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FUEL–OPTIMAL ATTITUDE MANEUVERING OF SPACE VEHICLE IN A TWISTING SLIDING MODE

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ABSTRACT. The current permanent rotation of a space vehicle in an inertial frame often needs to be changed. Attitude maneuvers of the so-called "twisting" type are considered with unconstrained time and minimal fuel consumption. The minimal roll angular velocity constraint is maintained in order to keep the artificial gravity. The problem is solved analytically using a global optimization method based on the Krotov sufficient conditions.

1. Introduction

Different aspects of the problem of optimal attitude control for space vehicle rotation around its center of mass have been addressed in many papers over the years (see *e.g.* Lee, 1962; Lee and Marcus, 1966, Borschevsky and Ioslovich, 1966, 1985; Ioslovich, 1966, 1967, 2003a,b; Athans *et al.*, 1963; Ackermann, 1969; Windeknecht, 1963; Dixon *et al.*, 1970; Soloviev, 1969; Gurman, 1965, Gurman et al., 1970; Lavrovskii, 1970; Rahn and Barba, 1991; Scrivener and Thompson, 1994; Kumar, 1965, 1966; Seywald and Kumar, 1993, 1994; Krstić and Tsiotras, 1999; Prajna et al., 2004; Gurfil, 2005; and many others. However most of the researches considered the quadratic objectives which have not applicable to the fuel optimality with control by reactive jets.

The rotation of a non-symmetric rigid body around its center of mass (De Valée Poussin, 1925) is described in the inertial frame as

$$\frac{d\mathbf{K}}{dt} = \mathbf{M} \tag{1.1}$$

where $\mathbf{K}$ is the vector of the angular momentum, and $\mathbf{M}$ is vector of the total control torque.

Initial value of $\mathbf{K}(t_0) = \mathbf{K_0}$ in the inertial frame must be changed at the end to the final value $\mathbf{K_T}$.

Considering control torque provided by three pairs of reactive jets, this system in the body-fixed frame will has the well known form, i. e. *dynamic Euler equations*,

$$\begin{aligned} A\frac{d\omega_x}{dt} &= (B-C)\omega_y\omega_z + b_1u_1, \\ B\frac{d\omega_y}{dt} &= (C-A)\omega_x\omega_z + b_2u_2, \\ C\frac{d\omega_z}{dt} &= (A-B)\omega_x\omega_y + b_3u_3, \end{aligned} \tag{1.2}$$

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where ω $(\omega_x, \omega_y, \omega_z)$ is the angular velocity vector. A, B, C are the principal moments of inertia, b_1, b_2, b_3 are constant values dependent on the velocity of the reactive flow c and the length of the arm l_m of the correspondent control torque, $b_m = c \cdot l_m,\ m = 1, 2, 3.$

The control variables are u_1, u_2, u_3.

The value $\sum_m(\mid u_m \mid)$ corresponds to the rate of fuel consumption, [kg/s]. The sign of u_m corresponds to direct (+) or reversed (-) direction of the component of the control torque.

The body-fixed axis z, (not the middle axis) corresponds to the axis of the initial permanent rotation of the body.

The control functions are assumed to be constrained

$$\mid u_m \mid \leq \overline{u}_m$$

For zero values of the control, the passive trajectory is initially just the self-rotation (roll) around the z-axis .

The functional to be minimized is given by

$$J = \int_{t_0}^{T} \sum_m \mid u_m(t) \mid dt. \tag{1.3}$$

The integral is taken over the total non-fixed time of the transfer from the given initial to the given end point in the inertial state space.

With a short pulse control of e.g. u_1, a small change of the position of the angular momentum will be achieved in the fixed frame, and a small change of the angle between initial and final values of the angular momentum in the inertial frame will be obtained as well.

For this operation, the control torque vector must be located into a close proximity of the plane containing the initial and final positions of the angular momentum, $\mathbf{K}_0$ and $\mathbf{K}_T$. Next time the control pulse is given at the appropriate position to catch the precessing system in an appropriate state, the deviation of the angular momentum from the axis z will be compensated.

The difference of the angle between initial and final angular momenta in the inertial frame will continue to change.

2. The twisting sliding mode maneuvering

These short control pulses must be given at the moments when the control torque is approximately coinciding with the plane of two vectors: vector K and vector K_T, and also with the plane $\omega_y = 0$.

The *ergodic property* (Arnold, 1989) of the two-periodic movement guarantees this approximate coincidence.

On the figure (2) one can see the angle I between vectors K and K_T that must be zero at the end of the process, and angle S between positive direction x of the control pulse and the plane containing vectors K and K_T. Using variable $K =\mid \mathbf{K} \mid$

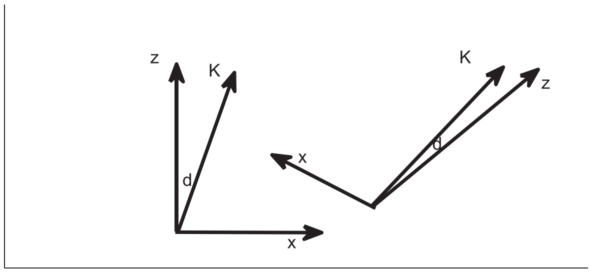

FIGURE 1. Angular momentum is changing by application of two consecutive control pulses $u_1 > 0$ and $u_1 < 0$ in the plane containing the current angular momentum K, the final angular momentum K_T, the axis z and the vector of the direction of the control torque x. At the moment of the control pulse $\omega_y = 0$. The small nutation angle d is obtained as the deviation with the first control pulse.

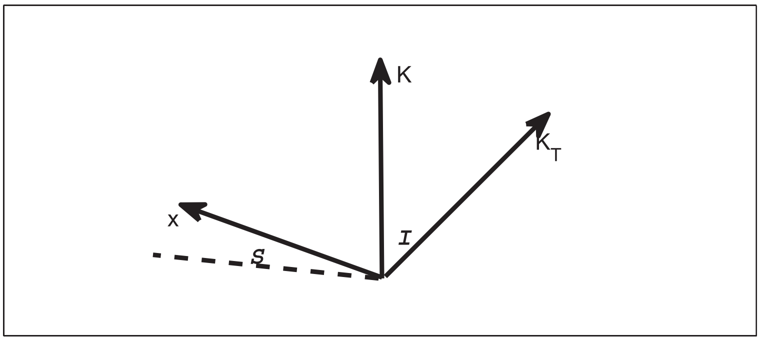

FIGURE 2. Vectors K, K_T, and angles S and I.

and angles I, S, the equations of this *twisting mode* process are

$$\begin{aligned} \frac{dK}{dt} &= b_3 u_3, \\ \frac{dI}{dt} &= u_1 b_1 \frac{cosS}{K} + u_2 b_2 \frac{sinS}{K}. \end{aligned} \tag{2.1}$$

Here K and I are the state variables and $u_m, m = 1, 2, 3$ are controls. Angle S is also considered as a control.
In addition the state constraint is considered

$$K_M \leq K,$$

which means that the angular velocity ω_z around axe z is bounded from below.

To simplify the calculations variables K, I are normalized according to transformations

$$k = \frac{K}{b_3},\ i = NI,\ N = \frac{b_3}{b_1},$$

assuming $L = \frac{b_2}{b_1}$, and for certainty that $L \leq 1$ Here $0 \leq i \leq N\pi$. The state equations now have form

$$\begin{aligned} \frac{dk}{dt} &= u_3, \\ \frac{di}{dt} &= \frac{u_1 cosS + Lu_2 sinS}{k} \end{aligned} \tag{2.2}$$

3. Optimal control determination

Following Hamilton-Bellman-Jacobi formalism (Krotov, 1988, 1996) the nonlinear PDE equation must be solved for the Krotov function $V(t,k,i)$ and system (2.2) as follows

$$\begin{aligned} \sup_{S,u_m,m=1,2,3} & \{V_k u_3 + V_i[(u_1 cosS + u_2 LsinS)/k] + V_t \\ & - \sum_m \mid u_m \mid)\} = 0. \end{aligned} \tag{3.1}$$

Here V_k, V_i, V_t are the correspondent partial derivatives of the function $V(t,k,i)$.

The figure (3) shows different regions of determination of the function V and different behavior of the optimal trajectories. The function V is assumed to be not

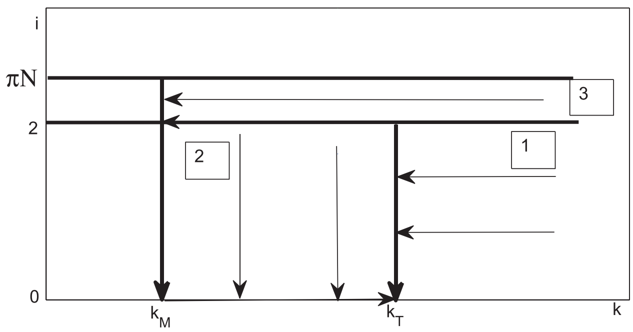

Figure 3. Regions $1, 2, 3$ in the state space k, i.

dependent of t and is sought in the form

$$V(i,k) = P \cdot i + Q \cdot k + D \cdot ik + U,$$

where P, Q, D, U are unknown constants. Function V must be continuous on all the borders of different regions. It is composed as a bundle of the first integrals of the "passive" movement. The unknown constants P, Q, D, U are determined separately for each region.

The regions are as follows:

1. $k_T \leq k,\ 0 \leq i \leq 2$
2. $k_m \leq k \leq k_T,\ 0 \leq i \leq 2$
3. $k_M \leq k,\ 2 \leq i \leq N\pi.$

The Krotov equation (3.1) with the assumptions on the function V can be represented by the system of three equations of the form

$$\sup_{u_i,S}[d_i(i,k,S)u_m - \mid u_i \mid] = 0,\ m = 1,2,3. \tag{3.2}$$

Here one has

$$d_1 = \frac{V_i}{k}cosS,\ d_2 = \frac{V_i}{k}LsinS,\ d_3 = V_k. \tag{3.3}$$

It follows (Grigoriev and Ioslovich, 1985) that the inequalities must be satisfied

$$\mid d_m \mid \leq 1. \tag{3.4}$$

To switch on the pair of jets for u_1 it requires that

$$\sup_{S,i,k} \mid d_1 \mid = 1. \tag{3.5}$$

Therefore the relation is found that
$u_1 = 0,\ u_2 = 0$, if $\mid V_i \mid < 0$,
and if $\mid V_i \mid = 1$ then
$u_2 = 0,\ sign(u_1) = sign(V_i cosS), \mid cosS \mid = 1$.
One always has for $L < 1$ that
$\sup_{S,i,k} \mid d_2 \mid < \sup_{S,i,k} \mid d_1 \mid$, thus always $u_2 = 0$.

3.1. **Case 1.** $k_T \leq k,\ 0 \leq i \leq 2$.

For case 1 the coefficients of the function V are chosen as

$$P = -k_T,\ Q = -1,\ D = 0,\ U = k_T.$$

Thus function V is

$$V = -k_T i - k + k_T.$$

The Krotov equation is presented as only two inequalities:

$$\mid d_1 \mid \leq 1, ; \mid d_3 \mid \leq 1$$

taking into account that it has been shown that from $sup_{S,i,k} d_1 \leq 1$ it follows $\mid d_2 \mid < 1$ and thus $u_2 = 0$. Here

$$d_1 = \frac{P + Dk}{k}cosS = \frac{-k_T}{k}cosS,\ d_3 = Q + Di = -1.$$

It follows that in this region $sign(u_3) = -1$ until $k_T \leq k$, and $u_1 = 0$ until $k_T \leq k$. When $k = k_T$, one takes $u_3 = 0$, and $sign(u_1) = -sign(cosS)$.
Thus along this line k decreases to k_T and when $k = k_T$ then the value of i decreases to zero.

3.2. **Case 2.** $k_m \leq k \leq k_T,\ 0 \leq i \leq 2$. For case 2 the constants are chosen as $P = 0,\ Q = 1,\ D = -1,\ U = -k_T$. Thus function V has form

$$V = k - ik - k_T.$$

It follows that

$$d_1 = \frac{Dk}{k}cosS = -cosS,\ d_3 = Q + Di = 1 - i.$$

Thus $u_2 = 0$, $sign(u_1) = -sign(cosS)$, $u_3 = 0$, if $i > 0$, and if $i = 0$, then $sign(u_3) = 1$.
It means that i decreases until zero, and then k increases until $k = k_T$.

3.3. **Case 3.** $k_M \leq k,\ 2 \leq i \leq N\pi$. Here the constants are chosen

$$P = -k_M,\ Q = -1,\ D = 0,\ U = 2k_M - k_T.$$

Thus function V has form $V = -k_M i - k + 2k_M - k_T$. It follows that

$$d_1 = \frac{-k_M}{k} cosS,\ d_3 = -1.$$

Thus $u_2 = 0$, $sign(u_3) = -1$, $u_1 = 0$ if $k > k_M$, and $sign(u_1) = -cosS$ if $k = k_M$.
It means that first k decreases to minimal value $k = k_M$, and then i decreases to the border value $i = 2$ and enters to the region 2.
The continuity of the function V is easily verified.

4. Approximate time evaluation

During this cycling sliding mode optimal process the fuel consumption tends to the optimum while time tends to infinity. The direct simulation performs some difficulties because of multiple revolutions on any approximate solution. However a reasonable evaluation of the time of the process can be done based of the *ergodic theory*. First let us note that only the changing of the inclination of angular momentum leads to the sliding mode approximation and increasing time. Therefore only this part of solution needs to be evaluated. This evaluation can be done as follows. According to the ergodic property of our passive (without control) dynamical system, Arnold, 1989, for almost all trajectories the time average is equal to the average over the space. Let say we designed an approximate-optimal system where the control torque is switched on only inside the angle of plus/minus 10 degrees around of the axis $w_y = 0$ and plus/minus 10 degrees around the plane containing vectors $\mathbf{K_0}$ and $\mathbf{K_T}$. The full state space is $360 \cdot 360$ $degrees^2$ and the part of the space for the working engines is $(10+10) \cdot (10+10) \cdot 2$. Thus the ratio of the "working time" to the full time is approximately 1 to 162. It means that if the continuous working time of pair of jets in order to change the inclination angle is $1min$ then the total time of the approximately optimal process is $162min$. This is not a short time but not so critical in many cases.The percent of deviation from the optimum can be evaluated to be less then $1 - cos(10°)$ which means less then 1.5%.

5. Conclusions

The optimal synthesis for minimal fuel consumption with control torque provided by reactive jets in the class of two-periodic cyclic sliding modes in the "twisting mode" regime is found. The initial permanent rotation is transferred to another permanent rotation in the inertial frame while preserving minimal rotation and thus artificial gravity. The optimal synthesis satisfies the Krotov's sufficient conditions. The Krotov function $V(i, k)$ is found in explicit form. It has different forms in three different regions and it is continuous and piece-wise smooth. The author is convinced that no other method other than Krotov's global optimality approach can be used to solve this problem analytically or numerically. The optimal control is non-unique but conditions of optimality determine it the explicit form. Altogether three regions of different solutions are described. The solution has properties of the two-periodical cycling mode movement. This is a generalization of the cycling mode solutions considered in Grigoriev and Ioslovich, 1985.

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