Applied Analysis and Optimization
Volume 3, Number 2, 2019, 241–251

Yokohama Publishers
ISSN 2189-1664 Online Journal

STRONG CONVERGENCE OF PROXIMAL POINT ALGORITHM WITH INERTIAL TERMS AND ERRORS

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Abstract. In this paper, we introduce a proximal point algorithm with both inertial terms and errors for solving a monotone inclusion problem and obtain strong convergence result in real Hilbert spaces under some reasonable assumptions on the sequence of parameters. Our result unifies many existing results in the literature on both proximal point algorithm with errors and proximal point algorithm with inertial terms which have been subject of intense research by many authors.

1. Introduction

Let H be a real Hilbert space with scalar product $\langle .,. \rangle$ and the norm $\|.\|$. A set-valued mapping $A : H \to 2^H$ is said to be monotone if

$$\langle v - w, x - y \rangle \geq 0$$

for all $v \in A(x), w \in A(y)$. A monotone operator A is said to be maximal monotone if its graph

$$\text{Gr } A = \{(x, v) \in H \times H | v \in A(x)\}$$

is not properly contained in the graph of any other monotone operator. It is well known that if A is maximal monotone and $\gamma > 0$, then the resolvent of A, the operator $J_\gamma^A : H \to H$ defined by $J_\gamma^A(x) := (I + \gamma A)^{-1}(x)$, is single-valued and nonexpansive, i.e., for all $x, y \in H$, $\|J_\gamma^A(x) - J_\gamma^A(y)\| \leq \|x - y\|$.

We are interested in the inclusion problem: Find $x \in H$ such that

$$0 \in A(x), \tag{1.1}$$

which appears in a wide variety of equilibrium problems such as convex programming and variational inequalities. A classical way to solve problem (1.1) is the proximal point algorithm (PPA) (see [29]). For any given initial guess $x_1 \in H$, the PPA generates an iterative sequence as

$$x_{n+1} = J_{\gamma_n}^A(x_n + e_n), \tag{1.2}$$

where $\{e_n\}$ is the error sequence. It is well known that the PPA (1.2) converges weakly to a zero of A provided that $\gamma_n \geq \gamma > 0$ for all $n \geq 1$.

However, as shown by Güler (see [17]), the PPA (1.2) does not necessarily converge strongly. Since then, many authors have carried out many worthwhile research findings on modifying the PPA (1.2) so that the strong convergence is guaranteed

2010 *Mathematics Subject Classification.* 47H06, 47H09, 47J05, 47J25.
Key words and phrases. proximal point algorithm; inertial terms and errors; strong convergence; Hilbert spaces.

(see for instance [6,8,9,13,19,25,32,33,36–38]). In particular, Xu [25,36] introduced the following iteration (called contraction-proximal point algorithm (CPPA)):

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J^A_{\gamma_n} x_n + e_n, \tag{1.3}$$

where $u \in H$ is fixed and $\{\alpha_n\} \subset (0,1)$ is real sequence. It was proved in [18,25,36] that $\{x_n\}$ converges strongly to a solution of (1.1) under the assumptions that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \|e_n\| < \infty$. Several authors have obtained strong convergence results with errors for solving (1.1) (see, for example). One of these recent results is the result of Boikanyo and Morosanu [8], where it was proved that $\{x_n\}$ generated by (1.3) converges to a solution of (1.1) under the conditions that (see also [34,35]) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and either $\sum_{n=1}^{\infty} \|e_n\| < \infty$ or $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$. It was also shown in [8] that (1.3) and the following regularization for proximal point algorithm:

$$x_{n+1} = J^A_{\gamma_n}(\alpha_n u + (1-\alpha_n)x_n + e_n) \tag{1.4}$$

are equivalent.

In [2], Alvarez and Attouch introduced and inertial proximal point algorithm for solving (1.1):

$$x_{n+1} = J^A_{\gamma_n}(x_n + \beta_n(x_n - x_{n-1})) \tag{1.5}$$

and proved that the sequence $\{x_n\}$ generated by (1.5) converges weakly, under certain conditions on $\{\beta_n\}$ and $\{\gamma_n\}$, to a solution of (1.1) in real Hilbert spaces. Their result extend the classical convergence results concerning the standard proximal method. Alvarez in [1] introduced a general implicit iterative method for solving (1.1) in a Hilbert space which unified relaxation, inertial type extrapolation and projection step and obtained weak convergence result under appropriate assumptions on the algorithm parameters.

Let us recall that the term $\beta_n(x_n - x_{n-1})$ in (1.5) is called the inertial. The algorithm (1.5) is based upon a discrete version of a second order dissipative dynamical system [3,4] and can be regarded as a procedure of speeding up the convergence properties (see, e.g., [2,7,15,23,24,28,31]). It is worth mentioning that the scheme (1.5) reduces to (1.2) when $\beta_n = 0$. Recently, there have been increasing interests in studying inertial type algorithms. See, for example, inertial forward-backward splitting methods [5,20,27], inertial Douglas-Rachford splitting method [10], inertial ADMM [11,14], and inertial forward-backward-forward method [12]. These results and other related ones analyzed the convergence properties of inertial extrapolation type algorithms and demonstrated their performance numerically on some imaging and data analysis problems. It is based on this recent trend that our contribution in solving (1.1) in this paper lies.

Our aim in this paper is to introduce contraction-proximal point algorithm (1.3) with inertial terms and obtain strong convergence of the generated sequence to a solution of (1.1) in real Hilbert spaces. Our convergence analysis is obtained under some reasonable assumptions on the sequence of parameters and our result serves a unification of many existing results on proximal point algorithm with errors ([8, 25, 35, 36]) and proximal point algorithm with inertial terms ([1, 2]).

2. Preliminaries

We state the following well-known lemmas which will be used in our convergence analysis in the sequel.

Lemma 2.1. *The following well-known results hold in a real Hilbert space:*

(i) $||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2, \forall x, y \in H.$
(ii) $||x+y||^2 \leq ||x||^2 + 2\langle y, x+y\rangle, \forall x, y \in H.$

Lemma 2.2 ([21,36])**.** *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n + \gamma_n,\ n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in (0,1) and $\{\sigma_n\}$ is a real sequence. Assume $\sum \gamma_n < \infty$. Then the following results hold:

(i) *If $\sigma_n \leq \alpha M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
(ii) *If $\sum \alpha_n = \infty$ and $\limsup \frac{\sigma_n}{\alpha_n} \leq 0$, then $\lim a_n = 0$.*

Lemma 2.3. *Let A be a maximal monotone operator and J_γ^A be its resolvent with $\gamma > 0$. Then, we have*

(i) *$J_\gamma^A : H \to H$ is single-valued and firmly nonexpansive (i.e., $\|J_\gamma^A(x) - J_\gamma^A(y)\|^2 \leq \|x-y\|^2 - \|(I-J_\gamma^A)(x) - (I-J_\gamma^A)(y)\|^2, \forall x, y \in H$;*
(ii) *$F(J_\gamma^A) = A^{-1}(0) := \{x \in H : 0 \in Ax\}$, where $F(J_\gamma^A) := \{x \in H | J_\gamma^A(x) = x\}$ is the seet of fixed points of J_γ^A;*
(iii) *$\|x - J_\gamma^A x\| \leq 2\|x - J_\delta^A x\|$ for all $0 < \gamma \leq \delta$ and all $x \in H$ (see [25]).*

Lemma 2.4. *([16]) Let $T : H \to H$ be a nonexpansive operator. Let $\{x_n\}$ be a sequence in H and x be a point in H. Suppose that $x_n \rightharpoonup x, n \to \infty$ (i.e., $\{x_n\}$ converges weakly to x) and that $x_n - Tx_n \to 0, n \to \infty$. Then, $x \in F(T)$.*

3. Proposed Method

In this section, we give a precise statement of our method and give some discussions. Its convergence analysis is postponed to the next section. We first state the assumptions that we will assume to hold through the rest of this paper.

Assumption 3.1. (a) *$A : H \to 2^H$ is a maximal monotone operator.*
(b) *$A^{-1}(0) \neq \emptyset$.*

Assumption 3.2. *Suppose $\{\alpha_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$ and $\{\epsilon_n\}_{n=1}^\infty$ are positive sequences and sequence of errors $\{e_n\}_{n=1}^\infty \subset H$ satisfying the following conditions:*

(a) *$\alpha_n \in (0,1), \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty, \epsilon_n = o(\alpha_n)$, where $\epsilon_n = o(\alpha_n)$ means $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$.*
(b) *$\gamma_n \geq \gamma > 0$ for all $n \geq 1$.*
(c) *either $\sum_{n=1}^\infty \|e_n\| < \infty$ or $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$.*

We next give a precise statement of our method as follows.

Algorithm 3.3. (S.0) *Choose sequences* $\{\alpha_n\}_{n=1}^\infty$, $\{\gamma_n\}_{n=1}^\infty$, $\{\epsilon_n\}_{n=1}^\infty$ *and* $\{e_n\}_{n=1}^\infty$ *such that the conditions from Assumption 3.2 hold. Select arbitrary points* $x_0, x_1 \in H$ *and* $\theta \in [0,1)$. *Let* u *be arbitrary but fixed in* H. *Set* $n := 1$.

(S.1) *Given the iterates* x_{n-1} *and* x_n $(n \geq 1)$, *choose* β_n *such that* $0 \leq \beta_n \leq \bar{\beta}_n$, *where*

$$\bar{\beta}_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1} \\ \theta, \quad \text{otherwise.} & \end{cases}$$

(S.2) *Compute*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{\gamma_n}^A (x_n + \beta_n(x_n - x_{n-1})) + e_n. \tag{3.1}$$

Remark 3.4. Observe that from Assumption 3.2 and Algorithm 3.3 we have that

$$\lim_{n\to\infty} \beta_n \|x_n - x_{n-1}\| = 0$$

and

$$\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0.$$

We remark also here that the Step (S.1) in our Algorithm 3.3 is easily implemented in numerical computation since the value of $\|x_n - x_{n-1}\|$ is a priori known before choosing β_n.

4. Convergence Analysis

We give our main result in this paper in the next theorem.

Theorem 4.1. *Let Assumptions 3.1 and 3.2 hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.3 converges strongly to* $z := P_{A^{-1}(0)} u$.

Proof. Let $y_n := x_n + \beta_n(x_n - x_{n-1})$. Then

$$\begin{aligned} \|x_{n+1} - z\| &\leq \alpha_n \|u - z\| + (1-\alpha_n)\|J_{\gamma_n}^A y_n - z\| + \|e_n\| \\ &\leq \alpha_n \|u - z\| + (1-\alpha_n)\|y_n - z\| + \|e_n\| \\ &\leq \alpha_n \|u - z\| + (1-\alpha_n)(\|x_n - z\| + \beta_n \|x_n - x_{n-1}\|) + \|e_n\| \\ &= (1-\alpha_n)\|x_n - z\| + \alpha_n\|u - z\| + (1-\alpha_n)\beta_n\|x_n - x_{n-1}\| \\ &\quad + \|e_n\|. \end{aligned} \tag{4.1}$$

If $\lim_{n\to\infty} \frac{\|e_n\|}{\alpha_n} = 0$, then $\{\frac{\|e_n\|}{\alpha_n}\}$ is bounded. Furthermore, $\sup_{n\geq 1} \frac{\|e_n\|}{\alpha_n}$ and $\sup_{n\geq 1} \frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\|$ both exist by Lemma 3.4. Take

$$M := 3 \max\left\{\|u - z\|, \sup_{n\geq 1}(1-\alpha_n)\frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\|, \sup_{n\geq 1} \frac{\|e_n\|}{\alpha_n}\right\}.$$

Then (4.1) becomes

$$\|x_{n+1} - z\| \leq (1-\alpha_n)\|x_n - z\| + \alpha_n M.$$

By Lemma 2.2 , we get that $\{x_n\}$ is bounded.

Suppose $\sum_{n=1}^{\infty} \|e_n\| < \infty$. We have from (4.1) that

$$\begin{aligned} \|x_{n+1} - z\| &\leq (1-\alpha_n)\|x_n - z\| + \alpha_n\|u - z\| + (1-\alpha_n)\beta_n\|x_n - x_{n-1}\| + \|e_n\| \\ &= (1-\alpha_n)\|x_n - z\| + \alpha_n(\|u - z\| + (1-\alpha_n)\frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\|) \\ &\quad +\|e_n\|. \end{aligned} \tag{4.2}$$

Take

$$M_1 := 2\max\{\|u - z\|, \sup_{n\geq 1}(1-\alpha_n)\frac{\beta_n}{\alpha_n}\|x_n - x_{n-1}\|\}.$$

Then (4.2) becomes

$$\|x_{n+1} - z\| \leq (1-\alpha_n)\|x_n - z\| + \alpha_n M_1 + \|e_n\|.$$

By Lemma 2.2, we have that $\{x_n\}$ is bounded.

Now, by Lemma 2.1 (i) and the fact that $\beta_n \in [0,1)$, we get

$$\begin{aligned} \|y_n - z\| &= \|x_n - z + \beta_n(x_n - x_{n-1})\|^2 \\ &\leq \|x_n - z\|^2 + 2\beta_n\langle x_n - x_{n-1}, x_n - z\rangle + \beta_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{4.3}$$

Using Lemma 2.1 (i) again, we get

$$\langle x_n - x_{n-1}, x_n - z\rangle = -\frac{1}{2}\|x_{n-1} - z\|^2 + \frac{1}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2. \tag{4.4}$$

Substituting (4.4) into (4.3), we get

$$\begin{aligned} \|y_n - z\|^2 &= \|x_n - z\|^2 + \beta_n(-\|x_{n-1} - z\|^2 + \|x_n - z\|^2 + \|x_n - x_{n-1}\|^2) \\ &\quad +\beta_n\|x_n - x_{n-1}\|^2 \\ &= \|x_n - z\|^2 + \beta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) \\ &\quad +2\beta_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{4.5}$$

Using Lemma 2.1 (ii) in (3.1) (noting that $J^A_{\gamma_n}$ is firmly nonexpansive), we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(u - z) + (1-\alpha_n)(J^A_{\gamma_n}y_n - z) + e_n\|^2 \\ &= \|\alpha_n(u - z + \frac{e_n}{\alpha_n}) + (1-\alpha_n)(J^A_{\gamma_n}y_n - z)\|^2 \\ &\leq (1-\alpha_n)\|J^A_{\gamma_n}y_n - z\|^2 + 2\alpha_n\langle u - z + \frac{e_n}{\alpha_n}, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)\|y_n - z\|^2 - (1-\alpha_n)\|J^A_{\gamma_n}y_n - y_n\|^2 \\ &\quad +2\alpha_n\langle u - z + \frac{e_n}{\alpha_n}, x_{n+1} - z\rangle. \end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\alpha_n)\|y_n-z\|^2-(1-\alpha_n)\|J^A_{\gamma_n}y_n-y_n\|^2\\ &\quad +2\alpha_n\langle u-z-\frac{e_n}{\alpha_n},x_{n+1}-z\rangle\\ &= (1-\alpha_n)\|x_n-z\|^2-(1-\alpha_n)\|J^A_{\gamma_n}y_n-y_n\|^2\\ &\quad +\beta_n(1-\alpha_n)(\|x_n-z\|^2-\|x_{n-1}-z\|^2)\\ &\quad +2\beta_n(1-\alpha_n)\|x_n-x_{n-1}\|^2\\ &\quad +2\alpha_n\langle u-z+\frac{e_n}{\alpha_n},x_{n+1}-z\rangle. \end{aligned}\tag{4.7}$$

Set

$$\Gamma_n := \|x_n-z\|^2, \forall n\ge 1.$$

Then (4.7) implies

$$\begin{aligned}\Gamma_{n+1} &\le (1-\alpha_n)\Gamma_n-(1-\alpha_n)\|J^A_{\gamma_n}y_n-y_n\|^2+\beta_n(1-\alpha_n)(\Gamma_n-\Gamma_{n-1})\\ &\quad +2\beta_n(1-\alpha_n)\|x_n-x_{n-1}\|^2+2\alpha_n\langle u-z+\frac{e_n}{\alpha_n},x_{n+1}-z\rangle.\end{aligned}\tag{4.8}$$

We consider two cases for the rest of the proof.

Case 1: Suppose there exists a natural number n_0 such that $\Gamma_{n+1}\le\Gamma_n$ for all $n\ge n_0$. Therefore, $\lim_{n\to\infty}\Gamma_n$ exists. From (4.8), we have

$$\begin{aligned}(1-\alpha_n)\|J^A_{\gamma_n}y_n-y_n\|^2 &\le (\Gamma_n-\Gamma_{n+1})+\beta_n(1-\alpha_n)(\Gamma_n-\Gamma_{n-1})\\ &\quad +2\beta_n(1-\alpha_n)\|x_n-x_{n-1}\|^2\\ &\quad +2\alpha_n\langle u-z+\frac{e_n}{\alpha_n},x_{n+1}-z\rangle\\ &= (\Gamma_n-\Gamma_{n+1})+\beta_n(1-\alpha_n)(\Gamma_n-\Gamma_{n-1})\\ &\quad +2\beta_n(1-\alpha_n)\|x_n-x_{n-1}\|^2+2\alpha_n\langle u-z,x_{n+1}-z\rangle\\ &\quad +2\langle e_n,x_{n+1}-z\rangle.\end{aligned}\tag{4.9}$$

Using Assumption 3.2 (noting that $\lim_{n\to\infty}\beta_n\|x_n-x_{n-1}\|=0$ and $\{x_n\}$ is bounded), we have

$$\lim_{n\to\infty}\|J^A_{\gamma_n}y_n-y_n\|=0.$$

This together with Lemma 2.3 (iii) immediately implies that

$$\lim_{n\to\infty}\|J^A_{\gamma}y_n-y_n\|=0.$$

Since $\{x_n\}$ is bounded, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup p\in H$ and

$$\begin{aligned}\limsup_{n\to\infty}\langle u-z,x_n-z\rangle &= \lim_{k\to\infty}\langle u-z,x_{n_k}-z\rangle\\ &= \langle u-z,p-z\rangle.\end{aligned}\tag{4.10}$$

From $y_n=x_n+\beta_n(x_n-x_{n-1})$, we get

$$\|y_n-x_n\|=\beta_n\|x_n-x_{n-1}\|\to 0.$$

Since $x_{n_k} \rightharpoonup p$, then $y_{n_k} \rightharpoonup p$. Lemma 2.4 then guarantees that $p \in A^{-1}(0)$. Furthermore, we have from $z = P_{A^{-1}(0)}u$ that

$$\limsup_{n\to\infty}\langle u - z, x_n - z\rangle \le 0. \tag{4.11}$$

From (4.8), we have

$$\begin{aligned}
\Gamma_{n+1} &\le (1-\alpha_n)\Gamma_n + \beta_n(1-\alpha_n)(\Gamma_n - \Gamma_{n-1})\\
&\quad +2\beta_n(1-\alpha_n)\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle u - z + \frac{e_n}{\alpha_n}, x_{n+1} - z\rangle\\
&\le (1-\alpha_n)\Gamma_n + \beta_n(1-\alpha_n)\|x_n - x_{n-1}\|(\sqrt{\Gamma_n} + \sqrt{\Gamma_{n-1}})\\
&\quad +2\beta_n(1-\alpha_n)\|x_n - x_{n-1}\|^2 + 2\alpha_n\langle u - z + \frac{e_n}{\alpha_n}, x_{n+1} - z\rangle\\
&\le (1-\alpha_n)\Gamma_n + \beta_n\|x_n - x_{n-1}\|M_2\\
&\quad +2\alpha_n\langle u - z + \frac{e_n}{\alpha_n}, x_{n+1} - z\rangle,
\end{aligned} \tag{4.12}$$

for some $M_2 > 0$ such that $M_2 := \sup_{n\ge1}(1-\alpha_n)(\sqrt{\Gamma_n} + \sqrt{\Gamma_{n-1}} + 2\|x_n - x_{n-1}\|)$.

If $\sum_{n=1}^{\infty}\|e_n\| < \infty$, then we derive from (4.12) that

$$\begin{aligned}
\Gamma_{n+1} &\le (1-\alpha_n)\Gamma_n + \beta_n\|x_n - x_{n-1}\|M_2\\
&\quad +\alpha_n b_n + c_n,
\end{aligned} \tag{4.13}$$

where $b_n := 2\langle u - z, x_{n+1} - z\rangle$ and $c_n := M_3\|e_n\|$, for some $M_3 > 0$. Using Lemma 2.2 (ii) and Assumption 3.2 in (4.13), we get $\Gamma_n = \|x_n - z\| \to 0$ and thus $x_n \to z$ as $n \to \infty$.

If $\frac{\|e_n\|}{\alpha_n} \to 0$, we have from (4.12) that

$$\begin{aligned}
\Gamma_{n+1} &\le (1-\alpha_n)\Gamma_n + \beta_n\|x_n - x_{n-1}\|M_2\\
&\quad +\alpha_n b_n,
\end{aligned} \tag{4.14}$$

where $b_n = 2\langle u - z + \frac{e_n}{\alpha_n}, x_{n+1} - z\rangle$. Observe that $\limsup_{n\to\infty} b_n \le 0$ by (4.11). Using Lemma 2.2 and Assumption 3.2 in (4.14), we get that $x_n \to z$ as desired.

Case 2: Assume that there is no $n_0 \in \mathbb{N}$ such that $\{\Gamma_n\}_{n=n_0}^{\infty}$ is monotonically decreasing. The technique of proof used here is adapted from [22]. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \ge n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \le n, \Gamma_k \le \Gamma_{k+1}\},$$

i.e. $\tau(n)$ is the largest number k in $\{1, \dots, n\}$ such that Γ_k increases at $k = \tau(n)$; note that, in view of Case 2, this $\tau(n)$ is well-defined for all sufficiently large n. Clearly, τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \le \Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \quad \forall n \ge n_0.$$

After a similar to (4.9), it is easy to show that $\lim_{n\to\infty}\|J^A_{\gamma_{\tau(n)}}y_{\tau(n)} - y_{\tau(n)}\| = 0$, $\lim_{n\to\infty}\|J^A_{\gamma}y_{\tau(n)} - y_{\tau(n)}\| = 0$, $\lim_{n\to\infty}\|y_{\tau(n)} - x_{\tau(n)}\| = 0$, and $\lim_{n\to\infty}\|J^A_{\gamma_{\tau(n)}}y_{\tau(n)} - x_{\tau(n)}\| = 0$.

Furthermore, using the boundedness of $\{x_n\}$ and Assumption 3.2, we get

$$\begin{aligned} \|x_{\tau(n)+1} - x_{\tau(n)}\| &\leq \alpha_{\tau(n)}\|u - x_{\tau(n)}\| + (1-\alpha_{\tau(n)})\|J^A_{\gamma_{\tau(n)}} y_{\tau(n)} - x_{\tau(n)}\| \\ &\quad +\|e_{\tau(n)}\| \to 0, \ n \to \infty. \end{aligned} \tag{4.15}$$

Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence of $\{x_{\tau(n)}\}$, still denoted by $\{x_{\tau(n)}\}$, which converges weakly to some $p \in A^{-1}(0)$. Similarly, as in Case 1 above, we can show that $\limsup_{n\to\infty}\langle u - z, x_{\tau(n)+1} - z\rangle \leq 0$. Following (4.12), we obtain

$$\begin{aligned} \alpha_{\tau(n)}\Gamma_{\tau(n)} &\leq \beta_{\tau(n)}\|x_{\tau(n)} - x_{\tau(n)-1}\|M_2 \\ &\quad +2\alpha_{\tau(n)}\langle u - z + \frac{e_{\tau(n)}}{\alpha_{\tau(n)}}, x_{\tau(n)+1} - z\rangle, \end{aligned} \tag{4.16}$$

which shows

$$\begin{aligned} \Gamma_{\tau(n)} &\leq \frac{\beta_{\tau(n)}}{\alpha_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|M_2 \\ &\quad +2\langle u - z + \frac{e_{\tau(n)}}{\alpha_{\tau(n)}}, x_{\tau(n)+1} - z\rangle. \end{aligned} \tag{4.17}$$

We have from (4.17) that $\lim_{n\to\infty}\|x_{\tau(n)} - z\| = \lim_{n\to\infty}\Gamma_{\tau(n)} = 0$ which, in turn, implies $\lim_{n\to\infty}\|x_{\tau(n)+1} - z\| = 0$. Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ (observe that $\tau(n) \leq n$ for $n \geq n_0$ and consider the three cases: $\tau(n) = n, \tau(n) = n-1$ and $\tau(n) < n-1$. For the first and second cases, it is obvious that $\Gamma_n \leq \Gamma_{\tau(n)+1}$, for $n \geq n_0$. For the third case $\tau(n) \leq n-2$, we have from the definition of $\tau(n)$ and for any integer $n \geq n_0$ that $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n-1$. Thus, $\Gamma_{\tau(n)+1} \geq \Gamma_{\tau(n)+2} \geq \cdots \geq \Gamma_{n-1} \geq \Gamma_n$). As a consequence, we obtain for all sufficiently large n that $0 \leq \Gamma_n \leq \Gamma_{\tau(n)+1}$. Hence $\lim_{n\to\infty}\Gamma_n = 0$. Therefore, $\{x_n\}$ converges strongly to z.

□

Remark 4.2. (a) Our result compliment the strong convergence result of proximal point algorithm with errors already obtained in [8, 25, 35, 36]. In particular, our result reduces to the result in [8, 25, 35, 36] when $\beta_n = 0, \forall n \geq 1$.
(b) Our result can also be viewed as the strong convergence version of the already obtained weak convergence result of proximal point algorithm with inertial obtained in [1, 2].

5. Application

In this section, we apply Theorem 4.1 by establishing strong convergence result with inertial terms for the problem of finding a minimizer of a convex function.

Let us consider the problem of finding $z \in H$ such that $f(z) \leq f(x)\ \forall x \in H$, where $f : H \to (-\infty, +\infty]$ is a proper lower semicontinuous and convex function. Recall that z minimizes f if and only if $0 \in \partial f(z)$, where $\partial f(x) := \{s \mid f(y) \geq f(x) + \langle s, y - x\rangle\ \forall y\}$ denotes the subdifferential of f at x. Furthermore, ∂f is a maximal monotone operator; see [26, 30].

Corollary 5.1. *Let H be a real Hilbert space, f a proper lower semicontinuous convex function of H into $(-\infty, \infty]$. Let Assumptions 3.1 and 3.2 hold. Then the sequence $\{x_n\}$ generated by the following algorithm:*

Algorithm 5.2. (S.0) *Choose sequences $\{\alpha_n\}_{n=1}^{\infty}$, $\{\gamma_n\}_{n=1}^{\infty}$, $\{\epsilon_n\}_{n=1}^{\infty}$ and $\{e_n\}_{n=1}^{\infty}$ such that the conditions from Assumption 3.2 hold. Select arbitrary points $x_0, x_1 \in H$ and $\theta \in [0, 1)$. Let u be arbitrary but fixed in H. Set $n := 1$.*

(S.1) *Given the iterates x_{n-1} and x_n $(n \geq 1)$, choose β_n such that $0 \leq \beta_n \leq \bar{\beta_n}$, where*

$$\bar{\beta_n} = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1} \\ \theta, & \text{otherwise.} \end{cases}$$

(S.2) *Compute*

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}) \\ z_n = \text{argmin}\{f(z) + \frac{\|z-y_n\|^2}{2\gamma_n} : z \in H\} \\ x_{n+1} = \alpha_n u + (1-\alpha_n) z_n + e_n, \ n \geq 1, \end{cases} \tag{5.1}$$

Then $\{x_n\}$ converges strongly to the minimizer of f nearest to u.

Proof. We know that ∂f is a maximal monotone operator and $(\partial f)^{-1}(0)$ coincides with the set of minimizers of f. Suppose I denotes the identity mapping on H. It is known that

$$(I + \gamma_n(\partial f))^{-1} x = \text{argmin}\{f(z) + \frac{\|z - x\|^2}{2\gamma_n} : z \in H\}$$

for all $n \geq 1$ and $x \in H$. Therefore, (5.1) reduces to $x_{n+1} = \alpha_n u + (1-\alpha_n)(I + \gamma_n(\partial f))^{-1}(x_n + \beta_n(x_n - x_{n-1})) + e_n$ and we obtain the desired conclusion by applying Theorem 4.1. □

Acknowledgment

The research of this author is supported by the Alexander von Humboldt-Foundation.

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Manuscript received February 24 2018
revised June 22 2018

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