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REGULARIZED CONTINUOUS THIRD-ORDER METHOD FOR MONOTONE OPERATOR EQUATIONS IN HILBERT SPACE

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ABSTRACT. The equation $Ax = f$ (1) with arbitrary monotone continuous operator A is given in a Hilbert space. This is the ill-posed problem. We assume that solution set of (1) is not empty and propose the third-order regularization continuous method to find some point of this set. Our method is reduced to the Cauchy problem for the third-order operator differential equation with arbitrary initial conditions. It is linear with respect to all derivatives. Its coefficients are not constant and they can be selected by the special way. As before (see [1]), we study (1) with both the perturbed operator A and the right-hand side f. We give some sufficient conditions for strong convergence of this method to the normal solution of (1). Examples of parametric functions satisfying the conditions of convergence are also considered.

1. INTRODUCTION AND PRELIMINARIES

Let H be a real Hilbert space, (u, v) be a scalar product of elements $u, v \in H$, $A : H \to H$ be a nonlinear monotone continuous operator with $D(A) = H$. A monotonicity of A means that the condition

$$(Au - Av, u - v) \geq 0 \tag{1.1}$$

is satisfied for all $u, v \in H$. We consider the equation

$$Ax = f, \tag{1.2}$$

where f is a fixed element in H. Suppose that the equation (1.2) has a nonempty solution set N. It is well known that N is convex and closed set (see, for instance, [2], p. 31). Denote by $x^* \in N$ its normal solution, i.e., the solution with the minimum norm. The problem of finding x^* under the condition (1.1) belongs to the class of ill-posed problems. Therefore, to solve (1.2), it is required to apply any suitable regularizing procedure. For this goal in the present note we construct the continuous regularized method which is reduced to third-order differential equation.

The first-order continuous regularized method has been studied by many authors (see, for example, [1], [2]). The second-order continuous method was proposed in [3]. Convergence of the regularized second-order continuous method was proved in [6].

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A third-order method for the equation (1.2) with a strongly monotone operator A was constructed in [5]. As it is noted above, in this paper we study the equation (1.2) with an arbitrary monotone operator. Such sort of ill-posed problems are very difficult to implementation. Their exact solution is possible only in exceptional cases. Therefore in order to find an approximate solution of (1.2), it is necessary to have as wide a variety of methods as possible. It is especially important when in addition the approximations should satisfy the a priori known properties of sought solution. Let us also emphasize that the technique for the convergence proof of regularized continuous third-order method allows to investigate regularized continuous methods of order higher than three.

In accordance with the theory and applications of ill-posed problems, we assume in the sequel that in (1.2) A and f are known with errors, i.e.,
a) instead of f, the parametric approximations $f(t) \in H$ are given for all $t \geq t_0 \geq 0$ and there exists a positive function $\delta(t)$ such that $\delta(0) = 0$, $\delta(t) \to 0$ as $t \to +\infty$ and

$$\|f(t) - f\| \leq \delta(t) \quad \forall t \geq t_0, \tag{1.3}$$

b) instead of A, the parametric approximating monotone operators $A(t) : H \to H$ are given for all $t \geq t_0 \geq 0$ and there exists bounded positive functions $g(s), s \geq 0$, and $h(t)$ such that $h(0) = 0$, $h(t) \to 0$ as $t \to +\infty$, and

$$\|A(t)v - Av\| \leq h(t)g(\|v\|) \quad \forall t \geq t_0, \quad \forall v \in H. \tag{1.4}$$

It is known (see, for instance, [2], Theorem 2.1.2) that there exists a unique solution $x_\alpha(t)$ of equation

$$Ax_\alpha(t) + \alpha(t)x_\alpha(t) = f \tag{1.5}$$

for all $t \geq t_0$, and

$$\lim_{t\to\infty} \|x_\alpha(t) - x^*\| = 0, \tag{1.6}$$

if the continuous function $\alpha(t)$ satisfies the conditions

$$\alpha(t) > 0 \quad \forall t \geq t_0 \quad \text{and} \quad \lim_{t\to\infty} \alpha(t) = 0. \tag{1.7}$$

It follows from (1.6) that there exists a constant $M > 0$ such that

$$\|x_\alpha(t)\| \leq M \quad \forall t \geq t_0. \tag{1.8}$$

2. Regularized continuous method of the third-order

We construct a continuous regularized third-order method for the equation (1.2) in the form of the following Cauchy problem: For all $t \geq t_0 \geq 0$

$$y'''(t) + \varphi_1(t)y''(t) + \varphi_2(t)y'(t) + \varphi_3(t)[A(t)y(t) + \alpha(t)y(t) - f(t)] = 0, \tag{2.1}$$

$$y(t_0) = y_0, \;\; y'(t_0) = y_0', \;\; y''(t_0) = y_0'', \tag{2.2}$$

where y_0, y_0', y_0'' are some arbitrary fixed elements of H, $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ are positive continuous bounded functions.

Suppose that $\{A(t)\}$ is a family of continuous operators $A(t) : H \to H$ with respect to $t \in [t_0, +\infty)$ satisfying the Lipschitz condition with a constant $L > 0$, that is

$$\|A(t)u - A(t)v\| \leq L\|u - v\| \quad \forall u, v \in H, \quad \forall t \geq t_0. \tag{2.3}$$

Then the problem (2.1), (2.2) has a unique solution in the class of functions $C^3[t_0, +\infty)$ (see [7], p. 33.4). It is not difficulty to see that (2.3) and (1.4) imply inequality

$$\|Au - Av\| \leq L\|u - v\| \quad \forall u, v \in H.$$

Hence, we obtained the continuity of operator A.

Assume further that there exists a number $r_0 > 0$ such that at least for large enough t the following inequality is true:

$$\begin{aligned} \varphi_1(t)\|y''(t)\|^2 \quad &- \quad (y(t) + y''(t), y'(t)) + \varphi_2(t)(y'(t), y''(t)) \\ &+ \quad \varphi_3(t)(A(t)y(t) + \alpha(t)y(t) - f(t), y''(t)) \geq 0, \end{aligned} \tag{2.4}$$

if

$$\|y(t)\|^2 + \|y'(t)\|^2 + \|y''(t)\|^2 \geq r_0.$$

Similarly to [5], it is proved that (2.4) provides the existence of a constant $\tilde{r}_0 > 0$ such that

$$\|y(t)\| \leq \tilde{r}_0, \quad \|y'(t)\| \leq \tilde{r}_0, \quad \|y''(t)\| \leq \tilde{r}_0 \quad \forall t \geq t_0. \tag{2.5}$$

We can establish the conditions which guarantee the convergence $\|y(t) - x^*\| \to 0$ as $t \to \infty$. In view of (1.6), it is sufficient to present the conditions for convergence of $y(t)$ to $x_\alpha(t)$ as $t \to \infty$. Let $\lambda(t)$ and $\bar{\lambda}(t) = \dfrac{\lambda(t)}{\alpha(t)}$ be positive double continuously differentiable decreasing convex from below functions for all $t \geq t_0 \geq 0$, l and m be some positive numbers. Let

$$l > m > \lambda(t) > 0, \quad \lim_{t \to \infty} \lambda(t) = 0, \tag{2.6}$$

$$\lim_{t \to \infty} t\lambda(t) = +\infty, \tag{2.7}$$

$$\lim_{t \to \infty} \frac{\lambda(t)}{\alpha(t)} = 0. \tag{2.8}$$

By (1.7), we deduce from (2.8) that

$$\lim_{t \to \infty} \lambda(t) = 0. \tag{2.9}$$

Introduce now in (2.1) the functions $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ as it follows:

$$\varphi_1(t) = l + m + \lambda(t), \quad \varphi_2(t) = lm + (l + m)\lambda(t), \quad \varphi_3(t) = \frac{lm\lambda(t)}{2\alpha(t)}. \tag{2.10}$$

Let $\tau \geq t_0$ be an arbitrary fixed number and $x_\alpha(\tau)$ be a solution of the equation (1.5) with $t = \tau$, i.e.,

$$Ax_\alpha(\tau) + \alpha(\tau)x_\alpha(\tau) = f. \tag{2.11}$$

Multiplying (2.1) by $y(t)-x_\alpha(\tau)$ and using (2.11), we get for all $t\geq t_0$

$$
\begin{aligned}
(y'''(t),y(t) &- x_\alpha(\tau))+\varphi_1(t)(y''(t),y(t)-x_\alpha(\tau))\\
&+ \varphi_2(t)(y'(t),y(t)-x_\alpha(\tau))\\
&+ \varphi_3(t)(A(t)y(t)-Ax_\alpha(\tau)+\alpha(t)y(t)-\alpha(\tau)x_\alpha(\tau)\\
&+ f-f(t),y(t)-x_\alpha(\tau))=0. \qquad (2.12)
\end{aligned}
$$

We make the following transformations:

$$
\begin{aligned}
\Lambda(t,\tau)&=\varphi_3(t)(A(t)y(t)-Ax_\alpha(\tau)+\alpha(t)y(t)-\alpha(\tau)x_\alpha(\tau)+f-f(t),\\
y(t)-x_\alpha(\tau))&=\varphi_3(t)[(A(t)y(t)-Ay(t),y(t)-x_\alpha(\tau))\\
&+(Ay(t)-Ax_\alpha(\tau),y(t)-x_\alpha(\tau))+\varphi_3(\tau)\alpha(\tau)\|y(t)-x_\alpha(\tau)\|^2\\
&+[\varphi_3(t)\alpha(t)-\varphi_3(\tau)\alpha(\tau)]\,(y(t),y(t)-x_\alpha(\tau))\\
&+[\varphi_3(\tau)-\varphi_3(t)]\,\alpha(\tau)(x_\alpha(\tau),y(t)-x_\alpha(\tau))\\
&+\varphi_3(t)(f-f(t),y(t)-x_\alpha(\tau)).
\end{aligned}
$$

Since the operator A is monotone and bounded, the aggregate families $\{f(t)\}$ and $\{A(t)\}$ are bounded for all $t\geq t_0$ (see (1.3) and (1.4)), by (1.8), (2.10) and (2.5) we conclude that there exists a positive constant c_1 such that the following estimate is fulfilled:

$$
\begin{aligned}
\Lambda(t,\tau) &\geq c_1\Big\{-\varphi_3(t)[h(t)+\delta(t)]-|\lambda(t)-\lambda(\tau)|-|\bar\lambda(t)-\bar\lambda(\tau)|\alpha(\tau)\Big\}\\
&+ \varphi_3(\tau)\alpha(\tau)\|y(t)-x_\alpha(\tau)\|^2. \qquad (2.13)
\end{aligned}
$$

We introduce a scalar function

$$r(t,\tau)=\frac{\|y(t)-x_\alpha(\tau)\|^2}{2} \qquad (2.14)$$

and find

$$r'_t(t,\tau)=(y'(t),y(t)-x_\alpha(\tau)), \qquad (2.15)$$

$$r''_t(t,\tau)=\|y'(t)\|^2+(y''(t),y(t)-x_\alpha(\tau)), \qquad (2.16)$$

$$r'''_t(t,\tau)=(y'''(t),y(t)-x_\alpha(\tau))+3(y''(t),y'(t)). \qquad (2.17)$$

By make using (2.5) and (2.13) – (2.17) (2.12) leads to the inequality

$$
\begin{aligned}
r'''_t(t,\tau)&+\varphi_1(\tau)r''_t(t,\tau)+\varphi_2(\tau)r'_t(t,\tau)+2\varphi_3(\tau)\alpha(\tau)r(t,\tau)\\
&\leq c_1\left\{\varphi_3(t)[h(t)+\delta(t)]+|\lambda(t)-\lambda(\tau)|+|\bar\lambda(t)-\bar\lambda(\tau)|\alpha(\tau)\right\}\\
&+3(y''(t),y'(t))+\varphi_1(\tau)\|y'(t)\|^2. \qquad (2.18)
\end{aligned}
$$

To estimate the right-hand side of (2.18), firstly we multiply (2.1) by $y'(t)$ and get the following equality for scalar products:

$$
\begin{aligned}
(y'''(t),y'(t))&+\varphi_1(t)(y''(t),y'(t))+\varphi_2(t)\|y'(t)\|^2\\
&+\varphi_3(t)(A(t)y(t)+\alpha(t)y(t)-f(t),y'(t))=0. \qquad (2.19)
\end{aligned}
$$

Suppose that $\rho(t)=\|y'(t)\|^2/2$. Then

$$\rho'(t)=(y''(t),y'(t)) \quad\text{and}\quad \rho''(t)=(y'''(t),y'(t))+\|y''(t)\|^2.$$

Taking into account again that the families $\{f(t)\}$ and $\{A(t)\}$ are bounded, $y(t)$ is bounded and $\alpha(t)$ is continuous for all $t \geq t_0$, we assert from (2.19) that there exists a constant $c_2 > 0$ such that

$$\rho''(t) + \varphi_1(t)\rho'(t) + 2\varphi_2(t)\rho(t) \leq c_2\varphi_3(t)\|y'(t)\| + \|y''(t)\|^2.$$

Hence, we obtain the inequality

$$\begin{aligned} &\rho''(t) + (l+m)\rho'(t) + 2\varphi_2(t)\rho(t) \\ &\quad \leq c_2\varphi_3(t)\|y'(t)\| + \lambda(t)|\rho'(t)| + \|y''(t)\|^2. \end{aligned} \tag{2.20}$$

Secondly, we multiply (2.1) by $y''(t)$ and get following equality:

$$\begin{aligned} &(y'''(t), y''(t)) + \varphi_1(t)\|y''(t)\|^2 + \varphi_2(t)(y'(t), y''(t)) + \\ &\quad + \varphi_3(t)(A(t)y(t) + \alpha(t)y(t) - f(t), y''(t)) = 0. \end{aligned} \tag{2.21}$$

Let $R(t) = \|y''(t)\|^2/2$. Then $R'(t) = (y'''(t), y''(t))$, and from (2.21) it follows that there exists a constant $c_3 > 0$ such that

$$R'(t) + 2\varphi_1(t)R(t) + \varphi_2(t)\rho'(t) \leq c_3\varphi_3(t)\|y''(t)\|. \tag{2.22}$$

Using the numerical inequality $2ab \leq a^2 + b^2$ we can write the estimate

$$c_3\varphi_3(t)\|y''(t)\| \leq [\varphi_3(t)]^{\epsilon} R(t) + 2^{-1}c_3^2 [\varphi_3(t)]^{2-\epsilon}, \tag{2.23}$$

where $\epsilon \in (0, 2)$ is any fixed number. Since

$$|\rho'(t)| = |(y''(t), y'(t))| \leq \rho(t) + R(t), \tag{2.24}$$

we state from (2.22) and (2.23) that there exists a constant $c_4 > 0$ such that

$$\begin{aligned} &R'(t) + \{2\varphi_1(t) - [\varphi_3(t)]^{\epsilon} - \varphi_2(t)\}R(t) \\ &\quad \leq c_4 [\varphi_3(t)]^{2-\epsilon} + \varphi_2(t)\rho(t), \quad t \geq t_0. \end{aligned} \tag{2.25}$$

Assumptions (2.10), (2.6), (2.8) and property (2.9) of the function $\lambda(t)$ allow us to claim: there exist constants l, m, l_0 such that the relations

$$2\varphi_1(t) - [\varphi_3(t)]^{\epsilon} - \varphi_2(t) \geq l_0 > 1 \tag{2.26}$$

are satisfied at least with sufficiently large $t \geq t_0$. Therefore, (2.25) yields the inequality

$$R'(t) + l_0R(t) \leq c_4 [\varphi_3(t)]^{2-\epsilon} + \varphi_2(t)\rho(t).$$

Further we use Lemma 7.2.2 from [2], p. 389 and the L'Hopital's rule to obtain the following assertion: there exist constants $c_5 > 0$ and $\beta \in (0, 1)$ such that

$$\begin{aligned} R(t) \leq{}& R(t_0)\exp(-l_0(t - t_0)) \\ &+ \int_{t_0}^{t} \left\{c_4 [\varphi_3(s)]^{2-\epsilon} + \varphi_2(s)\rho(s)\right\} \exp(l_0(s - t))ds \\ \leq{}& c_5 \left\{\exp(-l_0t) + [\varphi_3(t)]^{2-\epsilon}\right\} + \beta\varphi_2(t)\rho(t). \end{aligned} \tag{2.27}$$

Taking into account (2.24), we conclude from (2.20) and (2.27) that there exists a constant $c_6 > 0$ such that

$$\begin{aligned} \rho''(t) + (l+m)\rho'(t) + 2\varphi_2(t)\rho(t) \\ \leq c_6 \left\{ \varphi_3(t)\|y'(t)\| + \exp(-l_0 t) + [\varphi_3(t)]^{2-\epsilon} \right\} \\ + \left[2\beta\varphi_2(t) + \lambda(t) + \beta\lambda(t)\varphi_2(t) \right] \rho(t). \end{aligned} \tag{2.28}$$

Similarly to (2.23), we have

$$c_6\varphi_3(t)\|y'(t)\| \leq [\varphi_3(t)]^{\epsilon}\rho(t) + 2^{-1}c_6^2 [\varphi_3(t)]^{2-\epsilon}.$$

Therefore, there exists a constant $c_7 > 0$ such that from (2.28) it follows the inequality

$$\begin{aligned} \rho''(t) + (l+m)\rho'(t) + \{2\varphi_2(t)(1-\beta) - \lambda(t)(1+\beta\varphi_2(t)) - [\varphi_3(t)]^{\epsilon}\}\rho(t) \\ \leq c_7 \left\{ \exp(-l_0 t) + [\varphi_3(t)]^{2-\epsilon} \right\}. \end{aligned} \tag{2.29}$$

Next we find the number m_0 such that

$$2\varphi_2(t)(1-\beta) - \lambda(t)(1+\beta\varphi_2(t)) - [\varphi_3(t)]^{\epsilon} > m_0 > 0 \tag{2.30}$$

at least for sufficiently large $t \geq t_0$. Such a choice m_0 is possible because under our conditions $\lambda(t)$ and $\varphi_3(t)$ are infinitely small as $t \to \infty$ and $\beta \in (0,1)$. Without loss of generality, we believe that (2.30) is true for all $t \geq t_0$. Then (2.29) leads to the inequality

$$\rho''(t) + (l+m)\rho'(t) + m_0\rho(t) \leq c_7 \left\{ \exp(-l_0 t) + [\varphi_3(t)]^{2-\epsilon} \right\}. \tag{2.31}$$

In virtue of (2.30) we can choose m_0 to guarantee real and negative roots $k_1 = \nu_1$ and $k_2 = \nu_2$ of the characteristic equation $k^2 + (l+m)k + m_0 = 0$ for (2.31). If $\nu_1 > \nu_2$, $\nu = \min\{l_0, -\nu_1\}$, then (2.31) and Lemma 2 of [4] allows us to write an estimate

$$\rho(t) \leq c_8 \Big\{ \exp(-\nu t) + \int_{t_0}^{t} [\varphi_3(s)]^{2-\epsilon} \exp(\nu_1(t-s))ds \Big\}.$$

Applying the L'Hopital's rule to the rightmost term of the previous inequality, we come to the final estimate: there exists a constant $c_9 > 0$ such that

$$\rho(t) \leq c_9 \left\{ \exp(-\nu t) + [\varphi_3(t)]^{2-\epsilon} \right\}. \tag{2.32}$$

Combining (2.32) and (2.27), we conclude that there exists a constant $c_{10} > 0$ such that

$$R(t) \leq c_{10} \left\{ \exp(-\nu t) + [\varphi_3(t)]^{2-\epsilon} \right\}. \tag{2.33}$$

The above functions $\lambda(t)$ and $\bar{\lambda}(t)$ and their properties permit to write the simple inequalities (see [9], p. 266):

$$0 \leq \lambda(s) - \lambda(t) \leq \lambda'(s)(s-t), \;\; 0 \leq \bar{\lambda}(s) - \bar{\lambda}(t) \leq \bar{\lambda}'(s)(s-t), \;\; s \leq t. \tag{2.34}$$

Returning now to (2.18), by (2.24), (2.32) – (2.34), one gets that there exists a constant $c_{11} > 0$ such that for all $t \geq t_0$, forall $\tau \geq t_0$, $t \leq \tau$ and $\epsilon \in (0, 2)$:

$$\begin{aligned} r_t'''(t,\tau) + \varphi_1(\tau) r_t''(t,\tau) + \varphi_2(\tau) r_t'(t,\tau) + lm\lambda(\tau) r(t,\tau) \\ \leq c_{11}\Big\{ \varphi_3(t)[h(t)+\delta(t)] + \lambda'(t)(t-\tau) + \bar{\lambda}'(t)(t-\tau) \\ + \exp(-\nu t) + [\varphi_3(t)]^{2-\epsilon} \Big\} = c_{11} F(t,\tau). \end{aligned} \tag{2.35}$$

As in [5], consider the cubic equation

$$k^3 + \varphi_1(\tau)k^2 + \varphi_2(\tau)k + \varphi\psi\lambda(\tau) = 0.$$

Its roots are real negative numbers $k_1 = -l$, $k_2 = -m$ and $k_3 = -\lambda(\tau)$. By Lemma 1 of [5], we deduce from (2.35) the following estimate:

$$\begin{aligned} r(t,\tau) \leq c_{12}\Big\{ & \exp(-\lambda(\tau)t) \\ & + \int_{t_0}^{t} F(u,\tau)\exp(-\lambda(\tau)(t-u))du \Big\},\ t \geq t_0,\ \tau \geq t_0,\ t \leq \tau. \end{aligned}$$

Hence, for $t = \tau$ we obtain

$$\begin{aligned} r(\tau,\tau) \leq c_{12}\Big\{ & \exp(-\lambda(\tau)\tau) \\ & + \int_{t_0}^{\tau} \varphi_3(u)[h(u)+\delta(u)]\exp(-\lambda(\tau)(\tau-u))du \\ & + \int_{t_0}^{\tau} \lambda'(u)(u-\tau)\exp(-\lambda(\tau)(\tau-u))du \\ & + \int_{t_0}^{\tau} \bar{\lambda}'(u)(u-\tau)\exp(-\lambda(\tau)(\tau-u))du \\ & + \int_{t_0}^{\tau} \exp(-\nu u)\exp(-\lambda(\tau)(\tau-u))du \\ & + \int_{t_0}^{\tau} [\varphi_3(u)]^{2-\epsilon}\exp(-\lambda(\tau)(\tau-u))du \Big\}. \end{aligned} \tag{2.36}$$

Now we introduce additional assumptions:

$$\Gamma_1(t) = (t\lambda(t))' > 0, \quad \Gamma_2(t) = \big[(t\lambda(t))'\big]^2 + (t\lambda(t))'' > 0 \qquad \forall t \geq t_0, \tag{2.37}$$

$$\lim_{t\to+\infty} \frac{h(t)+\delta(t)}{\alpha(t)} \frac{\lambda(t)}{(t\lambda(t))'} = 0, \tag{2.38}$$

$$\lim_{t\to\infty} \frac{[\varphi_3(t)]^{2-\epsilon}}{(t\lambda(t))'} = 0, \quad \epsilon \in (0,2), \tag{2.39}$$

$$\lim_{t\to+\infty} \frac{\lambda'(t) + \bar{\lambda}'(t)}{\Gamma_2(t)} = 0. \tag{2.40}$$

Let us show that the right-hand side of the inequality (2.36) tends to zero as $\tau \to +\infty$. Really, by (2.7), $\lim_{\tau\to\infty} \exp(-\lambda(\tau)\tau) = 0$. The convergence

$$\int_{t_0}^{\tau} \exp(-\nu u)\exp(-\lambda(\tau)(\tau-u))du \to 0$$

is proved by the direct computation and by (2.7) again. In order to obtain some sufficient conditions for convergence of the remaining terms, we use the L'Hopital's rule and decreasing property of $\lambda(t)$. This leads to the following result: there exists $a > 0$ such that

$$\int_{t_0}^{\tau} \varphi_3(u)[h(u)+\delta(u)]\exp(\lambda(\tau)(u-\tau))du \le a\frac{h(\tau)+\delta(\tau)}{\alpha(\tau)}\ \frac{\lambda(\tau)}{(\tau\lambda(\tau))'}\ \ \forall \tau \ge t_0.$$

This integral tends to zero by (2.38). Next it is not difficult to verify that if (2.39) is fulfilled, then

$$\int_{t_0}^{\tau} [\varphi_3(u)]^{2-\epsilon}\exp(-\lambda(\tau)(\tau-u))du \to 0.$$

Applying twice the L'Hopital's rule, we assert that there exit constants $a_1 > 0$ and $a_2 > 0$ such that

$$\int_{t_0}^{\tau} \lambda'(u)(u-\tau)\exp(-\lambda(\tau)(\tau-u))du \le a_1\frac{\lambda'(\tau)}{\Gamma_2(\tau)}$$

and

$$\int_{t_0}^{\tau} \bar{\lambda}'(u)(u-\tau)\exp(-\lambda(\tau)(\tau-u))du \le a_2\frac{\bar{\lambda}'(\tau)}{\Gamma_2(\tau)}.$$

Convergence to zero of these integrals is defined by (2.40). Thus, the assumptions (2.38)–(2.40) give us $\lim_{\tau\to\infty} r(\tau,\tau) = 0$. Finally, taking into account (1.6), we get the main result $\lim_{t\to\infty} y(t) = x^*$.

So, we have proved the following statement.

Theorem 2.1. *Suppose that H is a real Hilbert space, operator $A : H \to H$ satisfies (1.1), the equation (1.2) has a nonempty solution set, A and f in (1.2) are given approximately such that the conditions a) and b) hold. Assume further that the family of operators $\{A(t)\}$ is continuous with respect to parameter t and inequality (2.3) is fulfilled. Let functions $\varphi_k(t)$, $k = 1,2,3$, be defined by the equalities (2.10). Presume that (2.6)-(2.8) are valid, $\lambda(t)$ and $\bar{\lambda}(t)$ are positive, twice continuously differentiable, decreasing and convex from below functions, limit equalities (2.38)–(2.40) and inequalities (2.26), (2.30), (2.37) are true. Then the Cauchy problem (2.1), (2.2) defining a continuous regularized third-order method, is uniquely solvable in the class of functions $C^3[t_0,\infty)$ with any elements y_0, y_0' and y_0'' from H. If the condition (2.4) is satisfied, then a solution $y(t)$ of the problem (2.1), (2.2) strongly converges to the normal solution of the equation (1.2) as $t \to \infty$.*

We present examples of power functions satisfying the conditions of Theorem 2.1. Let $\alpha(t) = t^{-\alpha}$, $\lambda(t) = t^{-\lambda}$, $\delta(t) = t^{-\delta}$, $h(t) = t^{-h}$, where α, λ, δ, h are the positive numbers, $\alpha < \lambda$, $\lambda \in (0,1)$. Then $(t\lambda(t))' = (1-\lambda)\lambda(t)$ and (2.38) coincides with the classical convergence condition of the operator regularization method for monotone equations with approximate data (see, for instance, [2], p.118). So, we come to conditions $\alpha < \delta,\ \ \alpha < h$. The equality (2.39) has the form

$$\lim_{t\to\infty} \frac{[\lambda(t)]^{1-\epsilon}}{[\alpha(t)]^{2-\epsilon}} = 0.$$

Hence, we obtain the following conditions for ϵ: $\epsilon \in (0,1),\ \lambda(1-\epsilon) > \alpha(2-\epsilon)$. Inequalities (2.37) are fulfilled for all $\lambda \in (0,1)$ and at least for large enough t. The condition (2.40) is true as $\alpha+\lambda < 1,\ \lambda \in (0,1),\ \alpha \in (0,1)$. The set of $\alpha,\ \lambda,\ \delta,\ h,\ \epsilon$, satisfying the above conditions is nonempty, for example, $\alpha = 1/5,\ \lambda = 3/5,\ \delta = 2/5,\ h = 3/5,\ \epsilon = 1/3$.

In order to obtain any estimate of the convergence rate of $y(t)$ to x^*, it is necessary to have an upper bound for $\|x_\alpha(t) - x^*\|$. It can be an established under some strong assumptions (see [2], p. 129; [8]).

Theorem 2.1 can be a proved by use of the technique of [1], [2], p. 363, [9], p.269. For this we must construct two auxiliary Cauchy problems:

$$\begin{aligned} z_t'''(t,\tau) + \varphi_1(\tau) z_t''(t,\tau) + \varphi_2(\tau) z_t'(t,\tau) \\ + \varphi_3(\tau)[Az(t,\tau) + \alpha(\tau) z(t,\tau) - f] = 0, \end{aligned} \tag{2.41}$$

$$z(t_0,\tau) = y_0, \quad z_t'(t_0,\tau) = y_0', \ \ z_t''(t_0,\tau) = y_0'', \tag{2.42}$$

and

$$\begin{aligned} \tilde z_t'''(t,\tau) + \varphi_1(\tau) \tilde z_t''(t,\tau) + \varphi_2(\tau) \tilde z_t'(t,\tau) \\ + \varphi_3(\tau)[A(t)\tilde z(t,\tau) + \alpha(\tau)\tilde z(t,\tau) - f(t)] = 0, \end{aligned} \tag{2.43}$$

$$\tilde z(t_0,\tau) = y_0, \quad \tilde z_t'(t_0,\tau) = y_0', \ \ \tilde z_t''(t_0,\tau) = y_0''. \tag{2.44}$$

The convergences of $\|z(\tau,\tau) - x_\alpha(\tau)\|$, $\|\tilde z(\tau,\tau) - z(\tau,\tau)\|$ and $\|\tilde z(\tau,\tau) - y(\tau)\|$ to zero as $\tau \to \infty$ are established similarly to Theorem 2.1. The requirements for power functions are the same as in Theorem 2.1. However, there is a need for the implementation of inequalities of the form (2.5) for the solutions of the problems (2.41)-(2.44).

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