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THREE EXTENSIONS OF THE CARISTI'S THEOREM

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Abstract. In this paper we obtain three extensions of the well-known Caristi's theorem. Our first result is on single-valued mappings while in the second and the third results we deal with set-valued mappings. In particular, we are interesting in versions of the Caristi's theorem which take into account possible computational errors.

1. Introduction

During more than fifty years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [2, 4, 7, 8, 9, 10, 11, 12, 13, 14, 18, 19] and the references cited therein. This activity stems from Banach's classical theorem [1] regarding the existence of a unique fixed point for a strict contraction and covers also the convergence of (inexact) iterates of a mapping to one of its fixed points. Since that seminal result, many developments have taken place in this area containing, in particular, the studies on feasibility problems and on common fixed point problems which find important applications in engineering and medical sciences [3, 6, 16, 17, 18, 19].

In the present paper we establish three extensions of the well-know Caristi's fixed point theorem [5, 15]. In particular, we are interesting in its versions which take into account possible computational errors and which are concerned with the case where mappings take a nonempty and closed subset of a complete metric space X into X.

We begin by recalling the following two versions of Caristi's theorem.

Theorem 1.1 (Theorem 3.9 of [10]). *Suppose that (X, ρ) is a complete metric space and $T : X \to X$ is a continuous mapping which satisfies for some $\phi : X \to [0, \infty)$,*

$$\rho(x, Tx) \le \phi(x) - \phi(Tx),\ x \in X.$$

Then $\{T^n x\}_{n=1}^{\infty}$ converges to a fixed point of T for each $x \in X$.

Theorem 1.2 (Theorem 4.1 of [10]). *Suppose that (X, ρ) is a complete metric space, $\phi : X \to R^1$ is a lower semicontinuous function which is bounded from below, and $T : X \to X$ satisfies*

$$\rho(x, Tx) \le \phi(x) - \phi(Tx),\ x \in X.$$

Then T has a fixed point.

The following set-valued analog of Caristi's theorem was obtained in [4] (see also Theorem 9.37 of [14]).

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Theorem 1.3. *Assume that* (X,ρ) *is a complete metric space,* $T : X \to 2^X \setminus \{\emptyset\}$, *graph*$(T) := \{(x,y) \in X \times X : \ y \in T(x)\}$ *is closed,* $\phi : X \to R^1 \cup \{\infty\}$ *is bounded from below, and that for each* $x \in X$,

$$\inf\{\phi(y) + \rho(x,y) : \ y \in T(x)\} \le \phi(x).$$

Let $\{\epsilon_n\}_{n=0}^{\infty} \subset (0,\infty)$, $\sum_{n=0}^{\infty} \epsilon_n < \infty$, *and let* $x_0 \in X$ *satisfy* $\phi(x_0) < \infty$. *Assume that for each integer* $n \ge 0$,

$$x_{n+1} \in T(x_n)$$

and

$$\phi(x_{n+1}) + \rho(x_n, x_{n+1}) \le \inf\{\phi(y) + \rho(x_n, y) : \ y \in T(x_n)\} + \epsilon_n.$$

Then $\{x_n\}_{n=0}^{\infty}$ *converges to a fixed point of* T.

2. The first result

Let (X,ρ) be a complete metric space. For each $x \in X$ and each $r > 0$ set

$$B(x,r) = \{y \in X : \ \rho(x,y) \le r\}.$$

For each nonempty set Y and each function $h : Y \to R^1 \cup \{\infty\}$ set

$$\inf(h) = \inf\{h(y) : \ y \in Y\}.$$

Theorem 2.1. *Let* K *be a nonempty closed subset of* X, $\phi : X \to [0,\infty]$, $T : K \to X$ *be a mappings such that*

$$\textit{graph}(T) := \{(x, Tx) : \ x \in K\}$$

is a closed set in $X \times X$ *equipped with the product topology and such that for all* $x \in K$,

$$\phi(Tx) + \rho(x, Tx) \le \phi(x). \tag{2.1}$$

Assume that $x_0 \in K$ *and* $r > 0$ *satisfy*

$$B(x_0, r) \subset K \tag{2.2}$$

and

$$\phi(x_0) \le r. \tag{2.3}$$

Then the sequence $\{T^i x_0\}_{i=1}^{\infty}$ *is well defined and converges to a fixed point of* T.

Proof. Set $T^0 x = x$ for all $x \in K$. If

$$Tx_0 = x_0,$$

then the assertion of the theorem holds. Assume that

$$Tx_0 \ne x_0. \tag{2.4}$$

By (2.1)-(2.3),

$$\rho(x_0, Tx_0) \le \phi(x_0) - \phi(Tx_0) \le r$$

and

$$Tx_0 \in K. \tag{2.5}$$

Assume that $q \ge 1$ is an integer and that

$$T^i(x_0) \in K, \ i = 1, \dots, q.$$

(Note that in view of (2.5) our assumption holds for $q = 1$.) If an integer $i \in [1, q)$ satisfies

$$T^i x_0 = T^{i+1} x_0$$

then the assertion of the theorem holds.

Assume that

$$T^i x_0 \neq T^{i+1} x_0, \ i = 0, \dots, q-1. \tag{2.6}$$

If

$$T(T^q x_0) = T^q x_0,$$

then the assertion of the theorem holds. Therefore we consider only the case with

$$T(T^q x_0) \neq T^q x_0. \tag{2.7}$$

By (2.1), (2.6) and (2.7), for all $i = 0, \dots, q$,

$$\phi(T^{i+1} x_0) < \rho(T^{i+1} x_0, T^i x_0) + \phi(T^{i+1} x_0) \leq \phi(T^i x_0). \tag{2.8}$$

In view of (2.3) and (2.8),

$$\begin{aligned} \rho(x_0, T^{q+1} x_0) &\leq \sum_{i=0}^{q} \rho(T^i x_0, T^{i+1} x_0) \\ &\leq \sum_{i=0}^{q} (\phi(T^i x_0) - \phi(T^{i+1} x_0)) \leq \phi(x_0) \leq r. \end{aligned}$$

Together with (2.2) this implies that

$$T^{q+1} x_0 \in B(x_0, r) \subset K.$$

Thus by induction we showed that $T^i x_0$ is well defined and

$$T^i x_0 \in K$$

for all integers $i \geq 0$. In view of (2.1),

$$\sum_{i=0}^{\infty} \rho(T^i x_0, T^{i+1} x_0) \leq \sum_{i=0}^{\infty} (\phi(T^i x_0) - \phi(T^{i+1} x_0)) \leq \phi(x_0).$$

Therefore $\{T^i x_0\}_{i=1}^{\infty}$ is a Cauchy sequence and there exists

$$x_* = \lim_{i \to \infty} T^i x_0.$$

Since the graph of T is closed we have

$$T x_* = x_*.$$

Theorem 2.1 is proved. □

3. The second result

Suppose that (X, ρ) is a complete metric space.

Theorem 3.1. *Let K is a nonempty closed subset of X, $\phi : X \to [0, \infty]$ be a lower semicontinuous function, a mapping $T : K \to 2^X$ satisfy the following property*

(a) for each $x \in K$ and each $M > 0$ the set $B(x, M) \cap T(x)$ is compact.

Assume that

$$graph(T) := \{(x, y) \in K \times X : \ y \in T(x)\}$$

is a closed set in $X \times X$ equipped with the product topology and that for each $x \in K$,

$$\inf\{\rho(x, y) + \phi(y) : \ y \in T(x)\} \le \phi(x). \tag{3.1}$$

Let $x_0 \in K$, $r > 0$,

$$\phi(x_0) \le r, \tag{3.2}$$

$$B(x_0, r) \subset K. \tag{3.3}$$

Then the sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that for each integer $i \ge 0$,

$$x_{i+1} \in T(x_i), \tag{3.4}$$

$$\rho(x_i, x_{i+1}) + \phi(x_{i+1}) = \inf\{\rho(x_i, z) + \phi(x) : \ z \in T(x_i)\} \tag{3.5}$$

is well defined,

$$x_i \in K, \ i = 0, 1, \dots,$$

there exists

$$\lim_{i\to\infty} x_i$$

in X and

$$\lim_{i\to\infty} x_i \in T(\lim_{i\to\infty} x_i).$$

Proof. Since the function

$$\rho(x_0, z) + \phi(z), \ z \in X$$

is lower semicontinuous, property (i) implies that there exists

$$x_1 \in T(x_0) \tag{3.6}$$

such that

$$\rho(x_0, x_1) + \phi(x_1) = \inf\{\rho(x_0, z) + \phi(z) : \ z \in T(x_0)\} \tag{3.7}$$

By (3.1), (3.2) and (3.7),

$$\rho(x_0, x_1) + \phi(x_1) \le \phi(x_0) \le r. \tag{3.8}$$

In view of (3.3) and (3.8),

$$\rho(x_0, x_1) \le r, \tag{3.9}$$

$$x_1 \in K. \tag{3.10}$$

Assume that $k \ge 1$ is an integer,

$$x_i \in K, \ i = 0, \dots, k$$

and that for all $i = 0, \dots, k-1$, (3.4) and (3.5) hold. (Note that in view of (3.6), (3.7) and (3.10), our assumption holds for $k = 1$.)

By (3.1) and (3.5), for all $i = 0, \dots, k-1$,

$$\rho(x_{i+1}, x_i) + \phi(x_{i+1}) \leq \phi(x_i)$$

and

$$\rho(x_{i+1}, x_i) \leq \phi(x_i) - \phi(x_{i+1}). \tag{3.11}$$

It follows from (3.2) and (3.11) that

$$\begin{aligned} \sum_{i=0}^{k} \rho(x_{i+1}, x_i) &\leq \sum_{i=0}^{k} (\phi(x_i) - \phi(x_{i+1})) \\ &\leq \phi(x_0) - \phi(x_{k+1}) \leq \phi(x_0) \leq r. \end{aligned} \tag{3.12}$$

Since the function

$$\rho(x_k, z) + \phi(z),\ z \in X$$

is lower semicontinuous property (a) implies that there exists

$$x_{k+1} \in T(x_k) \tag{3.13}$$

such that

$$\rho(x_k, x_{k+1}) + \phi(x_{k+1}) = \inf\{\rho(x_k, z) + \phi(z) :\ z \in T(x_k)\}. \tag{3.14}$$

By (3.1) and (3.14),

$$\rho(x_k, x_{k+1}) + \phi(x_{k+1}) \leq \phi(x_k). \tag{3.15}$$

In view of (3.11) and (3.15),

$$\begin{aligned} \sum_{i=0}^{k} \rho(x_{i+1}, x_i) &\leq \sum_{i=0}^{k} (\phi(x_i) - \phi(x_{i+1})) \\ &\leq \phi(x_0) - \phi(x_{k+1}) \leq \phi(x_0). \end{aligned}$$

Together with (3.2) this implies that

$$\rho(x_0, x_{k+1}) \leq \sum_{i=0}^{k+1} \rho(x_{i+1}, x_i) \leq \phi(x_0) \leq r.$$

In view of (3.3),

$$x_{k+1} \in B(x_0, r) \subset K.$$

Therefore the assumption made for k also holds for $k+1$. Thus by induction we showed that the sequence $\{x_i\}_{i=0}^{\infty}$ is well defined and $x_i \in K$ for all integers $i \geq 0$. By (3.1) and (3.5), for all $i = 0, 1, \dots,$

$$\rho(x_{i+1}, x_i) + \phi(x_{i+1}) \leq \phi(x_i)$$

and

$$\rho(x_{i+1}, x_i) \leq \phi(x_i) - \phi(x_{i+1}). \tag{3.16}$$

It follows from (3.16) that for all natural numbers T,

$$\sum_{i=0}^{T-1} \rho(x_{i+1}, x_i) \le \sum_{i=0}^{T-1} (\phi(x_i) - \phi(x_{i+1}))$$
$$\le \phi(x_0) - \phi(x_T) \le \phi(x_0).$$

Therefore $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence and there exists

$$x_* = \lim_{i\to\infty} x_i$$

in X. Clearly,

$$(x_*, x_*) = \lim_{i\to\infty} (x_i, x_{i+1}) \in \text{ graph}(T)$$

and $x_* \in T(x_*)$. Theorem 3.1 is proved. □

4. The third result

Suppose that (X, ρ) is a complete metric space.

Theorem 4.1. *Let K is a nonempty closed subset of X, $\phi : X \to [0, \infty]$ and $T : K \to 2^X$ be a mapping. Assume that*

$$graph(T) := \{(x, y) \in K \times X : \ y \in T(x)\}$$

is a closed set in $X \times X$ equipped with the product topology and that for each $x \in K$,

$$\inf\{\rho(x, y) + \phi(y) : \ y \in T(x)\} \le \phi(x). \tag{4.1}$$

Let $x_0 \in K$, $r > 0$,

$$\phi(x_0) < r, \tag{4.2}$$

$$B(x_0, r) \subset K, \tag{4.3}$$

$\{\epsilon_i\}_{i=0}^{\infty}$ satisfy

$$\sum_{i=0}^{\infty} \epsilon_i \le r - \phi(x_0). \tag{4.4}$$

Assume that a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies for each integer $i \ge 0$,

$$x_{i+1} \in T(x_i), \tag{4.5}$$

$$\rho(x_i, x_{i+1}) + \phi(x_{i+1}) \le \inf\{\rho(x_i, z) + \phi(x) : \ z \in T(x_i)\} + \epsilon_i. \tag{4.6}$$

Then it is well defined,

$$x_i \in K, \ i = 0, 1, \dots,$$

there exists

$$\lim_{i\to\infty} x_i$$

in X and

$$\lim_{i\to\infty} x_i \in T(\lim_{i\to\infty} x_i).$$

Proof. Assume that $q \geq 0$ is an integer and that ,

$$x_i \in K,\ i = 0, \dots, q$$

are already well defined by (4.5) and (4.6) hold. (Clearly, our assumption holds for $q = 0$.)

By (4.5) and (4.6),

$$x_{q+1} \in T(x_q), \tag{4.7}$$

$$\rho(x_{q+1}, x_q) + \phi(x_{q+1}) \leq \inf\{\rho(x_q, z) + \phi(z) :\ z \in T(x_q)\} + \epsilon_q. \tag{4.8}$$

It is clear that

$$\phi(x_i) < \infty \text{ for all } i = 0, \dots, q + 1.$$

In order to show that our assumption holds for $q+1$ too it is sufficient to show that

$$x_{q+1} \in K.$$

By (4.1) and (4.6), for $i = 0, \dots, q$,

$$\begin{aligned} \rho(x_{i+1}, x_i) &\leq \epsilon_i + \inf\{\rho(z, x_i) + \phi(x) :\ z \in T(x_i)\} - \phi(x_{i+1}) \\ &\leq \epsilon_i + \phi(x_i) - \phi(x_{i+1}). \end{aligned} \tag{4.9}$$

It follows from (4.4) and (4.9) that

$$\begin{aligned} \rho(x_0, x_{q+1}) &\leq \sum_{i=0}^{q} \rho(x_{i+1}, x_i) \\ &\leq \sum_{i=0}^{q} (\phi(x_i) - \phi(x_{i+1}) + \epsilon_i) \\ &\leq \sum_{i=0}^{q} \epsilon_i + \phi(x_0) - \phi(x_{q+1}) \\ &\leq \sum_{i=0}^{q} \epsilon_i + \phi(x_0) \\ &\leq \sum_{i=0}^{\infty} \epsilon_i + \phi(x_0) \leq r. \end{aligned}$$

Together with (4.3) the relation above implies that

$$x_{q+1} \in B(x_0, r) \subset K.$$

Thus by induction we showed that the sequence $\{x_i\}_{i=0}^{\infty}$ is well defined and $x_i \in K$ for all integers $i \geq 0$. By (4.1) and (4.6), for all $i = 0, 1, \dots,$

$$\rho(x_{i+1}, x_i) + \phi(x_{i+1}) \leq \phi(x_i) + \epsilon_i$$

and

$$\rho(x_{i+1}, x_i) \leq \phi(x_i) - \phi(x_{i+1}) + \epsilon_i. \tag{4.10}$$

It follows from (4.2), (4.4) and (4.10) that for all natural numbers T,

$$\sum_{i=0}^{T-1} \rho(x_{i+1}, x_i) \le \sum_{i=0}^{T-1} (\phi(x_i) - \phi(x_{i+1}) + \epsilon_i)$$
$$\le \phi(x_0) - \phi(x_T) + \sum_{i=0}^{T-1} \epsilon_i$$
$$\le \phi(x_0) + \sum_{i=0}^{\infty} \epsilon_i < \infty.$$

Therefore $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence and there exists

$$x_* = \lim_{i\to\infty} x_i$$

in X. Clearly,

$$(x_*, x_*) = \lim_{i\to\infty} (x_i, x_{i+1}) \in \text{ graph}(T)$$

and $x_* \in T(x_*)$. Theorem 4.1 is proved. □

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