

## FIXED POINTS OF METRICALLY NONSPREADING MAPPINGS IN HADAMARD SPACES

FUMIAKI KOHSAKA

ABSTRACT. We study the existence and approximation of fixed points of metrically nonspreading mappings and firmly metrically nonspreading mappings in Hadamard spaces. The resolvents of monotone operators satisfying range conditions are typical examples of firmly metrically nonspreading mappings. Applications to monotone operators in such spaces are also included.

### 1. INTRODUCTION

A number of nonlinear variational problems can be formulated as the problem of finding zero points of maximal monotone operators in Banach spaces. Among those problems are convex minimization problems [34, 35], variational inequality problems [36], saddle point problems [37], and equilibrium problems [5].

The class of nonspreading mappings first introduced by Kohsaka and Takahashi [27] is closely related to the problem of finding zero points of maximal monotone operators in Banach spaces. The authors of the papers [24, 27, 28] obtained some basic results on the fixed point problem for nonspreading mappings and applied them to maximal monotone operators in Banach spaces.

Recall that a mapping  $T$  of a nonempty subset  $C$  of a smooth real Banach space  $E$  into itself is said to be nonspreading if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all  $x, y \in C$ , where  $\phi$  is the two variable real function [3, 4, 19] on  $E^2$  defined by

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv \rangle + \|v\|^2$$

for all  $u, v \in E$  and  $J$  denotes the normalized duality mapping of  $E$  into  $E^*$ . It is known [27, 28] that if  $E$  is a smooth, strictly convex, and reflexive Banach space, then the following hold.

- The generalized projection  $\Pi_C$  [3, 4] of  $E$  onto a nonempty closed convex subset  $C$  of  $E$  is nonspreading and  $\mathcal{F}(\Pi_C) = C$ ;
- the resolvent  $Q_A$  [17, 18, 26] of a maximal monotone operator  $A: E \rightarrow 2^{E^*}$  defined by  $Q_A = (J + A)^{-1}J$  is nonspreading and  $\mathcal{F}(Q_A) = A^{-1}(0)$ .

We know the following results for nonspreading mappings in Banach spaces.

**Theorem 1.1** ([27, Theorem 4.1]). *Let  $C$  be a nonempty closed convex subset of a smooth, strictly convex, and reflexive real Banach space  $E$  and  $T$  a nonspreading mapping of  $C$  into itself. Then  $\mathcal{F}(T)$  is nonempty if and only if  $\{T^n x\}$  is bounded for some  $x \in C$ .*

2010 *Mathematics Subject Classification*. Primary: 47H10, 47J05; Secondary: 52A41, 90C25.

*Key words and phrases*. Fixed point, geodesic space, Hadamard space, metrically nonspreading mapping.

**Theorem 1.2** ([27, Proposition 3.2]). *Let  $C$  be a nonempty closed convex subset of a strictly convex real Banach space  $E$  with a uniformly Gâteaux differentiable norm,  $T$  a nonspreading mapping of  $C$  into itself, and  $u$  an element of  $C$  such that there exists a sequence  $\{x_n\}$  in  $C$  which is weakly convergent to  $u$  and satisfies  $\|x_n - Tx_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $u$  is an element of  $\mathcal{F}(T)$ .*

**Theorem 1.3** ([27, Theorem 4.6]). *Let  $C$  be a nonempty bounded closed convex subset of a smooth, strictly convex, and reflexive real Banach space  $E$  and  $\{T_\alpha\}_{\alpha \in A}$  a commutative family of nonspreading mappings of  $C$  into itself. Then  $\bigcap_{\alpha \in A} \mathcal{F}(T_\alpha)$  is nonempty.*

In particular, we know the following result on the asymptotic behavior of the Mann iteration [30] for nonspreading mappings in Banach spaces.

**Theorem 1.4** ([24, Theorem 4.1]). *Let  $C$  be a nonempty closed convex subset of a smooth, strictly convex, and reflexive real Banach space  $E$ ,  $J$  the normalized duality mapping of  $E$  into  $E^*$ ,  $\Pi_C$  the generalized projection of  $E$  onto  $C$ ,  $T$  a nonspreading mapping of  $C$  into itself,  $\{\alpha_n\}$  a sequence in  $(0, 1]$  such that  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , and both  $\{x_n\}$  and  $\{z_n\}$  sequences in  $C$  defined by  $x_1 \in C$  and*

$$x_{n+1} = \Pi_C J^{-1}((1 - \alpha_n)Jx_n + \alpha_n JT x_n);$$

$$z_n = \frac{1}{\sum_{l=1}^n \alpha_l} \sum_{k=1}^n \alpha_k T x_k$$

for all  $n \in \mathbb{N}$ . Then the following are equivalent.

- (i)  $\mathcal{F}(T)$  is nonempty;
- (ii)  $\{x_n\}$  is bounded;
- (iii)  $\{z_n\}$  is bounded;
- (iv)  $\{z_n\}$  has a bounded subsequence.

In this case, each subsequential weak limit of  $\{z_n\}$  belongs to  $\mathcal{F}(T)$ .

In the special case where  $E$  is a real Hilbert space, a mapping  $T$  of a nonempty subset  $C$  of  $E$  into itself is nonspreading if

$$(1.1) \quad 2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all  $x, y \in C$ . This condition is satisfied whenever  $T$  is firmly nonexpansive, i.e.,

$$(1.2) \quad \|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle$$

for all  $x, y \in C$ .

Motivated by (1.1) and (1.2), we say that a mapping  $T$  of a metric space  $(X, d)$  into itself is

- metrically nonspreading if

$$2d(Tx, Ty)^2 \leq d(Tx, y)^2 + d(Ty, x)^2$$

for all  $x, y \in X$ ;

- firmly metrically nonspreading if

$$2d(Tx, Ty)^2 + d(Tx, x)^2 + d(Ty, y)^2 \leq d(Tx, y)^2 + d(Ty, x)^2$$

for all  $x, y \in X$ .

Every firmly metrically nonspreading mapping is obviously metrically nonspreading. We can also see that every metrically nonspreading mapping with a fixed point is quasinonexpansive and that every firmly metrically nonspreading mapping  $T$  of  $X$  into itself with a fixed point satisfies

$$(1.3) \quad d(u, Tx)^2 + d(Tx, x)^2 \leq d(u, x)^2$$

for all  $u \in \mathcal{F}(T)$  and  $x \in X$ . If  $X$  is a nonempty subset of a real Hilbert space, then

$$\langle x - y, z - w \rangle = \frac{1}{2} (\|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2)$$

for all  $x, y, z, w \in X$ , where the left hand side is the inner product on the space. In this case,  $T$  is firmly metrically nonspreading if and only if it is firmly nonexpansive.

Metrically nonspreading mappings and firmly metrically nonspreading mappings are also called 1/2-nonexpansive mappings and firmly nonexpansive mappings by Naraghirad, Wong, and Yao [32, Definition 4.6] and Khatibzadeh and Ranjbar [20, Definition 3.5], respectively. The notion of  $\alpha$ -nonexpansive mapping was first introduced by Aoyama and Kohsaka [6, Definition 2.2] in the context of Banach spaces.

As we see in Sections 3 and 6, the metric projections onto nonempty closed convex sets, the proximity mappings of proper lower semicontinuous convex functions, and the resolvents of monotone operators satisfying range conditions in Hadamard spaces are firmly metrically nonspreading. Thus the fixed point problem for such mappings is closely related to convex analysis in Hadamard spaces.

The aim of this paper is to study the existence and approximation of fixed points of metrically nonspreading mappings and firmly metrically nonspreading mappings in Hadamard spaces. In particular, we obtain analogues of Theorems 1.1, 1.2, 1.3, and 1.4 for metrically nonspreading mappings in Hadamard spaces. We finally apply our results to monotone operators in Hadamard spaces.

## 2. PRELIMINARIES

Throughout this paper, we denote by  $\mathbb{N}$  and  $\mathbb{R}$  the sets of all positive integers and real numbers, respectively. The two dimensional Euclidean space and its norm are denoted by  $\mathbb{R}^2$  and  $|\cdot|_{\mathbb{R}^2}$ , respectively. Unless otherwise specified, we denote by  $X$  a metric space with a metric  $d$ . The set of all fixed points of a mapping  $T$  of  $X$  into itself is denoted by  $\mathcal{F}(T)$ . A mapping  $T$  of  $X$  into itself is said to be

- asymptotically regular if  $\lim_n d(T^{n+1}x, T^n x) = 0$  for all  $x \in X$ ;
- nonexpansive if  $d(Tx, Ty) \leq d(x, y)$  for all  $x, y \in X$ ;
- quasinonexpansive if  $\mathcal{F}(T)$  is nonempty and  $d(u, Tx) \leq d(u, x)$  for all  $u \in \mathcal{F}(T)$  and  $x \in X$ .

The product space  $X \times X$  and its element  $(x, y)$  are denoted by  $X^2$  and  $\overrightarrow{xy}$ , respectively. The quasilinearization  $\langle \cdot, \cdot \rangle$  on  $X^2$  introduced by Berg and Nikolaev [10] is a real function on  $X^2 \times X^2$  defined by

$$(2.1) \quad \langle \overrightarrow{xy}, \overrightarrow{zw} \rangle = \frac{1}{2} (d(x, w)^2 + d(y, z)^2 - d(x, z)^2 - d(y, w)^2)$$

for all  $\overrightarrow{xy}, \overrightarrow{zw} \in X^2$ . If  $X$  is particularly a real Hilbert space, then

$$\langle \overrightarrow{xy}, \overrightarrow{zw} \rangle = \langle x - y, z - w \rangle$$

for all  $x, y, z, w \in X$ . It is clear that

- $\langle \vec{xy}, \vec{xy} \rangle = d(x, y)^2$ ;
- $\langle \vec{xy}, \vec{zw} \rangle = \langle \vec{zw}, \vec{xy} \rangle = -\langle \vec{yx}, \vec{zw} \rangle$ ;
- $\langle \vec{xp}, \vec{zw} \rangle + \langle \vec{py}, \vec{zw} \rangle = \langle \vec{xy}, \vec{zw} \rangle$ ;
- $d(x, y)^2 = d(x, z)^2 + d(z, y)^2 + 2\langle \vec{xz}, \vec{zy} \rangle$

for all  $x, y, z, w, p \in X$ .

A metric space  $X$  is said to be uniquely geodesic if for each  $x, y \in X$ , there exists a unique mapping  $\gamma$  of  $[0, l]$  into  $X$  such that  $\gamma(0) = x$ ,  $\gamma(l) = y$ , and

$$d(\gamma(s), \gamma(t)) = |s - t|$$

for all  $s, t \in [0, l]$ , where  $l = d(x, y)$ . The mapping  $\gamma$  is called a geodesic from  $x$  to  $y$  and the point  $\gamma(\alpha l)$  is denoted by  $(1 - \alpha)x \oplus \alpha y$  for all  $\alpha \in [0, 1]$ . A metric space  $X$  is said to be a CAT(0) space if it is uniquely geodesic and the following CAT(0) inequality

$$d((1 - \alpha)x \oplus \alpha y, (1 - \beta)x \oplus \beta z) \leq |(1 - \alpha)\bar{x} + \alpha\bar{y} - ((1 - \beta)\bar{x} + \beta\bar{z})|_{\mathbb{R}^2}$$

holds whenever  $x, y, z \in X$ ,  $\bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^2$ ,

$$d(x, y) = |\bar{x} - \bar{y}|_{\mathbb{R}^2}, \quad d(y, z) = |\bar{y} - \bar{z}|_{\mathbb{R}^2}, \quad d(z, x) = |\bar{z} - \bar{x}|_{\mathbb{R}^2},$$

and  $\alpha, \beta \in [0, 1]$ . It follows from [10, Corollary 3] and [9, Theorem 1.3.3 (v)] that a uniquely geodesic metric space  $X$  is a CAT(0) space if and only if the following Cauchy–Schwarz inequality

$$(2.2) \quad |\langle \vec{xy}, \vec{zw} \rangle| \leq d(x, y)d(z, w)$$

holds for all  $\vec{xy}, \vec{zw} \in X^2$ .

It is obvious that if  $X$  is a CAT(0) space, then

- $d(z, (1 - \alpha)x \oplus \alpha y) \leq (1 - \alpha)d(z, x) + \alpha d(z, y)$ ;
- $d(z, (1 - \alpha)x \oplus \alpha y)^2 \leq (1 - \alpha)d(z, x)^2 + \alpha d(z, y)^2 - \alpha(1 - \alpha)d(x, y)^2$

for all  $x, y, z \in X$  and  $\alpha \in [0, 1]$ . A complete CAT(0) space is called an Hadamard space. Among typical examples of Hadamard spaces are nonempty closed convex subsets of real Hilbert spaces, open unit balls of complex Hilbert spaces with hyperbolic metrics, and simply connected complete Riemannian manifolds with non-positive sectional curvature; see [9, 11, 12] on geodesic metric spaces and CAT(0) spaces for more details.

It is well known that if  $X$  is a CAT(0) space and  $T$  is a quasinonexpansive mapping of  $X$  into itself, then  $\mathcal{F}(T)$  is closed and convex. Hence the fixed point set of every metrically nonspreading mapping with a fixed point is closed and convex.

The concept of  $\Delta$ -convergence, first introduced by Lim [29] and later applied to the study of CAT(0) spaces by Kirk and Panyanak [23], is a generalization of weak convergence in the context of Hilbert spaces to metric spaces. The asymptotic center  $\mathcal{A}(\{x_n\})$  of a sequence  $\{x_n\}$  in a metric space  $X$  is defined by

$$\mathcal{A}(\{x_n\}) = \left\{ u \in X : \limsup_n d(u, x_n) = \inf_{y \in X} \limsup_n d(y, x_n) \right\},$$

which coincides with the whole space  $X$  if  $\{x_n\}$  is unbounded. The sequence  $\{x_n\}$  is said to be  $\Delta$ -convergent to  $p \in X$  if

$$\mathcal{A}(\{x_{n_i}\}) = \{p\}$$

for each subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$ , in which case  $p$  is said to be the  $\Delta$ -limit of  $\{x_n\}$ . We denote by  $\omega_\Delta(\{x_n\})$  the set of all subsequential  $\Delta$ -limits of  $\{x_n\}$ . It is obvious that if  $\{x_n\}$  is  $\Delta$ -convergent to  $p$ , then  $\{x_n\}$  is bounded and  $\omega_\Delta(\{x_n\}) = \{p\}$ .

The following lemmas are of fundamental importance.

**Lemma 2.1** ([15, Proposition 7]; see also [9, Section 3.1]). *The asymptotic center of every bounded sequence in an Hadamard space is a singleton.*

**Lemma 2.2** ([23, Section 3]; see also [9, Proposition 3.1.2]). *Every bounded sequence in an Hadamard space has a  $\Delta$ -convergent subsequence.*

**Lemma 2.3** ([21, Lemma 2.6]; see also [22, Proposition 3.1]). *If  $\{x_n\}$  is a bounded sequence in an Hadamard space such that  $\{d(z, x_n)\}$  is convergent for each  $z$  in  $\omega_\Delta(\{x_n\})$ , then  $\{x_n\}$  is  $\Delta$ -convergent.*

A subset  $C$  of a CAT(0) space  $X$  is said to be convex if  $(1 - \alpha)x \oplus \alpha y \in C$  whenever  $x, y \in C$  and  $\alpha \in [0, 1]$ . A function  $f$  of  $X$  into  $(-\infty, \infty]$  is said to be proper if  $f(p)$  is finite for some  $p \in X$ . It is also said to be convex if

$$f((1 - \alpha)x \oplus \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y)$$

whenever  $x, y \in X$  and  $\alpha \in (0, 1)$ . The set of all minimizers of a function  $f$  of  $X$  into  $(-\infty, \infty]$  is denoted by  $\operatorname{argmin}_X f$  or  $\operatorname{argmin}_{y \in X} f(y)$ . If  $\operatorname{argmin}_X f$  is a singleton  $\{p\}$  for some  $p \in X$ , we sometimes identify  $\operatorname{argmin}_X f$  with  $p$ .

We know the following minimization theorem in Hadamard spaces.

**Theorem 2.4** ([21, Theorem 4.1]). *Let  $\{z_n\}$  be a bounded sequence in an Hadamard space  $X$ ,  $\{\beta_n\}$  a sequence of positive real numbers such that  $\sum_{n=1}^\infty \beta_n = \infty$ , and  $g$  the real function on  $X$  defined by*

$$g(y) = \limsup_n \frac{1}{\sum_{l=1}^n \beta_l} \sum_{k=1}^n \beta_k d(y, z_k)^2$$

for all  $y \in X$ . Then  $g$  is a continuous and convex function such that  $\operatorname{argmin}_X g$  is a singleton.

We also know the following lemmas.

**Lemma 2.5** ([38, Lemma 11]). *Let  $A$  be a bounded function of  $\mathbb{N} \times \mathbb{N}$  into  $[0, \infty)$  such that  $A(n, n) = 0$  for all  $n \in \mathbb{N}$  and*

$$2A(n + 1, m + 1) \leq A(n + 1, m) + A(n, m + 1)$$

for all  $n, m \in \mathbb{N}$ . Then  $\lim_n A(n, n + 1) = 0$ .

**Lemma 2.6** ([25, Lemma 2.5]). *Let  $I$  be a nonempty closed subset of  $\mathbb{R}$ ,  $\{t_n\}$  a bounded sequence in  $I$ , and  $f$  a nondecreasing and continuous real function on  $I$ . Then*

$$f\left(\limsup_n t_n\right) = \limsup_n f(t_n).$$

## 3. EXAMPLES OF METRICALLY NONSPREADING MAPPINGS

In this section, we discuss some examples of metrically nonspreading mappings and firmly metrically nonspreading mappings in Hadamard spaces.

Using (2.1) and (2.2), we readily obtain the following. We note that (3.1) is equivalent to (1.2) when  $X$  is a nonempty subset of a real Hilbert space.

**Lemma 3.1.** *Let  $X$  be a metric space and  $T$  a mapping of  $X$  into itself. Then  $T$  is firmly metrically nonspreading if and only if*

$$(3.1) \quad d(Tx, Ty)^2 \leq \left\langle \overline{(Tx)(Ty)}, \overline{xy} \right\rangle$$

for all  $x, y \in X$ . If  $X$  is a CAT(0) space and  $T$  is firmly metrically nonspreading, then  $T$  is nonexpansive.

The metric projections and the proximity mappings in Hadamard spaces are two particularly important examples of firmly metrically nonspreading mappings.

Let  $X$  be an Hadamard space. If  $C$  is a nonempty closed convex subset of  $X$ , then the metric projection  $P_C$  of  $X$  onto  $C$  given by

$$P_C(x) = \operatorname{argmin}_{y \in C} d(y, x)$$

for all  $x \in X$  is a well-defined nonexpansive mapping of  $X$  onto  $C$  and  $\mathcal{F}(P_C) = C$ ; see [9, 11] for more details. More generally, if  $f$  is a proper lower semicontinuous convex function of  $X$  into  $(-\infty, \infty]$ , then the proximity mapping  $\operatorname{Prox}_f$  of  $f$  given by

$$(3.2) \quad \operatorname{Prox}_f(x) = \operatorname{argmin}_{y \in X} \left\{ f(y) + \frac{1}{2}d(y, x)^2 \right\}$$

for all  $x \in X$  is a well-defined nonexpansive mapping of  $X$  into itself and  $\mathcal{F}(\operatorname{Prox}_f) = \operatorname{argmin}_X f$ ; see [9, 16, 31]. It is also known [7, Proposition 3.3] that  $\operatorname{Prox}_f$  is firmly nonexpansive, i.e.,

$$d(\operatorname{Prox}_f x, \operatorname{Prox}_f y) \leq d(\alpha x \oplus (1 - \alpha) \operatorname{Prox}_f x, \alpha y \oplus (1 - \alpha) \operatorname{Prox}_f y)$$

for all  $\alpha \in [0, 1]$  and  $x, y \in X$ . The proximity mappings in Hadamard spaces were originally studied by Jost [16] and Mayer [31]; see also [8, 13, 14, 21] on minimization algorithms based on the proximity mappings in Hadamard spaces. It follows from [21, Corollary 3.2] and Lemma 6.1 that the following holds.

**Example 3.2.** *If  $X$  is an Hadamard space, then the following hold.*

- (i) *The metric projection  $P_C$  of  $X$  onto a nonempty closed convex subset  $C$  is firmly metrically nonspreading;*
- (ii) *the proximity mapping  $\operatorname{Prox}_f$  of a proper lower semicontinuous convex function  $f$  of  $X$  into  $(-\infty, \infty]$  is firmly metrically nonspreading.*

Motivated by [6, Example 2.4], we show the following result.

**Example 3.3.** *Let  $X$  be a metric space, both  $S$  and  $T$  metrically nonspreading mappings such that  $S(X)$  and  $T(X)$  are contained by a closed ball  $\overline{B}_r(a)$  for some*

$a \in X$  and  $r > 0$ ,  $\delta$  a positive real number satisfying  $\delta \geq (1 + 2\sqrt{2})r$ , and  $U$  the mapping of  $X$  into itself defined by

$$Ux = \begin{cases} Sx & (x \in \overline{B}_\delta(a)); \\ Tx & (\text{otherwise}). \end{cases}$$

Then  $U$  is metrically nonspreading.

*Proof.* Let  $x, y \in X$  be given. If either  $x, y \in \overline{B}_\delta(a)$  or  $x, y \in X \setminus \overline{B}_\delta(a)$ , then we have

$$2d(Ux, Uy)^2 \leq d(Ux, y)^2 + d(Uy, x)^2$$

since both  $S$  and  $T$  are metrically nonspreading. If  $x \in \overline{B}_\delta(a)$  and  $y \in X \setminus \overline{B}_\delta(a)$ , then we have

$$\begin{aligned} & d(Ux, y)^2 + d(Uy, x)^2 \\ & \geq d(Ux, y)^2 = d(Sx, y)^2 \geq (d(y, a) - d(Sx, a))^2 > (\delta - r)^2 \geq 8r^2 \end{aligned}$$

and

$$8r^2 \geq 2(d(Sx, a) + d(a, Ty))^2 \geq 2d(Sx, Ty)^2 = 2d(Ux, Uy)^2.$$

Hence we have

$$(3.3) \quad d(Ux, y)^2 + d(Uy, x)^2 > 2d(Ux, Uy)^2.$$

If  $x \in X \setminus \overline{B}_\delta(a)$  and  $y \in \overline{B}_\delta(a)$ , then we also obtain (3.3). Therefore, the mapping  $U$  is metrically nonspreading.  $\square$

*Remark 3.4.* It follows from Lemma 3.1 and Example 3.3 that there exists a metrically nonspreading mapping which is not firmly metrically nonspreading. In fact, let  $X$  be an unbounded Hadamard space. Then we have  $a \in X$  and  $\delta > 0$  such that  $X \setminus \overline{B}_\delta(a)$  is nonempty. Let  $r$  be the same as in Example 3.3, both  $S$  and  $T$  the metric projections of  $X$  onto  $\{a\}$  and the closed ball  $\overline{B}_r(a)$ , respectively, and  $U$  the mapping defined as in Example 3.3. Then it follows from Example 3.3 that  $U$  is metrically nonspreading. On the other hand,  $U$  is discontinuous at any  $x \in X$  with  $d(x, a) = \delta$ . Lemma 3.1 implies that  $U$  is not firmly metrically nonspreading.

#### 4. FIXED POINTS OF METRICALLY NONSPREADING MAPPINGS

In this section, we study some fundamental properties of metrically nonspreading mappings in Hadamard spaces.

**Lemma 4.1.** *Let  $X$  be a metric space,  $T$  a metrically nonspreading mapping of  $X$  into itself, and  $\{x_n\}$  a sequence in  $X$  such that  $\mathcal{A}(\{x_n\}) = \{p\}$  for some  $p \in X$ . If*

- $\limsup_n d(x_n, p) = \limsup_n d(Tx_n, p)$ ;
- $\limsup_n d(x_n, Tp) = \limsup_n d(Tx_n, Tp)$ ,

*then  $p$  is an element of  $\mathcal{F}(T)$ .*

*Proof.* Since  $\mathcal{A}(\{x_n\}) = \{p\}$ , the sequence  $\{x_n\}$  is bounded. In fact, if  $X$  is a singleton, then  $\{x_n\}$  is obviously bounded. In the other case, we have  $q \in X$  which is distinct from  $p$  and hence

$$\limsup_n d(x_n, p) < \limsup_n d(x_n, q).$$

This implies the boundedness of  $\{x_n\}$ .

On the other hand, since  $T$  is metrically nonspreading, we have

$$2d(Tx_n, Tp)^2 \leq d(Tx_n, p)^2 + d(Tp, x_n)^2.$$

Taking the upper limit gives us that

$$2 \limsup_n d(Tx_n, Tp)^2 \leq \limsup_n d(Tx_n, p)^2 + \limsup_n d(Tp, x_n)^2.$$

By assumptions, we have

$$(4.1) \quad 2 \limsup_n d(x_n, Tp)^2 \leq \limsup_n d(x_n, p)^2 + \limsup_n d(x_n, Tp)^2.$$

Since  $\{d(x_n, Tp)^2\}$  is a bounded sequence in  $[0, \infty)$ , it follows from (4.1) and Lemma 2.6 that

$$\begin{aligned} \left( \limsup_n d(x_n, Tp) \right)^2 &= \limsup_n d(x_n, Tp)^2 \\ &\leq \limsup_n d(x_n, p)^2 = \left( \limsup_n d(x_n, p) \right)^2 \end{aligned}$$

and hence

$$\limsup_n d(x_n, Tp) \leq \limsup_n d(x_n, p).$$

It then follows from  $\mathcal{A}(\{x_n\}) = \{p\}$  that  $Tp = p$ .  $\square$

Using Lemma 4.1, we first obtain the following fixed point theorem for metrically nonspreading mappings in Hadamard spaces. This result also follows from the result [32, Lemma 4.7]. We note that the proof of [32, Lemma 4.7] is valid to the case where  $1 - 2\alpha \geq 0$ .

**Theorem 4.2.** *Let  $X$  be an Hadamard space and  $T$  a metrically nonspreading mapping of  $X$  into itself. Then  $\mathcal{F}(T)$  is nonempty if and only if  $\{T^n x\}$  is bounded for some  $x \in X$ .*

*Proof.* Since the only if part is obvious, it is sufficient to prove the if part. Suppose that  $\{T^n x\}$  is bounded for some  $x \in X$  and let  $\{x_n\}$  be the sequence in  $X$  defined by  $x_n = T^n x$  for all  $n \in \mathbb{N}$ . It then follows from Lemma 2.1 that  $\mathcal{A}(\{x_n\}) = \{p\}$  for some  $p \in X$ . By the definition of  $\{x_n\}$ , we have

$$\limsup_n d(x_n, y) = \limsup_n d(Tx_n, y)$$

for all  $y \in X$ . Thus it follows from Lemma 4.1 that  $p$  is an element of  $\mathcal{F}(T)$ .  $\square$

As a direct consequence of Theorem 4.2, we obtain the following corollary.

**Corollary 4.3.** *Every metrically nonspreading mapping from a bounded Hadamard space into itself has a fixed point.*

We can also show the following common fixed point theorem.

**Theorem 4.4.** *Let  $X$  be a bounded Hadamard space and  $\{T_k\}_{k=1}^m$  a commutative finite family of metrically nonspreading mappings of  $X$  into itself. Then  $\bigcap_{k=1}^m \mathcal{F}(T_k)$  is nonempty.*



*Proof.* The proof is given by induction on  $m$ . Theorem 4.2 implies that the conclusion holds if  $m = 1$ . Suppose that the conclusion holds for some  $m = l \in \mathbb{N}$  and let  $\{T_k\}_{k=1}^{l+1}$  be a commutative family of metrically nonspreading mappings of  $X$  into itself. Then the set  $Y$  given by  $Y = \bigcap_{k=1}^l \mathcal{F}(T_k)$  is a nonempty closed convex subset of  $X$ . Hence  $Y$  is an Hadamard space. We next show that  $T_{l+1}Y$  is contained by  $Y$ . In fact, if  $v \in Y$  and  $k \in \{1, 2, \dots, l\}$ , then it follows from  $T_kv = v$  and  $T_{l+1}T_k = T_kT_{l+1}$  that

$$T_{l+1}v = T_{l+1}T_kv = T_kT_{l+1}v$$

and hence  $T_{l+1}v \in \mathcal{F}(T_k)$ . Thus  $T_{l+1}Y$  is a subset of  $Y$ . Accordingly, the restriction of  $T_{l+1}$  to the Hadamard space  $Y$  is a metrically nonspreading self mapping on  $Y$ . Then Theorem 4.2 ensures that there exists  $u \in Y$  such that  $T_{l+1}u = u$ , and hence  $u \in \bigcap_{k=1}^{l+1} \mathcal{F}(T_k)$ . Therefore, the set  $\bigcap_{k=1}^{l+1} \mathcal{F}(T_k)$  is nonempty.  $\square$

Using Lemma 4.1, we next obtain the following demiclosed principle for metrically nonspreading mappings.

**Theorem 4.5.** *Let  $X$  be a metric space,  $T$  a metrically nonspreading mapping of  $X$  into itself,  $\{x_n\}$  a sequence in  $X$  such that  $\mathcal{A}(\{x_n\}) = \{p\}$  for some  $p \in X$  and  $d(Tx_n, x_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Then  $p$  is an element of  $\mathcal{F}(T)$ .*

*Proof.* Since  $d(Tx_n, x_n) \rightarrow 0$  as  $n \rightarrow \infty$ , we have

$$\limsup_n d(x_n, y) = \limsup_n d(Tx_n, y)$$

for all  $y \in X$ . Thus it follows from Lemma 4.1 that  $p$  is an element of  $\mathcal{F}(T)$ .  $\square$

Using Lemma 2.5, we next show the asymptotic regularity of metrically nonspreading mappings.

**Lemma 4.6.** *Let  $X$  be a metric space and  $T$  a metrically nonspreading mapping of  $X$  into itself such that  $\mathcal{F}(T)$  is nonempty. Then  $T$  is asymptotically regular.*

*Proof.* Let  $x \in X$  be given. Since  $\mathcal{F}(T)$  is nonempty and  $T$  is quasinonexpansive,  $\{T^n x\}$  is bounded. Let  $A$  be the bounded function of  $\mathbb{N} \times \mathbb{N}$  into  $[0, \infty)$  defined by

$$A(n, m) = d(T^n x, T^m x)^2$$

for all  $n, m \in \mathbb{N}$ . It is clear that  $A(n, n) = 0$  for all  $n \in \mathbb{N}$ . Since  $T$  is metrically nonspreading, we have

$$2d(T^{n+1}x, T^{m+1}x)^2 \leq d(T^{n+1}x, T^m x)^2 + d(T^{m+1}x, T^n x)^2$$

and hence

$$2A(n + 1, m + 1) \leq A(n + 1, m) + A(n, m + 1)$$

for all  $n, m \in \mathbb{N}$ . It then follows from Lemma 2.5 that

$$d(T^n x, T^{n+1}x) = \sqrt{A(n, n + 1)} \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore, the mapping  $T$  is asymptotically regular.  $\square$

We can directly show the asymptotic regularity for firmly metrically nonspreading mappings as follows.

**Lemma 4.7.** *Let  $X$  be a metric space and  $T$  a firmly metrically nonspreading mapping of  $X$  into itself such that  $\mathcal{F}(T)$  is nonempty. Then  $T$  is asymptotically regular.*

*Proof.* Let  $x \in X$  be given. By assumption, there exists  $u \in \mathcal{F}(T)$ . Since  $T$  is firmly metrically nonspreading, it follows from (1.3) that

$$d(u, T^{n+1}x)^2 \leq d(u, T^{n+1}x)^2 + d(T^{n+1}x, T^n x)^2 \leq d(u, T^n x)^2.$$

This implies that  $\{d(u, T^n x)^2\}$  is convergent and hence

$$0 \leq d(T^{n+1}x, T^n x)^2 \leq d(u, T^n x)^2 - d(u, T^{n+1}x)^2 \rightarrow 0$$

as  $n \rightarrow \infty$ . Consequently, the mapping  $T$  is asymptotically regular.  $\square$

**Theorem 4.8.** *Let  $X$  be an Hadamard space and  $T$  a metrically nonspreading mapping of  $X$  into itself such that  $\mathcal{F}(T)$  is nonempty. Then  $\{T^n x\}$  is  $\Delta$ -convergent to an element of  $\mathcal{F}(T)$  for all  $x \in X$ .*

*Proof.* Let  $z$  be an element of  $\omega_\Delta(\{T^n x\})$ . Then we have a subsequence  $\{T^{n_i} x\}$  of  $\{T^n x\}$  which is  $\Delta$ -convergent to  $z$ . In particular, we have

$$\mathcal{A}(\{T^{n_i} x\}) = \{z\}.$$

Since  $\mathcal{F}(T)$  is nonempty, it follows from Lemma 4.6 that

$$d(T(T^{n_i} x), T^{n_i} x) = d(T^{n_i+1} x, T^{n_i} x) \rightarrow 0$$

as  $i \rightarrow \infty$ . Lemma 4.5 then ensures that  $z$  is an element of  $\mathcal{F}(T)$ . It also follows from  $d(z, T^{n+1}x) \leq d(z, T^n x)$  that  $\{d(z, T^n x)\}$  is convergent. Thus the sequence  $\{d(z, T^n x)\}$  is convergent for each  $z$  in  $\omega_\Delta(\{T^n x\})$ . By Lemma 2.3, we conclude that  $\{T^n x\}$  is  $\Delta$ -convergent to some  $u \in X$ . Since

$$\{u\} = \omega_\Delta(\{T^n x\}) \subset \mathcal{F}(T),$$

we conclude that  $u$  is an element of  $\mathcal{F}(T)$ .  $\square$

## 5. ASYMPTOTIC BEHAVIOR OF THE MANN ITERATION

In this section, we study the asymptotic behavior of sequences generated by the Mann iteration [30] for metrically nonspreading mappings in Hadamard spaces.

Motivated by [24, Lemma 3.1], we first show the following equivalence.

**Lemma 5.1.** *Let  $X$  be a metric space and  $T$  a mapping of  $X$  into itself. Then  $T$  is metrically nonspreading if and only if*

$$0 \leq d(Ty, y)^2 + 2 \left\langle \overrightarrow{(Tx)(Ty)}, \overrightarrow{(Ty)y} \right\rangle + d(Ty, x)^2 - d(Tx, Ty)^2$$

for all  $x, y \in X$ .

*Proof.* Let  $x, y \in X$  be given. Then we have

$$\begin{aligned} & d(Tx, y)^2 + d(Ty, x)^2 - 2d(Tx, Ty)^2 \\ &= d(Tx, Ty)^2 + d(Ty, y)^2 + 2 \left\langle \overrightarrow{(Tx)(Ty)}, \overrightarrow{(Ty)y} \right\rangle + d(Ty, x)^2 \\ &\quad - 2d(Tx, Ty)^2 \end{aligned}$$

$$= d(Ty, y)^2 + 2 \left\langle \overrightarrow{(Tx)(Ty)}, \overrightarrow{(Ty)y} \right\rangle + d(Ty, x)^2 - d(Tx, Ty)^2$$

and hence the result follows. □

Motivated by [24, Lemma 3.2], we next show that every metrically nonspreading mapping is bounded on every bounded subset.

**Lemma 5.2.** *Let  $X$  be a metric space and  $T$  a metrically nonspreading mapping of  $X$  into itself. Then  $T(U)$  is bounded for each nonempty bounded subset  $U$  of  $X$ .*

*Proof.* If the conclusion does not hold, then there exists a bounded sequence  $\{x_n\}$  such that  $\{Tx_n\}$  is unbounded. Fix  $p \in X$ . Since

$$d(Tx_k, Tx_l) \leq d(Tx_k, p) + d(p, Tx_l) \leq 2 \sup_n d(Tx_n, p)$$

for all  $k, l \in \mathbb{N}$  and  $\{Tx_n\}$  is unbounded, we then have  $\sup_n d(Tx_n, p) = \infty$ . This implies that there exists a subsequence  $\{Tx_{n_i}\}$  of  $\{Tx_n\}$  such that  $\{d(Tx_{n_i}, p)\}$  is divergent to  $\infty$  as  $i \rightarrow \infty$ . This gives us that

$$\lim_{i \rightarrow \infty} d(Tx_{n_i}, z) = \infty$$

for all  $z \in X$ . Let  $y \in X$  be given. It follows from Lemma 5.1 that

$$\begin{aligned} & d(Tx_n, Ty)^2 \\ & \leq d(Ty, y)^2 + 2 \left\langle \overrightarrow{(Tx_n)(Ty)}, \overrightarrow{(Ty)y} \right\rangle + d(Ty, x_n)^2 \\ & = d(Ty, y)^2 + (d(Tx_n, y)^2 - d(Tx_n, Ty)^2 - d(Ty, y)^2) + d(Ty, x_n)^2 \\ & \leq (d(Tx_n, y) + d(Tx_n, Ty))d(y, Ty) + d(Ty, x_n)^2 \\ & \leq (2d(Tx_n, Ty) + d(Ty, y))d(y, Ty) + d(Ty, x_n)^2. \end{aligned}$$

Letting  $i \rightarrow \infty$  in

$$d(Tx_{n_i}, Ty) \leq \left( 2 + \frac{d(Ty, y)}{d(Tx_{n_i}, Ty)} \right) d(y, Ty) + \frac{d(x_{n_i}, Ty)^2}{d(Tx_{n_i}, Ty)}$$

gives us a contradiction. Thus the set  $T(U)$  is bounded. □

We finally show the following result on the Mann iteration for metrically nonspreading mappings.

**Theorem 5.3.** *Let  $X$  be an Hadamard space,  $T$  a metrically nonspreading mapping of  $X$  into itself,  $\{\alpha_n\}$  a sequence in  $(0, 1]$  such that  $\sum_{n=1}^\infty \alpha_n = \infty$ , and  $\{x_n\}$  a sequence defined by  $x_1 \in X$  and*

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_nTx_n$$

for all  $n \in \mathbb{N}$ . Then the following hold.

- (i)  $\mathcal{F}(T)$  is nonempty if and only if  $\{x_n\}$  is bounded;
- (ii) if  $\mathcal{F}(T)$  is nonempty and  $\inf_n \alpha_n(1 - \alpha_n) > 0$ , then  $\{x_n\}$  is  $\Delta$ -convergent to an element of  $\mathcal{F}(T)$ .

*Proof.* We first show the only if part of (i). Suppose that  $\mathcal{F}(T)$  is nonempty and fix  $u \in \mathcal{F}(T)$ . Since  $T$  is quasinonexpansive, we have

$$\begin{aligned} d(u, x_{n+1}) &= d(u, (1 - \alpha_n)x_n \oplus \alpha_n Tx_n) \\ &\leq (1 - \alpha_n)d(u, x_n) + \alpha_n d(u, Tx_n) \leq d(u, x_n) \end{aligned}$$

and hence  $\{d(u, x_n)\}$  is convergent. This implies that  $\{x_n\}$  is bounded.

We next show the if part of (i). Suppose that  $\{x_n\}$  is bounded and set

$$\omega_n = \sum_{l=1}^n \alpha_l$$

for all  $n \in \mathbb{N}$ . Then it follows from Lemma 5.2 that  $\{Tx_n\}$  is bounded. Let  $g$  be the real function on  $X$  defined by

$$g(y) = \limsup_n \frac{1}{\omega_n} \sum_{k=1}^n \alpha_k d(y, Tx_k)^2$$

for all  $y \in X$ . It then follows from Theorem 2.4 that  $g$  has a unique minimizer  $p \in X$ . By the definition of  $\{x_n\}$ , we have

$$(5.1) \quad \begin{aligned} d(Tp, x_{k+1})^2 &= d(Tp, (1 - \alpha_k)x_k \oplus \alpha_k Tx_k)^2 \\ &\leq (1 - \alpha_k)d(Tp, x_k)^2 + \alpha_k d(Tp, Tx_k)^2 \end{aligned}$$

Since  $T$  is metrically nonspreading, we have

$$(5.2) \quad 2d(Tp, Tx_k)^2 \leq d(Tp, x_k)^2 + d(Tx_k, p)^2.$$

Using (5.1) and (5.2), we have

$$\begin{aligned} &\alpha_k d(Tp, Tx_k)^2 \\ &\leq \alpha_k d(Tx_k, p)^2 + \alpha_k (d(Tp, x_k)^2 - d(Tp, Tx_k)^2) \\ &= \alpha_k d(Tx_k, p)^2 + d(Tp, x_k)^2 - ((1 - \alpha_k)d(Tp, x_k)^2 + \alpha_k d(Tp, Tx_k)^2) \\ &\leq \alpha_k d(Tx_k, p)^2 + d(Tp, x_k)^2 - d(Tp, x_{k+1})^2. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \sum_{k=1}^n \alpha_k d(Tp, Tx_k)^2 &\leq \sum_{k=1}^n \alpha_k d(p, Tx_k)^2 + (d(Tp, x_1)^2 - d(Tp, x_{n+1})^2) \\ &\leq \sum_{k=1}^n \alpha_k d(p, Tx_k)^2 + d(Tp, x_1)^2 \end{aligned}$$

and hence

$$\frac{1}{\omega_n} \sum_{k=1}^n \alpha_k d(Tp, Tx_k)^2 \leq \frac{1}{\omega_n} \sum_{k=1}^n \alpha_k d(p, Tx_k)^2 + \frac{1}{\omega_n} d(Tp, x_1)^2.$$

Since  $\omega_n \rightarrow \infty$  as  $n \rightarrow \infty$ , we obtain  $g(Tp) \leq g(p)$ . Since  $p$  is the unique minimizer of  $g$ , we conclude that  $Tp = p$ .

We finally show (ii). Suppose that  $\mathcal{F}(T)$  is nonempty and  $\inf_n \alpha_n(1 - \alpha_n) > 0$ . Fix  $v \in \mathcal{F}(T)$ . Since  $X$  is a CAT(0) space, we have

$$\begin{aligned} d(v, x_{n+1})^2 &= d(v, (1 - \alpha_n)x_n \oplus \alpha_nTx_n)^2 \\ &\leq (1 - \alpha_n)d(v, x_n)^2 + \alpha_nd(v, Tx_n)^2 - \alpha_n(1 - \alpha_n)d(x_n, Tx_n)^2 \\ &\leq d(v, x_n)^2 - \alpha_n(1 - \alpha_n)d(x_n, Tx_n)^2. \end{aligned}$$

This gives us that  $\{d(v, x_n)^2\}$  is convergent and hence

$$d(x_n, Tx_n)^2 \leq \frac{1}{\inf_m \alpha_m(1 - \alpha_m)} (d(v, x_n)^2 - d(v, x_{n+1})^2) \rightarrow 0$$

as  $n \rightarrow \infty$ . Consequently, we obtain  $d(x_n, Tx_n) \rightarrow 0$  as  $n \rightarrow \infty$ . If  $z$  is an element of  $\omega_\Delta(\{x_n\})$ , then there exists a subsequence  $\{x_{n_i}\}$  of  $\{x_n\}$  which is  $\Delta$ -convergent to  $z$ . Since

$$\lim_i d(x_{n_i}, Tx_{n_i}) = 0,$$

Lemma 4.5 ensures that  $z$  is an element of  $\mathcal{F}(T)$ . Hence  $\omega_\Delta(\{x_n\})$  is a subset of  $\mathcal{F}(T)$ . Thus, the sequence  $\{d(z, x_n)\}$  is convergent for each  $z$  in  $\omega_\Delta(\{x_n\})$ . Then, Lemma 2.3 implies that  $\{x_n\}$  is  $\Delta$ -convergent to some  $x_\infty \in X$ . Since

$$\{x_\infty\} = \omega_\Delta(\{x_n\}) \subset \mathcal{F}(T),$$

we conclude that  $x_\infty$  is an element of  $\mathcal{F}(T)$ . □

### 6. APPLICATIONS TO MONOTONE OPERATORS IN HADAMARD SPACES

In this section, we obtain two corollaries of our results for the problem of finding zero points of monotone operators in Hadamard spaces.

Before obtaining them, we first summarize the concepts of dual spaces and monotone operators in CAT(0) spaces. These concepts were introduced by Ahmadi Kakavandi and Amini [2]; see also Ahmadi Kakavandi [1] on related results.

Let  $X$  be a CAT(0) space and  $\hat{L}(X)$  the real linear space of all Lipschitz continuous real functions on  $X$ . We denote by  $\|\cdot\|$  the Lipschitz seminorm on  $\hat{L}(X)$  defined by

$$\|f\| = \sup \left\{ \frac{|f(p) - f(q)|}{d(p, q)} : p, q \in X, p \neq q \right\}$$

for all  $f \in \hat{L}(X)$ . In other words,

$$\|f\| = \min \{ \lambda \in [0, \infty) : |f(p) - f(q)| \leq \lambda d(p, q) \quad (\forall p, q \in X) \}$$

for all  $f \in \hat{L}(X)$ . Then the following conditions

$$\|f\| \geq 0; \quad \|\alpha f\| = |\alpha| \|f\|; \quad \|f + g\| \leq \|f\| + \|g\|$$

hold for all  $f, g \in \hat{L}(X)$  and  $\alpha \in \mathbb{R}$ .

We can define an equivalence relation  $\sim$  on  $\hat{L}(X)$  by  $f \sim g$  if  $\|f - g\| = 0$ . It is clear that  $f \sim g$  if and only if  $f - g$  is a constant function. We denote by  $[f]$  the equivalence class of  $f \in \hat{L}(X)$  and let  $L(X)$  be the space defined by

$$L(X) = \{[f] : f \in \hat{L}(X)\}.$$

The space  $L(X)$  is a real Banach space under the addition, the scalar multiplication, and the norm given by

$$[f] + [g] = [f + g], \quad \alpha[f] = [\alpha f], \quad \|[f]\| = \|f\|$$

for all  $[f], [g] \in L(X)$  and  $\alpha \in \mathbb{R}$ ; see [33, Proposition 2.4.1] for the proof of the metric completeness of  $L(X)$ .

We denote by  $\alpha\vec{xy}$  and  $\vec{xy}$  the elements  $(\alpha, \vec{xy})$  and  $(1, \vec{xy})$  in  $\mathbb{R} \times X^2$ , respectively. Then we define the mapping  $\Phi$  of  $\mathbb{R} \times X^2$  into  $\hat{L}(X)$  by

$$\Phi(\alpha\vec{xy})(p) = \alpha \langle \vec{xy}, \vec{xp} \rangle$$

for all  $\alpha\vec{xy} \in \mathbb{R} \times X^2$  and  $p \in X$ . It is easy to see that

$$\|\Phi(\alpha\vec{xy})\| = |\alpha| d(x, y).$$

We also define a real function  $\hat{D}$  by

$$(6.1) \quad \hat{D}(\alpha\vec{xy}, \beta\vec{z\bar{w}}) = \|\Phi(\alpha\vec{xy}) - \Phi(\beta\vec{z\bar{w}})\|$$

for all  $\alpha\vec{xy}, \beta\vec{z\bar{w}} \in \mathbb{R} \times X^2$ , which is a pseudometric on  $\mathbb{R} \times X^2$ , that is,

- $\hat{D}(\alpha\vec{xy}, \beta\vec{z\bar{w}}) \geq 0$  and  $\hat{D}(\alpha\vec{xy}, \alpha\vec{xy}) = 0$ ;
- $\hat{D}(\alpha\vec{xy}, \beta\vec{z\bar{w}}) = \hat{D}(\beta\vec{z\bar{w}}, \alpha\vec{xy})$ ;
- $\hat{D}(\alpha\vec{xy}, \beta\vec{z\bar{w}}) \leq \hat{D}(\alpha\vec{xy}, \gamma\vec{u\bar{v}}) + \hat{D}(\gamma\vec{u\bar{v}}, \beta\vec{z\bar{w}})$

hold for all  $\alpha\vec{xy}, \beta\vec{z\bar{w}}, \gamma\vec{u\bar{v}} \in \mathbb{R} \times X^2$ . We can define an equivalence relation  $\sim$  on  $\mathbb{R} \times X^2$  by

$$\alpha\vec{xy} \sim \beta\vec{z\bar{w}} \iff \hat{D}(\alpha\vec{xy}, \beta\vec{z\bar{w}}) = 0.$$

We then have the following equivalence; see also [2, Lemma 2.1].

$$(6.2) \quad \begin{aligned} & \alpha\vec{xy} \sim \beta\vec{z\bar{w}} \\ & \iff \Phi(\alpha\vec{xy}) \sim \Phi(\beta\vec{z\bar{w}}) \\ & \iff \Phi(\alpha\vec{xy})(p) - \Phi(\beta\vec{z\bar{w}})(p) = \Phi(\alpha\vec{xy})(q) - \Phi(\beta\vec{z\bar{w}})(q) \quad (\forall p, q \in X) \\ & \iff \alpha \langle \vec{xy}, \vec{xp} \rangle - \alpha \langle \vec{xy}, \vec{xq} \rangle = \beta \langle \vec{z\bar{w}}, \vec{z\bar{p}} \rangle - \beta \langle \vec{z\bar{w}}, \vec{z\bar{q}} \rangle \quad (\forall p, q \in X) \\ & \iff \alpha \langle \vec{xy}, \vec{p\bar{q}} \rangle = \beta \langle \vec{z\bar{w}}, \vec{p\bar{q}} \rangle \quad (\forall \vec{p\bar{q}} \in X^2). \end{aligned}$$

The dual space  $X^*$  of  $X$  in the sense of Ahmadi Kakavandi and Amini [2] is the metric space given by

$$X^* = \{[\alpha\vec{xy}] : \alpha\vec{xy} \in \mathbb{R} \times X^2\}$$

with the metric  $D$  defined by

$$D([\alpha\vec{xy}], [\beta\vec{z\bar{w}}]) = \hat{D}(\alpha\vec{xy}, \beta\vec{z\bar{w}})$$

$[\alpha\vec{xy}], [\beta\vec{z\bar{w}}] \in X^*$ , where  $[\alpha\vec{xy}]$  denotes the equivalence class of  $\alpha\vec{xy} \in \mathbb{R} \times X^2$ . The origin  $0$  of  $X^*$  is given by

$$0 = [\vec{a\bar{a}}],$$

where  $a$  is a fixed element of  $X$ . It is obvious that

$$0 = \{0\vec{xy} : x, y \in X\} \cup \{\alpha\vec{x\bar{x}} : \alpha \in \mathbb{R}, x \in X\}.$$

It is known [2, pp. 3451–3452] that if  $X$  is a closed convex subset of a real Hilbert space  $H$  with a nonempty interior, then  $X$  is isometric to  $X^*$ . In particular, if  $X = H$ , then the isometric bijection  $\tau$  of  $H$  onto  $H^*$  is given by

$$\tau(x) = [(1, 0, x)]$$

for all  $x \in H$ . For each  $u^* = [\alpha \overrightarrow{xy}] \in X^*$ , we define

$$\langle u^*, \overrightarrow{pq} \rangle = \alpha \langle \overrightarrow{xy}, \overrightarrow{pq} \rangle$$

for all  $\overrightarrow{pq} \in X^2$ . It follows from (6.2) that this is independent of the choice of  $\alpha \overrightarrow{xy}$ .

We next recall the concept of monotone operators in CAT(0) spaces. Let  $X$  be a CAT(0) space and  $X^*$  the dual space of  $X$ . Then an operator  $A: X \rightarrow 2^{X^*}$  is said to be monotone if

$$\langle u^*, \overrightarrow{vu} \rangle - \langle v^*, \overrightarrow{vu} \rangle \geq 0$$

whenever  $u^* \in Au$  and  $v^* \in Av$ . A monotone operator  $A: X \rightarrow 2^{X^*}$  is said to satisfy a range condition if for each  $x \in X$ , there exists  $z \in X$  such that

$$[\overrightarrow{zx}] \in Az.$$

If  $A: X \rightarrow 2^{X^*}$  is a monotone operator satisfying a range condition, then the resolvent  $J_A$  of  $A$  defined by

$$J_A(x) = \{z \in X : [\overrightarrow{zx}] \in Az\}$$

for all  $x \in X$  is a single-valued mapping of  $X$  into itself. The zero point set  $A^{-1}(0)$  is defined by

$$A^{-1}(0) = \{u \in X : 0 \in Au\}.$$

For a proper lower semicontinuous convex function  $f$  of a CAT(0) space  $X$  into  $(-\infty, \infty]$ , the subdifferential mapping  $\partial f: X \rightarrow 2^{X^*}$  of  $f$  in the sense of Ahmadi Kakavandi and Amini [2, Definition 4.1] is defined by

$$\partial f(x) = \{u^* \in X^* : f(x) + \langle u^*, \overrightarrow{xy} \rangle \leq f(y) \quad (\forall y \in X)\}$$

for all  $x \in X$ . It is known [2, Theorem 4.2] that if  $X$  is an Hadamard space, then  $\partial f: X \rightarrow 2^{X^*}$  is a monotone operator satisfying a range condition and

$$(\partial f)^{-1}(0) = \{u \in X : f(u) = \inf f(X)\}.$$

It is also known [20, Proposition 5.3] that the resolvent  $J_{\partial f}$  of  $\partial f$  coincides with the proximity mapping  $\text{Prox}_f$  of  $f$  defined by (3.2).

We know the following fundamental result.

**Lemma 6.1** ([20, Theorem 3.9]). *Let  $X$  be a CAT(0) space,  $A: X \rightarrow 2^{X^*}$  a monotone operator satisfying a range condition, and  $J_A$  the resolvent of  $A$ . Then  $J_A$  is a firmly metrically nonspreading mapping of  $X$  into itself such that  $\mathcal{F}(J_A) = A^{-1}(0)$ .*

For the sake of completeness, we give the proof.

*Proof.* Put  $T = J_A$ . The definition of  $T$  gives us that

$$u = Tu \iff 0 = [\overrightarrow{u\bar{u}}] \in Au$$

and hence  $\mathcal{F}(T) = A^{-1}(0)$ .

We next show that  $T$  is firmly metrically nonspreading. Let  $x, y \in X$  be given. By the definition of  $T$ , we have

$$\left[ \overrightarrow{(Tx)x} \right] \in A(Tx) \quad \text{and} \quad \left[ \overrightarrow{(Ty)y} \right] \in A(Ty).$$

The monotonicity of  $A$  implies that

$$\left\langle \left[ \overrightarrow{(Tx)x} \right], \overrightarrow{(Ty)(Tx)} \right\rangle - \left\langle \left[ \overrightarrow{(Ty)y} \right], \overrightarrow{(Ty)(Tx)} \right\rangle \geq 0$$

and hence

$$\left\langle \overrightarrow{(Tx)x}, \overrightarrow{(Ty)(Tx)} \right\rangle - \left\langle \overrightarrow{(Ty)y}, \overrightarrow{(Ty)(Tx)} \right\rangle \geq 0.$$

By the definition of quasilinearization, we have

$$d(x, Ty)^2 - d(Tx, Ty)^2 - d(x, Tx) - (d(Ty, Tx)^2 + d(y, Ty)^2 - d(y, Tx)^2) \geq 0$$

and hence

$$d(Tx, y)^2 + d(Ty, x)^2 - d(Tx, x)^2 - d(Ty, y)^2 \geq 2d(Tx, Ty)^2.$$

Therefore, the mapping  $T$  is firmly metrically nonspreading.  $\square$

Using Theorems 4.2, 4.8, and Lemma 6.1, we obtain the following corollary. The part (ii) also follows from a more general result in [20, Theorem 4.3].

**Corollary 6.2.** *Let  $X$  be an Hadamard space,  $A: X \rightarrow 2^{X^*}$  a monotone operator satisfying a range condition, and  $J_A$  the resolvent of  $A$ . Then the following hold.*

- (i) *The set  $A^{-1}(0)$  is nonempty if and only if  $\{(J_A)^n x\}$  is bounded for some  $x \in X$ ;*
- (ii) *if  $A^{-1}(0)$  is nonempty, then  $\{(J_A)^n x\}$  is  $\Delta$ -convergent to an element of  $A^{-1}(0)$  for all  $x \in X$ .*

Using Theorem 5.3 and Lemma 6.1, we obtain the following corollary.

**Corollary 6.3.** *Let  $X$  be an Hadamard space,  $A: X \rightarrow 2^{X^*}$  a monotone operator satisfying a range condition,  $J_A$  the resolvent of  $A$ ,  $\{\alpha_n\}$  a sequence in  $(0, 1]$  such that  $\sum_{n=1}^{\infty} \alpha_n = \infty$ , and  $\{x_n\}$  a sequence defined by  $x_1 \in X$  and*

$$x_{n+1} = (1 - \alpha_n)x_n \oplus \alpha_n J_A x_n$$

for all  $n \in \mathbb{N}$ . Then the following conditions hold.

- (i) *The set  $A^{-1}(0)$  is nonempty if and only if  $\{x_n\}$  is bounded;*
- (ii) *if  $A^{-1}(0)$  is nonempty and  $\inf_n \alpha_n(1 - \alpha_n) > 0$ , then  $\{x_n\}$  is  $\Delta$ -convergent to an element of  $A^{-1}(0)$ .*

#### ACKNOWLEDGMENT

This work was supported by JSPS KAKENHI Grant Number 17K05372.



## REFERENCES

- [1] B. Ahmadi Kakavandi, *Weak topologies in complete CAT(0) metric spaces*, Proc. Amer. Math. Soc. **141** (2013), 1029–1039.
- [2] B. Ahmadi Kakavandi and M. Amini, *Duality and subdifferential for convex functions on complete CAT(0) metric spaces*, Nonlinear Anal. **73** (2010), 3450–3455.
- [3] Y. I. Alber, *Metric and generalized projection operators in Banach spaces: properties and applications*, Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math., vol. 178, Dekker, New York, 1996, pp. 15–50.
- [4] Y. I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operator equations in Banach spaces*, Panamer. Math. J. **4** (1994), 39–54.
- [5] K. Aoyama, Y. Kimura, and W. Takahashi, *Maximal monotone operators and maximal monotone functions for equilibrium problems*, J. Convex Anal. **15** (2008), 395–409.
- [6] K. Aoyama and F. Kohsaka, *Fixed point theorem for  $\alpha$ -nonexpansive mappings in Banach spaces*, Nonlinear Anal. **74** (2011), 4387–4391.
- [7] D. Ariza-Ruiz, L. Leuştean, and G. López-Acedo, *Firmly nonexpansive mappings in classes of geodesic spaces*, Trans. Amer. Math. Soc. **366** (2014), 4299–4322.
- [8] M. Bačák, *The proximal point algorithm in metric spaces*, Israel J. Math. **194** (2013), 689–701.
- [9] ———, *Convex analysis and optimization in Hadamard spaces*, De Gruyter, Berlin, 2014.
- [10] I. D. Berg and I. G. Nikolaev, *Quasilinearization and curvature of Aleksandrov space*, Geom. Dedicata **133** (2008), 195–218.
- [11] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Springer-Verlag, Berlin, 1999.
- [12] D. Burago, Y. Burago, and S. Ivanov, *A course in metric geometry*, American Mathematical Society, Providence, RI, 2001.
- [13] P. Chaipunya and P. Kumam, *On the proximal point method in Hadamard spaces*, Optimization **66** (2017), 1647–1665.
- [14] P. Chulamjiak, *The modified proximal point algorithm in CAT(0) spaces*, Optim. Lett. **9** (2015), 1401–1410.
- [15] S. Dhompongsa, W. A. Kirk, and B. Sims, *Fixed points of uniformly Lipschitzian mappings*, Nonlinear Anal. **65** (2006), 762–772.
- [16] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv. **70** (1995), 659–673.
- [17] S. Kamimura, *The proximal point algorithm in a Banach space*, Nonlinear analysis and convex analysis, Yokohama Publishers, Yokohama, 2004, pp. 143–148.
- [18] S. Kamimura, F. Kohsaka, and W. Takahashi, *Weak and strong convergence theorems for maximal monotone operators in a Banach space*, Set-Valued Anal. **12** (2004), 417–429.
- [19] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.
- [20] H. Khatibzadeh and S. Ranjbar, *Monotone operators and the proximal point algorithm in complete CAT(0) metric spaces*, J. Aust. Math. Soc. **103** (2017), 70–90.
- [21] Y. Kimura and F. Kohsaka, *Two modified proximal point algorithms for convex functions in Hadamard spaces*, Linear Nonlinear Anal. **2** (2016), 69–86.
- [22] Y. Kimura, S. Saejung, and P. Yotkaew, *The Mann algorithm in a complete geodesic space with curvature bounded above*, Fixed Point Theory Appl. (2013), 2013:336, 1–13.
- [23] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces*, Nonlinear Anal. **68** (2008), 3689–3696.
- [24] F. Kohsaka, *Averaged sequences for nonspreading mappings in Banach spaces*, Banach and function spaces IV (ISBFS 2012), Yokohama Publishers, Yokohama, 2014, pp. 313–323.
- [25] ———, *Existence and approximation of fixed points of vicinal mappings in geodesic spaces*, Pure Appl. Funct. Anal. **3** (2018), 91–106.
- [26] F. Kohsaka and W. Takahashi, *Strong convergence of an iterative sequence for maximal monotone operators in a Banach space*, Abstr. Appl. Anal. (2004), 239–249.
- [27] ———, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.

- [28] ———, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.
- [29] T. C. Lim, *Remarks on some fixed point theorems*, Proc. Amer. Math. Soc. **60** (1976), 179–182.
- [30] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
- [31] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom. **6** (1998), 199–253.
- [32] E. Naraghirad, N.-C. Wong, and J.-C. Yao, *Approximating fixed points of  $\alpha$ -nonexpansive mappings in uniformly convex Banach spaces and CAT(0) spaces*, Fixed Point Theory Appl. (2013), 2013:57, 20 pp.
- [33] D. Pallaschke and S. Rolewicz, *Foundations of mathematical optimization. Convex analysis without linearity*, Kluwer Academic Publishers Group, Dordrecht, 1997.
- [34] R. T. Rockafellar, *Characterization of the subdifferentials of convex functions*, Pacific J. Math. **17** (1966), 497–510.
- [35] ———, *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. **33** (1970), 209–216.
- [36] ———, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.
- [37] ———, *Monotone operators associated with saddle-functions and minimax problems*, Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 1, Chicago, Ill., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 241–250.
- [38] T. Suzuki, *Fixed point theorems for a new nonlinear mapping similar to a nonspreading mapping*, Fixed Point Theory Appl. (2014), 2014:47, 1–13.

*Manuscript received November 29 2018*

*revised January 14 2018*

F. KOHSAKA

Department of Mathematical Sciences, Tokai University, Kitakaname, Hiratsuka, Kanagawa 259-1292, Japan

*E-mail address:* f-kohsaka@tsc.u-tokai.ac.jp