}$$

Since x and $\overline{x}$ are arbitrarily taken in S_p and $S_{\overline{p}}$ respectively, we clearly see that

$$e(S_{\overline{p}}, S_p) = \sup_{\overline{x} \in S_{\overline{p}}} d(\overline{x}, S_p) \leqslant \Big(\frac{\theta}{\frac{1}{4}\overline{\alpha} - \mu}\Big)^{\frac{1}{2-\delta}} \Big\|p - \overline{p}\Big\|^{\frac{\gamma}{2-\delta}}. \tag{4.21}$$

Similarly, we also have

$$e(S_p, S_{\overline{p}}) = \sup_{x \in S_p} d(x, S_{\overline{p}}) \leqslant \Big(\frac{\theta}{\frac{1}{4}\overline{\alpha} - \mu}\Big)^{\frac{1}{2-\delta}} \Big\|p - \overline{p}\Big\|^{\frac{\gamma}{2-\delta}}. \tag{4.22}$$

From (4.21) and (4.22) we immediately obtain

$$haus(S_p, S_{\overline{p}}) = \max\Big(e(S_{\overline{p}}, S_p), e(S_p, S_{\overline{p}})\Big) \leqslant \Big(\frac{\theta}{\frac{1}{4}\overline{\alpha} - \mu}\Big)^{\frac{1}{2-\delta}} \Big\|p - \overline{p}\Big\|^{\frac{\gamma}{2-\delta}}.$$

□

We end this subsection by a quantitative satability result for the case of uniqueness of solutions under additional assumptions.

Corollary 4.12. *Assume that the assumptions of Theorem 4.11 are satisfied and suppose moreover that $f_{\overline{p}}$ is μ-weakly pseudomonotone and has the μ-upper sign property, $\mu > 0$ and for all $p \in N\backslash\{\overline{p}\}$, f_p is $\frac{\mu}{\tau}$-strongly quasiconvex, for some $\tau \in]0, \frac{1}{4}[$, and $f_p(x,x) = 0$ for all $x \in K$.*
Then, for all $p \in N$, S_p is reduced to a singleton $\{x_p\}$ and the single-valued solutions map $p \mapsto x_p$ is Hölder continuous around $\overline{p}$ i.e.,

$$\|x_p - x_{p'}\| \leqslant \kappa \|p - p'\|^{\tilde{\gamma}},\ \forall p, p' \in N. \tag{4.23}$$

Here, as before, $\kappa = \Big(\frac{\theta}{\frac{1}{4}\overline{\alpha} - \mu}\Big)^{\frac{1}{2-\delta}}$ and $\tilde{\gamma} = \frac{\gamma}{2-\delta}$.

Proof. For $p = \overline{p}$, the conditions on $f_{\overline{p}}$ ensure that $S_{\overline{p}} = M^{\mu}_{\overline{p}}$. But, by Lemma 4.5, $M^{\mu}_{\overline{p}}$ is reduced to a singleton $\{x_{\overline{p}}\}$. For $p \neq \overline{p}$, S_p is reduced to a singleton thanks to Proposition 3.7 and Remark 3.8 (second point). The required estimate is then direct from Theorem 4.11. This completes the proof.

□

4.2. **Perturbation with respect to the feasibility set K.** In this section, we deal with the perturbation at the level of the feasibility set. Then, assume that the set of constraints K depends on a parameter $\lambda \in \Lambda$, i.e., $K : \Lambda \rightrightarrows X$, $\lambda \longmapsto K_\lambda$, while the objective function f is supposed to be a fixed with a relaxation of the domain to the whole space X i.e., $f : X \times X \to \mathbb{R}$. Then, for all $\lambda \in \Lambda$, the corresponding perturbed equilibrium problem is as follows: find $x_\lambda \in K_\lambda$ such that

$$EP(f, K_\lambda) \quad f(x_\lambda, y) \geqslant 0,\ \forall y \in K_\lambda.$$

Recall that the set of solutions to $EP(f, K_\lambda)$ is denoted by S_λ and K_λ is convex and closed for any $\lambda \in \Lambda$.

The following hypotheses will be considered in the sequel for our stability results around an initial value $\overline{\lambda}$ of the parameter λ:

($\mathcal{A}_2$) The set-valued map K defined by $K(\lambda) := K_\lambda$ is Lipschitz continuous around $\bar{\lambda}$, that is, for a neighborhood $\overline{\Lambda}$ of $\bar{\lambda}$ and some constants $L > 0$ and $1 \geqslant \xi > 0$,

$$K_\lambda \subset K_{\lambda'} + L \left\|\lambda - \lambda'\right\|^\xi B_X, \quad \text{for all} \quad \lambda, \lambda' \in \overline{\Lambda}.$$

($\mathcal{A}_3$) For some $\alpha > 0$, f is α-strongly quasiconvex in y, and $f(x, x) = 0$ for all $x \in X$.

($\mathcal{A}_4$) $-f$ is strictly pseudomonotone i.e., for all $x, y \in X$, $x \neq y$,

$$f(x, y) \leqslant 0 \Longrightarrow f(y, x) > 0.$$

($\mathcal{A}_5$) For some $m > 0$, f is m-strongly quasimonotone i.e., for all $x, y \in X$,

$$f(x, y) > 0 \Longrightarrow f(y, x) \leqslant -m \left\|x - y\right\|^2.$$

($\mathcal{A}_6$) For some $\beta > 0$, there exists $R > 0$ such that for all $x, y, y' \in X$ we have

$$|f(x, y) - f(x, y')| \leqslant R \left\|y - y'\right\|^\beta.$$

In [3, Theorem 3.5], the authors presented a quantitative stability result with respect to perturbed constraints for the particular problem of Stampacchia variational inequality by using ($\mathcal{A}_4$) and ($\mathcal{A}_5$). This double monotonicity assumption seems to be restrictive and interests only the subdifferential of some examples of functions as indicated in [3]. In the next result, we are able to remove this too demanding monotonicity condition in the framework of the more general abstract equilibrium formulation.

Theorem 4.13. *Assume that for all $\lambda \in \overline{\Lambda}$, S_λ is nonempty and the following conditions are satisfied:*

i) ($\mathcal{A}_2$) *with $\xi = 1$ holds and there exists a convex subset $D \subset X$ such that $K_\lambda \subset D$;*
ii) ($\mathcal{A}_3$) *holds;*
iii) ($\mathcal{A}_6$) *is verified with $\beta = 1$;*
iv) *f is quasimonotone on $D \times D$.*

Then, for all $\lambda \in \overline{\Lambda}$, S_λ is reduced to a singleton $\{x_\lambda\}$ and for some $\tau \in]0, \frac{1}{4}[$ such that $\alpha\tau < LR$ there exists a neighborhood $\underline{\Lambda}$ of $\overline{\lambda}$, such that

$$\|x_\lambda - x_{\lambda'}\| \leq \varrho_\tau \|\lambda - \lambda'\|^{\frac{1}{2}},\ \forall \lambda, \lambda' \in \underline{\Lambda}. \tag{4.24}$$

where, $\varrho_\tau = \left(\frac{LR}{\alpha\tau}\right)^{\frac{1}{2}} + L.$

Proof. Let $\tau \in]0, \frac{1}{4}[$ such that $\alpha\tau < LR$. Take $\underline{\Lambda} = \overline{\Lambda} \cap B(\overline{\lambda}, \frac{\alpha\tau}{2LR})$, where

$$B\big(\overline{\lambda}, \frac{\alpha\tau}{2LR}\big) = \big\{\lambda \in \Lambda : \|\lambda - \overline{\lambda}\| < \frac{\alpha\tau}{2LR}\big\}$$

is the open ball centred at $\overline{\lambda}$ with radius $\frac{\alpha\tau}{2LR}$. Let $\lambda, \lambda' \in \underline{\Lambda}$, and put $x = x_\lambda$ and $x' = x_{\lambda'}$ such that $x \neq x'$. There are two cases to treat:

`First case`: $f(x, x') > 0$, then by the assumption iv),

$$f(x', x) \leqslant 0 \quad \text{or else} \quad -f(x', x) \geqslant 0. \tag{4.25}$$

Now by assumption $(\mathcal{A}_2)$, there exists $y' \in K_{\lambda'}$ such that

$$\|x - y'\| \leqslant L\left\|\lambda - \lambda'\right\|. \tag{4.26}$$

Then, since $x' \in S_{\lambda'}$, from the assumption ii) and Proposition 3.4, it follows that

$$f(x', y') \geqslant \tau\alpha\|x' - y'\|^2. \tag{4.27}$$

Thus, (4.25) and (4.27) together imply

$$\alpha\tau\left\|x' - y'\right\|^2 \leqslant f(x', y') - f(x', x). \tag{4.28}$$

Now involve assumption iii) ($(\mathcal{A}_6)$ with $\beta = 1$) in (4.28) and see that

$$\alpha\tau\left\|x' - y'\right\|^2 \leqslant f(x', y') - f(x', x) \leqslant R\|y' - x\|. \tag{4.29}$$

Accordingly, from (4.26) and (4.29), we obtain

$$\alpha\tau\|x' - y'\|^2 \leqslant LR\|\lambda - \lambda'\|. \tag{4.30}$$

Thus,

$$\|x' - y'\| \leqslant \Big(\frac{LR}{\alpha\tau}\Big)^{\frac{1}{2}}\left\|\lambda - \lambda'\right\|^{\frac{1}{2}}. \tag{4.31}$$

Therefore, by (4.26) and (4.31), it follows that

$$\begin{aligned}\|x' - x\| &\leqslant \|x' - y'\| + \|y' - x\| \\ &\leqslant \Big(\frac{LR}{\alpha\tau}\Big)^{\frac{1}{2}}\left\|\lambda - \lambda'\right\|^{\frac{1}{2}} + L\left\|\lambda - \lambda'\right\|.\end{aligned}$$

Hence, since $\alpha\tau < LR$ and $\lambda, \lambda' \in \underline{\Lambda}$, a fortiori $\|\lambda - \lambda'\| < 1$. Then, from the previous inequality we deduce that

$$\|x' - x\| \leqslant \Big[\Big(\frac{LR}{\alpha\tau}\Big)^{\frac{1}{2}} + L\Big]\left\|\lambda - \lambda'\right\|^{\frac{1}{2}}. \tag{4.32}$$

`Second case`: $f(x, x') \leqslant 0$. By assumption $(\mathcal{A}_2)$, there exists $y \in K_\lambda$ such that

$$\|x' - y\| \leqslant L\left\|\lambda - \lambda'\right\|. \tag{4.33}$$

In addition, the same argument as in the justification of (4.27) enables us to claim that

$$f(x, y) \geqslant \alpha\tau\|x - y\|^2. \tag{4.34}$$

Hence, using the assumption of the case, we see that

$$f(x,y) - f(x,x') \geqslant \alpha\tau\|x-y\|^2. \tag{4.35}$$

Next, by assumption $(\mathcal{A}_6)$, (4.33) and (4.35), it results that

$$LR\left\|\lambda-\lambda'\right\| \geqslant R\|x'-y\| \geqslant \alpha\tau\|x-y\|^2. \tag{4.36}$$

This means that

$$\|x-y\| \leqslant \left(\frac{LR}{\alpha\tau}\right)^{\frac{1}{2}}\left\|\lambda-\lambda'\right\|^{\frac{1}{2}}. \tag{4.37}$$

Now with the same justification as in (4.32), thanks to (4.33) and (4.37), we conclude that

$$\begin{aligned} \|x-x'\| &\leqslant \|x-y\| + \|x'-y\| \\ &\leqslant \left[\left(\frac{LR}{\alpha\tau}\right)^{\frac{1}{2}} + L\right]\left\|\lambda-\lambda'\right\|^{\frac{1}{2}}. \end{aligned}$$

Put $\varrho_\tau = \left[\left(\frac{LR}{\alpha\tau}\right)^{\frac{1}{2}} + L\right]$ and obtain in both the two cases the following

$$\|x-x'\| \leqslant \varrho_\tau\left\|\lambda-\lambda'\right\|^{\frac{1}{2}}.$$

This finishes the proof.

□

Remark 4.14. Theorem 4.13 relaxes the strong quasimonotonicity on f considered in [5, Theotem 11] to standard quasimonotonicity.

In the following, we give an example of a bifunction satisfying the conditions of Theorem 4.13 under perturbed constraints.

Example 4.15. Let $X = \Lambda = \mathbb{R}^2$. Consider the polyhedral convex mapping $K : \mathbb{R}^2_+ \rightrightarrows \mathbb{R}^2$ defined for $\lambda \in \mathbb{R}^2$ by

$$K(\lambda) := \{x \in \mathbb{R}^2_+ \ : \ Ax \leq \lambda\},$$

where A is a fixed 2×2 matrix. Recall that any polyhedral convex mapping is Lipschitz continuous relatively to its domain (here $dom\, K \subset \mathbb{R}^2_+$), see ([22, pp 150]). Then, there exists $L > 0$ such that

$$K_\lambda \subset K_{\lambda'} + L\left\|\lambda-\lambda'\right\|\bar{B}_X, \text{ for all } \lambda,\lambda' \in \overline{\Lambda},$$

where $\overline{\Lambda}$ is a neighborhood of $\overline{\lambda} := \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ with $\overline{\Lambda} \subset dom\, K$ (the domain of K). Let us take

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}, \quad A := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \overline{\Lambda} := B(\overline{\lambda}, 1),$$

where $B(\overline{\lambda},1)$ is the open ball of center $\overline{\lambda}$ and radius 1 in the vector space $(\mathbb{R}^2, \|.\|)$, $\|.\|$ is the Euclidean norm. Then $K(\lambda) = \left\{x \in \mathbb{R}^2_+ \ : \ x \leq \lambda\right\}$, where $x \leq \lambda$ means that $x_1 \leq \lambda_1$ and $x_2 \leq \lambda_2$. Therefore, consider the bifunction $f : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by:

$$f(x, y) := \|y\| - \|x\|.$$

If $\lambda \in \overline{\Lambda}$ we have $\lambda_1 \leq 2$ and $\lambda_2 \leq 2$, then $\lambda \leq 2$, hence $K(\lambda) \subset \{x \in \mathbb{R}^2_+ : \ x \leq 2\}$ for all $\lambda \in \overline{\Lambda}$. Let us take $D := \{x \in \mathbb{R}^2_+ : \ x \leq 2\}$. The bifunction f satisfies all assumptions of Theorem 4.13. Indeed, it is straightforward to see that f is null on the diagonal, quasimonotone on $D \times D$ and satisfies the condition $(\mathcal{A}_6)$ with $R = \beta = 1$. Moreover, by [23, Theorem 2], f is α-strongly quasiconvex in the second argument on the bounded and closed subset D with $\alpha = 1$. Choose finally some $\tau \in]0, \frac{1}{4}[$ such that $\tau \leq L$ and $\underline{\Lambda} = B(1, \frac{\tau}{2L})$ and conclude that all the requirements of Theorem 4.13 are satisfied.

5. Application to parametric quasiconvex programming

In this last section, we turn our attention to a perturbed quasiconvex programming problem, and prove a Hölder-type estimate on the set of solutions for two situations: The case of the perturbation on the constraint and the case of the perturbation on the objective function.
As before, X, M, and Λ are normed vector spaces. Let $g : X \longrightarrow \mathbb{R}$ be a quasiconvex function, D be a closed convex subset of X, and $K : \Lambda \rightrightarrows X$ be a set-valued map for which we assume that for all $\lambda \in \Lambda$, $K_\lambda := K(\lambda)$ is a convex and closed subset of D. The notation $\underset{K_\lambda}{Argmin}\ g$ denotes the set of solutions of the following perturbed quasiconvex program:

$$(PQP)_\lambda \qquad \inf_{x \in K_\lambda} g(x).$$

The analysis will be done around an initial point $\overline{\lambda}$.

Definition 5.1. Let $g : X \longrightarrow \mathbb{R}$ be a real-valued function and let $\tau \geq 0$ be fixed. A point $\overline{x} \in X$ will be said a τ-*strong minimizer* of g over a constraints set $D \subset X$ if, and only if

$$g(x) \geqslant g(\overline{x}) + \tau\|x - \overline{x}\|^2, \ \ \forall x \in D. \tag{5.1}$$

We denote by S^τ_g the set of all τ-strong minimizers of g over a constraints set $D \subset X$.

Definition 5.2. Let $g : X \longrightarrow \mathbb{R}$ be a real-valued function and let $\mu \geq 0$ be fixed. A point $\overline{x} \in X$ will be called a *local weak* μ-*minimizer* of g over a set of constraints $D \subset X$ if, and only if there exists a neighborhood $V_{\overline{x}}$ of $\overline{x}$ such that

$$g(\overline{x}) \leq g(x) + \mu\|x - \overline{x}\|^2, \ \ \forall x \in D \cap V_{\overline{x}}. \tag{5.2}$$

We denote by $L - S^\mu_g$ the set of all local weak μ-minimizers of g over a constraints set $D \subset X$.

Remark 5.3.

- Let us consider the bifunction f_g defined on $X \times X$ by $f_g(x, y) := g(y) - g(x), \ \ x, y \in X$, then a point $\overline{x} \in X$ is a τ-strong minimizer for g over a set D if, and only if $\overline{x} \in X$ is a τ-strong solution to $EP(f_g, D)$.
  A point $\overline{x} \in X$ is a local weak μ-minimizer for g over a set D if, and only

if $\overline{x} \in X$ is a local weak μ-Minty solution to $EP(f_g, D)$. In this regard, the reader may also find valuable information in [7].
- τ-strong minimizers are known in the literature under the name "strict minimizer " of order 2 or else 2-order global sharp minimizers with modulus τ, see [30]. Here, we prefer to adopt the terminology of 'strong minimizer' since it is closed to the concept of strong equilibrium point proposed above in Definition 3.1.

Theorem 5.4. *Let $g : X \longrightarrow \mathbb{R}$ be a Lipschitz function with a Lipschitz coefficient $c > 0$. Assume that:*

i) *g is α-strongly quasiconvex, with $\alpha > 0$;*
ii) *the map K satisfies $(\mathcal{A}_2)$ with $\xi = 1$;*
iii) *for all $\overline{\Lambda}$, $\underset{K_\lambda}{Argmin}\ g$ is nonempty; where $\overline{\Lambda}$ is the neighborhood of $\overline{\lambda}$ provided in $(\mathcal{A}_2)$.*

Then, the following assertions are satisfied:

(1) *For all $\tau \in [0, \frac{1}{4}]$, and all $\lambda \in \overline{\Lambda}$, g admits a $\tau\alpha$-strong minimizer over K_λ.*
(2) *For all $\lambda \in \overline{\Lambda}$, $\underset{K_\lambda}{Argmin}\ g$ is reduced to a singleton $\{x_\lambda\}$ and for some $\tau \in]0, \frac{1}{4}[$ such that $\alpha\tau < cL$ there exists a neighborhood $\underline{\Lambda}$ of $\overline{\lambda}$, such that*

$$\|x_\lambda - x_{\lambda'}\| \leq \varrho_\tau \|\lambda - \lambda'\|^{\frac{1}{2}},\ \forall \lambda, \lambda' \in \underline{\Lambda}, \tag{5.3}$$

where, $\varrho_\tau = \left(\frac{cL}{\alpha\tau}\right)^{\frac{1}{2}} + L$, *$L$ being the constant provided in $(\mathcal{A}_2)$.*

Proof. Let f_g be the bifunction defined on $X \times X$ by $f_g(x, y) = g(y) - g(x)$, $x, y \in X$. Clearly, the first point of the conclusion comes from Proposition 3.4 by the use of assumptions i) and iii). The second point follows immediately from Theorem 4.13 applied to the bifunction f_g.

□

Remark 5.5.
- Theorem 5.4 improves [3, Theorem 4.3] by removing the following assumptions:
- $0 \notin \partial^C g(X)$, where $\partial^C g(X)$ is the Clarke subdifferential of g.
- For any $a > \inf_X g$, the interior of any adjusted sublevel set of g is nonempty.
- Our approach is a direct one and doesn't necessitate the recourse to Stampacchia variational inequalities formulated with the normal operator to adjusted sublevels sets of g. Moreover, in our case, the minimum value is achieved at a unique point.

We now come back into the case when the objective function g is perturbed by a parameter $p \in M$, here $g : X \times M \longrightarrow \mathbb{R}$ is assumed to be quasiconvex in X. We will write $g(., p) = g_p$ for any $p \in M$. The analysis will also be done around an initial value $\overline{p}$ of the parameter p. The set $\underset{K}{\mathrm{A}rgmin}\ g_p$ denotes the solutions set of the following perturbed quasiconvex program:

$$(PQP)_p \qquad \inf_{x\in K} g_p(x),$$

where K is a convex and closed subset of X. Consider the family of bifunctions $\{f_{g_p}, p \in M\}$ defined by $f_{g_p}(x,y) = g_p(y) - g_p(x)$ for all $x, y \in X$ and all $p \in M$.

Proposition 5.6. *Let $\alpha > 0$ be fixed, K be a nonempty convex and closed subset of X and N be a neighborhood of $\overline{p} \in M$. If the function g_p is α-strongly quasiconvex for all $p \in N$ then the bifunction f_{g_p} has the α-upper sign property at x for all $p \in N$.*

Proof. Let $x, y \in K$, $p \in N$ and consider $z_t = (1-t)x + ty$ for all $t \in]0,1[$. Assume that $g_p(y) < g_p(x)$. Hence the α-strongly quasiconvexity of g_p implies that

$$g_p(z_t) \leq g_p(x) - \alpha t(1-t)\|x-y\|^2, \ \ \forall t \in]0,1[,$$

thus

$$g_p(x) - g_p(z_t) \geq \alpha t(1-t)\|x-y\|^2, \ \ \forall t \in]0,1[.$$

Take $k \in]0,1[$ such that $k(1-k) > k^2$ and see that

$$g_p(x) - g_p(z_k) > \alpha k^2\|x-y\|^2,$$

this means that

$$g_p(x) - g_p(z_k) > \alpha\|z_k - x\|^2.$$

Hence,

$$g_p(y) < g_p(x) \Longrightarrow \exists\, k \in]0,1[\ \textit{ such that } \ g_p(x) - g_p(z_k) > \alpha\|z_k - x\|^2.$$

Accordingly,

$$g_p(x) - g_p(z_t) \leq \alpha\|z_t - x\|^2, \ \ \forall t \in]0,1[\Longrightarrow g_p(y) \geq g_p(x), \ \ \forall x, y \in K, \ \ \forall p \in N.$$

Therefore, the bifunction f_{g_p} has the (global) α-upper sign property at x, completing the proof.

□

Remark 5.7. For all $\mu \leq \alpha$, every bifunction with the α-upper sign property has the μ-upper sign property.

In the following, we extend the result of [21, Proposition 4] to local weak μ-minimizers.

Proposition 5.8. *Let $\alpha > 0$ and $\mu > 0$ such that $\mu \leq \alpha$. Let K be a nonempty convex and closed subset of X and let $\psi : X \longrightarrow \mathbb{R}$ be a real-valued function. Suppose that ψ is α-strongly quasiconvex. Then $\overline{x} \in K$ is a local weak μ-minimizer of ψ over K if and only if $\overline{x}$ is a (global) minimizer for ψ over K.*

Proof. A global minimizer of ψ over K is trivially a local weak μ-minimizer of ψ over K. We shall only show the converse. Suppose that a point $\overline{x} \in K$ is a local weak μ-minimizer of ψ over K and take $\varphi : K \times K \longrightarrow \mathbb{R}$ defined by $\varphi(x, y) := \psi(y) - \psi(x)$ for $x, y \in K$. Then, on the one hand, $\overline{x}$ is a local weak μ-Minty solution to $EP(\varphi, K)$. On the other hand, from Proposition 5.6 it results that φ has the α-upper sign property at x. Thus, according to Remark 5.7, the function φ has also the μ-upper sign property at x. Moreover, φ is null on the diagonal, then from Lemma 4.4, it follows that $\overline{x}$ is a (standard) solution to $EP(\varphi, K)$, which means that $\psi(y) - \psi(\overline{x}) \geq 0$ for all $y \in K$. This finishes the proof.

□

Now, we are ready to state our stability result for the parametric quasiconvex program $(PQP)_p$.

Theorem 5.9. *Let $\alpha > 0$, $\mu > 0$ and let $\overline{p} \in M$ be a fixed value of the parameter. Let K be a closed and convex subset of X. Assume that the following assumptions are satisfied:*

i) *There exist $\theta > 0$, $1 \geqslant \gamma > 0$ and a neighborhood N of $\bar{p}$ such that for all $x \in K$ and all $p, p' \in N$,*
$$|g_p(x) - g_{p'}(x)| \leqslant \vartheta \left\|p - p'\right\|^{\gamma};$$
ii) *For all $p \in N$, $\underset{K}{Argmin}\ g_p$ is nonempty;*

iii) *for all $p \in N$, g_p is α-strongly quasiconvex;*

iv) *$\mu < \frac{1}{4}\alpha$.*

Then, for all $p \in N$, $\underset{K}{Argmin}\ g_p$ is reduced to a singleton $\{x_p\}$ and the single valued map $p \mapsto x_p$ is Hölder continuous around $\overline{p}$, i.e.,

$$\|x_p - x_{p'}\| \leqslant \tau \left\|p - p'\right\|^{\frac{\gamma}{2}}, \tag{5.4}$$

where $\tau = \left(\frac{2\vartheta}{\frac{1}{4}\alpha - \mu}\right)^{\frac{1}{2}}$.

Proof. Recall that with the bifunction f_{g_p} (defined by $f_{g_p}(x, y) = g_p(y) - g_p(x)$, $x, y \in K$), $\underset{K}{Argmin}\ g_p$ is nothing else but the set of standard equilibrium points of f_{g_p} which coincide with its μ-Minty ones since f_{g_p} is trivially continuous and monotone. Now, with the assumption i), the bifunction f_{g_p} satisfies the condition $(\mathcal{A}_1)$ with $\delta = 0$ and $\theta = 2\vartheta$. Moreover, for any p be fixed in N, by assumption *ii)*, $\underset{K}{Argmin}\ g_p$ is nonempty, then there exists $\overline{x} \in K$ such that $g_p(\overline{x}) \leq g_p(y)$ for all $y \in K$, hence $f_{g_p}(y, \overline{x}) \leq 0 \leq \mu \parallel \overline{x} - y \parallel^2$. Moreover, for all $x \in K$, $f_{g_p}(x, x) = 0$ and f_{g_p} is α-strongly quasiconvexity in the second argument since g_p is so with respect to its variable. On the other hand, thanks to Proposition 5.6 and Remark 5.7, the bifunction f_{g_p} satisfies the μ-upper sign property at x. Thus, Lemma 4.5 ensures that $\underset{K}{Argmin}\ g_p$ is reduced to a singleton $\{x_p\}$. Therefore, the required estimate in (5.4) follows from Corollary 4.8.

□

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