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REVEALED PREFERENCE THEORY

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ABSTRACT. We investigate many results on classical revealed preference theory, and prove these results rigorously. Many proofs of these results are almost self-contained. Results treated in this paper include the necessary and sufficient condition for demand function, an example of a function that obeys the weak axiom and violates the strong axiom, and the relationships between revealed preference axioms and requirements of the Slutsky matrix.

1. INTRODUCTION

In economics, consumer's behavior is described by the following parametrized maximization problem:

$$\begin{aligned} \max\quad & u(x) \\ \text{subject to.}\quad & x\in\Omega, \\ & p\cdot x\le m, \end{aligned} \tag{1.1}$$

where the function u is called the **utility function**, and is considered to represent the consumer's preference. The set Ω is called the **consumption set** and describes the set of all capable consumption plans. The vector p denotes **prices** of commodities while the value m denotes **money**. Under several requirements of u, there is exactly one solution $f^u(p,m)$ to the above problem, and the function f^u is called the **demand function**.

In the 1930's, Paul Anthony Samuelson considered the following problem. The only observable of a consumer from the point of view of economists is his/her purchasing behavior, which relates to f^u. We can therefore observe f^u only. Can we observe the information of u from f^u? Samuelson's answer is as follows. Consider two different consumption plans x and y, and suppose that x is bought in a price-money system (p,m) and y can be bought in the same price-money system. Then, y can be chosen but x is chosen, and thus x is probably preferred to y. He said that in this case, x is **revealed** to be preferred to y. This is the origin of revealed preference theory.

Today, the use of the term "revealed preference theory" has spread widely, and at least three different areas of economics uses this term. The first use is the most classical one, which considers the following problem. *What choice function can be described by a solution function of some parametrized optimization problem?* The second use is in Afriat-Varian's nonparametric test theory, which treats the finite data of purchase behavior and asks whether these data can be explained by the above problem. The third use is in the menu problem, which treats the data of choice for

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restaurants, and tries to reveal the preference of menu items. The present paper only treats the first theory.

The purpose of this paper is to collect all important results relevant to this theory, and to prove these results rigorously. In this research area, many results have been separated in many papers, and several important papers are old and thus their proofs of important results are often unreadably rough. We therefore present readable proofs for their results.

Results to be treated in this paper are as follows. First, we show the most important theorem proved by Richter [18]. This theorem asserts that a candidate of demand f is a demand function if and only if f satisfies the strong axiom of revealed preference. Next, we show that if the dimension of the consumption space is two, then the weak axiom of revealed preference implies the strong axiom of revealed preference. This result was obtained by Rose [19]. If the dimension of the consumption space is more than two, then a counterexample of the above fact was obtained by Gale [5]. Thirdly, we consider the relationships between requirements of the Slutsky matrix and axioms of revealed preference. The first result is as follows: f satisfies the strong axiom of revealed preference if and only if the Slutsky matrix is negative semi-definite and symmetric. This result was proved by Hosoya [8]. Next, we introduce the result of Kihlstrom, Mas-Colell, and Sonnenschein [13], which showed that f satisfies the weak weak axiom of revealed preference if and only if the Slutsky matrix is negative semi-definite. Finally, we show the result of Hurwicz and Richter [11, 12], which asserts that the inverse demand function satisfies Ville's axiom of revealed preference if and only if the Slutsky matrix is symmetric. The author believes that all proofs of results are improved from the original one.

In section 2, we define several notations and axioms. In section 3, we present Houthakker-Uzawa-Richter's theorem. Section 4 is devoted to explain Rose's result and Gale's example. In section 5, we treat the relationships between revealed preference theory and the Slutsky matrix.

2. Preliminary

2.1. **Definitions of Notations.** Let X be a set and $\rhd$ be a binary relation on X, that is, $\rhd \subset X^2$. We write $x \rhd y$ if $(x,y) \in \rhd$ and $x \not\rhd y$ if $(x,y) \notin \rhd$. We say that $\rhd$ is

- **complete** if for any $x, y \in X$, either $x \rhd y$ or $y \rhd x$,
- **transitive** if for any $x, y, z \in X$, $x \rhd y$ and $y \rhd z$ imply $x \rhd z$,
- **asymmetric** if $x \rhd y$ implies $y \not\rhd x$.

Choose any family $(\rhd_i)_i$ of transitive binary relations on X. Then, $\cap_i \rhd_i$ is also transitive. For every binary relation $\rhd$ on X, the intersection $\rhd^*$ of all transitive binary relations including $\rhd$ is also a transitive binary relation including $\rhd$.[1] Of course, $\rhd^*$ is the least transitive binary relation including $\rhd$. This $\rhd^*$ is called the **transitive closure** of $\rhd$.

Actually, we can define $\rhd^*$ directly. That is, $x \rhd^* y$ if and only if there exists a finite sequence $z_1, ..., z_k \in X$ such that $z_1 = x, z_k = y$ and $z_i \rhd z_{i+1}$ for $i \in \{1, ..., k-1\}$. The proof of 'if' part is trivial, because $\rhd \subset \rhd^*$. To prove 'only if'

[1] Note that X^2 itself is clearly transitive, and thus $\rhd^*$ is well-defined.

part, define $\rhd^+$ as follows: $x \rhd^+ y$ if and only if there exists the above finite sequence. Then, clearly $\rhd^+$ is a transitive binary relation including $\rhd$, and thus $\rhd^* \subset \rhd^+$. Therefore, if $x \rhd^* y$, then $x \rhd^+ y$, and thus our claim is correct.

Let the notation Ω denote the set of all possible consumption vector, and assume that $\Omega = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x \geq 0\}$, where $n \geq 2$. We write $x \gg y$ if $x_i > y_i$ for any i, and define $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n | x \gg 0\}$.

Choose any binary relation $\succsim$ on Ω. Then, we say that $\succsim$ is

- **continuous** if $\succsim$ is closed in Ω^2,
- **upper semi-continuous** if for any $x \in \Omega$, the set $\{y \in \Omega | y \succsim x\}$ is closed in Ω,
- **monotone** if for any $x, y \in \Omega$, $x \succsim y$ and $y \not\succsim x$ when $x \gg y$.

We call a binary relation $\succsim$ on Ω a **preference relation** if it is complete and transitive. If $\succsim$ is a preference relation, then we write $x \succ y$ if $x \succsim y$ and $y \not\succsim x$, and $x \sim y$ if $x \succsim y$ and $y \succsim x$. We can easily show that $\succ$ is transitive and asymmetric.

Suppose that $u : \Omega \to \mathbb{R}$ satisfies the following condition:

$$u(x) \geq u(y) \Leftrightarrow x \succsim y.$$

Then, we say that u represents $\succsim$, or u is a **utility function** of $\succsim$. Note that if some function u represents $\succsim$, then $\succsim$ is a preference relation, and $\succsim$ is continuous (resp. upper semi-continuous) if u is continuous. (resp. upper semi-continuous.)[2]

Next, we call a function $f : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to \Omega$ a **candidate of demand** (CoD) if it satisfies the budget inequality: that is,

$$p \cdot f(p, m) \leq m,$$

for any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$. If

$$p \cdot f(p, m) = m$$

for any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$, then this CoD is said to satisfy **Walras' law**.

Let $\succsim$ be a binary relation on Ω and define

$$f^{\succsim}(p, m) = \{x \in \Omega | p \cdot x \leq m, \text{ and if } y \in \Omega \text{ and } p \cdot y \leq m, \text{ then } x \succsim y\}.$$

If $f^{\succsim}$ is a single-valued function, then $f^{\succsim}$ is a CoD. If $\succsim$ is a preference relation, then we call $f^{\succsim}$ a **demand function** of $\succsim$. For a CoD f, if $f = f^{\succsim}$ for some preference relation $\succsim$, then we say that $\succsim$ corresponds with f (or, f corresponds with $\succsim$). If u represents $\succsim$, then $f^{\succsim}$ is sometimes written as f^u. We also say that u corresponds with f (or, f corresponds with u) if $f^u = f$. Note that if $\succsim$ is monotone, then $f^{\succsim}$ satisfies Walras' law.

Suppose that $f : P \to \mathbb{R}^m$ and $P \subset \mathbb{R}^k \times \mathbb{R}^\ell$. Note that f is possibly not a CoD function. This function $f(x, y)$ is said to be **locally Lipschitz in** x if and only if for every compact set $C \subset P$, there exists $L > 0$ such that for any $y \in \mathbb{R}^\ell$ and $x_1, x_2 \in \mathbb{R}^k$ with $(x_i, y) \in C$,

$$\|f(x_1, y) - f(x_2, y)\| \leq L\|x_1 - x_2\|.$$

[2]Conversely, if a preference relation $\succsim$ is continuous, (resp. upper semi-continuous,) then there is a continuous (resp. upper semi-continuous) function u that represents $\succsim$. This result is obtained by the second countability of Ω. See Debreu [2].

Similarly, f is said to be **locally Lipschitz** if for any compact set $C \subset P$, there exists $L > 0$ such that for any $(x_1, y_1), (x_2, y_2) \in C$,

$$\|f(x_1, y_1) - f(x_2, y_2)\| \le L\|(x_1, y_1) - (x_2, y_2)\|.$$

Note that if f is a CoD, the local Lipschitz condition in m is called the **income-Lipschitzian property**.[3]

Finally, suppose that $f : P \to \mathbb{R}^n$ and $P \subset \mathbb{R}^n \times \mathbb{R}$ is open, and f is differentiable at (p, m). We define

$$S_f(p, m) = D_p f(p, m) + D_m f(p, m) f^T(p, m).$$

That is, the (i, j)-th element $s_{ij}(p, m)$ of $S_f(p, m)$ is

$$\frac{\partial f_i}{\partial p_j}(p, m) + \frac{\partial f_i}{\partial m}(p, m) f_j(p, m).$$

This matrix-valued function $S_f(p, m)$ is called the **Slutsky matrix**. We say that f satisfies (S) (resp. (NSD)) if and only if f is differentiable at everywhere and $S_f(p, m)$ is always symmetric (resp. negative semi-definite). Moreover, we say that f satisfies (R) if and only if the rank of $S_f(p, m)$ is always $n-1$, and (ND) if for every $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$ and $v \in \mathbb{R}^n$ that is not proportional to p, $v^T S_f(p, m) v < 0$.

2.2. **Basic Knowledge on Revealed Preference Theory (1).** Choose any CoD f, and define a binary relation $\succ_r$ on Ω such that

$$x \succ_r y \Leftrightarrow \exists (p, m) \text{ s.t. } x = f(p, m),\ p \cdot y \le m \text{ and } x \ne y.$$

If $f = f^{\succsim}$ for some preference relation $\succsim$, then $x \succ_r y$ implies that $x \succ y$. Therefore,

$$x \succ_r y \Rightarrow x \succ y \Rightarrow y \not\succ x \Rightarrow y \not\succ_r x,$$

and thus $\succ_r$ is an asymmetric binary relation. This relation is called the **direct revealed preference relation**.

We say that a CoD f satisfies the **weak axiom of revealed preference** (abbreviated as (WA)) if and only if $\succ_r$ is asymmetric. If f is a demand function, then by the above argument, f satisfies (WA).

Next, consider the transitive closure $\succ_{ir}$ of $\succ_r$. This relation $\succ_{ir}$ is called the **indirect revealed preference relation**. As we argued above, $x \succ_{ir} y$ if and only if there exists a finite sequence $z_1, ..., z_k$ such that $x = z_1,\ y = z_k$ and $z_i \succ_r z_{i+1}$ for every $i \in \{1, ..., k-1\}$. Again, if $f = f^{\succsim}$ for some preference relation, then $x \succ_{ir} y$ implies that

$$x = z_1 \succ z_2 \succ ... \succ z_k = y,$$

and because $\succ$ is transitive, we have $x \succ y$. Therefore,

$$x \succ_{ir} y \Rightarrow x \succ y \Rightarrow y \not\succ x \Rightarrow y \not\succ_{ir} x,$$

and thus $\succ_{ir}$ is also asymmetric.

We say that a CoD f satisfies the **strong axiom of revealed preference** (abbreviated as (SA)) if and only if $\succ_{ir}$ is asymmetric. If f is a demand function, then again by the above argument, f satisfies (SA). Because $\succ_r \subset \succ_{ir}$, (SA) implies (WA).

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[3] This name was used by Mas-Colell [14].

2.3. **Basic Knowledge on Revealed Preference Theory (2).** Let X be a set and $\mathscr{B}$ be a nonempty subset of the power set of X that does not include the empty set. A multi-valued function $C : \mathscr{B} \twoheadrightarrow X$ is called a **choice correspondence** if $C(B) \neq \emptyset$ and $C(B) \subset B$ for all $B \in \mathscr{B}$.

A choice correspondence C satisfies the **congruence axiom of revealed preference** if for every finite sequence $z_1, ..., z_k \in X$ and $B_1, ..., B_k$ such that $z_i \in C(B_i)$, $z_{i+1} \in B_i$ for all $i \in \{1, ..., k\}$ (where $z_{k+1} = z_1$), z_{i+1} must be included in $C(B_i)$ for all $i \in \{1, ..., k\}$.

Choose any choice correspondence C on X. If there is a complete and transitive binary relation $\succsim$ on X such that $C(B) = \{x \in B | x \succsim y \text{ for all } y \in B\}$, then we say that C is rationalizable, and $\succsim$ rationalizes C.

If a CoD f satisfies (WA), then we can construct a choice correspondence C^f. Define $\Delta(p, m) = \{x \in \Omega | p \cdot x \leq m\}$, $\mathscr{B} = \{\Delta(p, m) | p \gg 0, m > 0\}$, and if $B = \Delta(p, m)$, then define $C^f(B) = \{f(p, m)\}$. To confirm the well-definedness of C^f, we must ensure that if $B = \Delta(p, m) = \Delta(q, w)$, then $f(p, m) = f(q, w)$. Suppose on the contrary that $x = f(p, m) \neq f(q, w) = y$. Then, because $\Delta(p, m) = \Delta(q, w)$, we have $x \succ_r y$ and $y \succ_r x$, and (WA) is violated, which is absurd. Moreover, we can easily verify that (SA) in f is equivalent to the congruence axiom of revealed preference in C^f.

3. First Result: Houthakker-Uzawa-Richter Theorem

3.1. **Richter's lemma.** First, we introduce and prove Richter's [18] monumental result. This is the fundamental theorem of classical revealed preference theory.

Theorem 3.1. *A choice correspondence $C : \mathscr{B} \twoheadrightarrow X$ is rationalizable if and only if it satisfies the congruence axiom of revealed preference.*

Proof. We will first show 'only if' part. Suppose that $\succsim$ rationalizes C, and choose any finite sequence $z_1, ..., z_k$ and $B_1, ..., B_k$ such that $z_i \in C(B_i)$, $z_{i+1} \in B_i$ for all $i \in \{1, ..., k\}$, where $z_{k+1} = z_1$. Then, we have

$$z_1 \succsim z_2 \succsim ... \succsim z_k \succsim z_1,$$

and by transitivity, we have

$$z_{i+1} \succsim z_i,$$

for all $i \in \{1, ..., k\}$. This implies that $z_{i+1} \succsim z_i \succsim z$ for all $z \in B_i$, and thus $z_{i+1} \in C(B_i)$, and hence C satisfies the congruence axiom of revealed preference.

Next, we will show 'if' part. First, we define an equivalence relation $\sim_{ir}$ such that $x \sim_{ir} y$ if and only if either $x = y$ or there exists a finite sequence $z_1, ..., z_k$ such that $z_1 = x, z_k = y$ and there exists B_i such that $\{z_i, z_{i+1}\} \subset C(B_i)$ for every $i \in \{1, ..., k-1\}$. Let $[x]$ be the equivalence class of x with $\sim_{ir}$, that is,

$$[x] = \{y \in X | x \sim_{ir} y\}.$$

Let $Y = \{[x] | x \in X\}$. Define a binary relation $\succ_r$ on Y such that $[x] \succ_r [y]$ if and only if there exists $z_1 \in [x]$, $z_2 \in [y]$ and $B \in \mathscr{B}$ such that $z_1 \in C(B)$ and $z_2 \in B \setminus C(B)$. Let $\succ_{ir}$ be the transitive closure of $\succ_r$. Then, $[x] \succ_{ir} [y]$ if and only if there exists a finite sequence $z_1, ..., z_k$ such that $[x] = [z_1], [y] = [z_k]$

and $[z_i] \succ_r [z_{i+1}]$ for all $i \in \{1, ..., k-1\}$. By the congruence axiom of revealed preference, we have that $\succ_{ir}$ is asymmetric.

Let $\mathscr{P}$ be the set of all binary relations on Y such that it includes $\succ_{ir}$ and it is asymmetric and transitive. Clearly $\succ_{ir} \in \mathscr{P}$ and thus $\mathscr{P}$ is nonempty. Moreover, it is obvious that the set inclusion relation defines a partial order on $\mathscr{P}$. If $\mathscr{C}$ is a chain of $\mathscr{P}$, define

$$\succ_{\mathscr{C}} = \cup_{\succ \in \mathscr{C}} \succ .$$

Then, we can easily show that $\succ_{\mathscr{C}} \in \mathscr{P}$, and thus $\succ_{\mathscr{C}}$ is an upper bound of $\mathscr{C}$. Therefore, by Zorn's lemma, there exists a maximal element $\succ^*$ of $\mathscr{P}$ with set inclusion. Define $\succsim = \{(x, y) \in X^2 | [x] \succ^* [y] \text{ or } [x] = [y]\}$. It is easy to show that $\succsim$ is transitive. Suppose that $\succsim$ is not complete. Then, there exist $x, y \in X$ such that $x \not\succsim y$ and $y \not\succsim x$. By definition of $\succsim$, we have $[x] \neq [y]$. Define

$$\succ^+ = \succ^* \cup \{([z], [w]) | z \succsim x, y \succsim w\}.$$

Then, $\succ^+$ is a binary relation on Y such that $\succ^* \subset \succ^+$. Because $([x], [y]) \in \succ^+ \setminus \succ^*$, we have $\succ^+ \neq \succ^*$. It is easy to show that $\succ^+$ is transitive. Suppose that $\succ^+$ is not asymmetric. Then, there exist $z, w \in X$ such that $([z], [w]) \in \succ^+$ and $([w], [z]) \in \succ^+$. If $([z], [w]), ([w], [z]) \in \succ^*$, then $\succ^*$ is not asymmetric, and thus $\succ^* \notin \mathscr{P}$, which is absurd. Therefore, we can assume that $([z], [w]) \notin \succ^*$. Then, $z \succsim x$ and $y \succsim w$. If $([w], [z]) \in \succ^*$, then $w \succsim z$, and thus $y \succsim x$ by transitivity of $\succsim$, which contradicts our initial assumption. Hence, we have $w \succsim x$, and thus again $y \succsim x$, a contradiction. Therefore, $\succ^+ \in \mathscr{P}$. However, this implies that $\succ^*$ is not maximal, a contradiction. Thus, we have that $\succsim$ is a complete and transitive binary relation on X. Suppose that $x \in C(B)$ and $y \in B$. If $y \in C(B)$, then $[x] = [y]$ and thus $x \sim y$. If $y \notin C(B)$, then $[x] \succ_{ir} [y]$, and thus $[x] \succ^* [y]$, which implies that $x \succ y$. Thus, we have if $x \in C(B)$, then $x \succsim y$ for all $y \in B$. Conversely, suppose that $x \in B$ and $x \succsim y$ for all $y \in B$. Because $C(B) \neq \emptyset$, there exists $z \in C(B)$. Then, $x \succsim z$. If $[x] \succ^* [z]$, then $z \not\succsim x$, which contradicts the above argument. Therefore, we must have $[x] = [z]$. Hence, there exists $z_1, ..., z_k \in X$ such that $[z_1] = [x]$, $[z_k] = [z]$ and for $i \in \{1, ..., k-1\}$, $\{z_i, z_{i+1}\} \subset C(B_i)$ for some $B_i \in \mathscr{B}$. If we define $B_k = B$, then by congruence axiom of revealed preference, we must have $x \in C(B)$. Thus, we can conclude that $C(B) = \{x \in B | x \succsim y \text{ for all } y \in B\}$, and hence C is rationalizable. This completes the proof. □

As its corollary, we can show the following result.

Theorem 3.2. *A CoD f is a demand function of some preference relation $\succsim$ if and only if it satisfies (SA).*

Proof. The 'only if' part had been shown in subsection 2.2. To prove 'if' part, note that if f satisfies (SA), then C^f can be defined, and $f = f^{\succsim}$ if and only if $\succsim$ rationalizes C^f. Therefore, it suffices to show that (SA) in f implies the congruence axiom of revealed preference in C^f. Suppose that f satisfies (SA), and for $z_1, ..., z_k \in \Omega$, $z_i \in C^f(B_i)$ and $z_{i+1} \in B_i$ for $i \in \{1, ..., k\}$, where $z_{k+1} = z_1$. By definition of C^f, there exists (p_i, m_i) such that $B_i = \Delta(p_i, m_i)$, and $z_i = f(p_i, m_i)$, $p_i \cdot z_{i+1} \le m_i$. If $z_{i+1} \neq z_i$ for some $i \in \{1, ..., k-1\}$, then $z_1 \succ_{ir} z_k$, and thus $z_k \not\succ_r z_1$. This implies that $z_1 = z_k$, and thus $z_1 \succ_{ir} z_1$, which is absurd. Therefore, we must have

$z_1 = z_2 = ... = z_k$, and thus $z_{i+1} \in C^f(B_i)$, which implies that C^f satisfies the congruence axiom of revealed preference. This completes the proof. $\square$

Remark 3.3. In Richter [18], he introduced a result of Szpilrajn [22] to define $\succsim$ indirectly. He said that to prove this result, we do not need Zorn's lemma, and only the existence theorem of the maximal ideal in Boolean algebra, which is weaker than the axiom of choice in Zermelo-Frankel's axiomatic set theory.[4] However, Szpilrajn [22] is written in not English but French, and thus the author cannot read directly. Thus, whether Richter's claim is correct is unknown.

Meanwhile, section 3.J of Mas-Colell, Whinston, and Green [15] includes the direct proof of theorem 3.2, which uses Zorn's lemma. However, this proof does not include the proof of theorem 3.1, and the author thinks that theorem 3.1 itself is also important. Therefore, we first showed theorem 3.1 by using Zorn's lemma simply, and then proved theorem 3.2 by using theorem 3.1.

Because of the use of Zorn's lemma, the proofs of the above theorems are not constructive. Meanwhile, Uzawa [23] showed that under several strong requirements in a CoD f, $f = f^{\succsim}$ for $\succsim = \{(x, y)|y \not\succ_{ir} x\}$ and $\succsim$ is a preference relation. Houthakker [10] also argued such a result. However, Houthakker's arguments are too rough, and thus the author could not understand his proof. In contrast, Uzawa's proof is very clear. Therefore, to the best understanding of the author, theorem 3.2 was first a conjecture of Houthakker, and then partially proved by Uzawa, and finally perfectly solved by Richter.

3.2. **Utility Maximization Hypothesis.** By theorem 3.2, we have for a CoD f, $f = f^{\succsim}$ if and only if f satisfies (SA). The next question is as follows: under what condition does $f = f^u$ for some real-valued function u defined on Ω? Because the classical consumer theory formulated by Walras is problem (1.1), this problem is also important.

Richter [18] claimed that (SA) also ensures the existence of such u. However, at least the author could not understand his proof. In this subsection, we will show that under some additional requirements in f, (SA) ensures the existence of u with $f = f^u$.

Before proving our result, we define an additional term. Let C be a subset of $\mathbb{R}^n$. Then, C is called **segmentally open** if and only if for every $x, y \in C$ with $x \neq y$, there exists $z \in [x, y]$ such that $z \in C$ and $x \neq z \neq y$. Note that C is segmentally open if C is either open or convex. Therefore, the requirement of the segmental openness is not so strict.

Theorem 3.4. *Suppose that f is a continuous and income-Lipschitzian CoD that satisfies Walras' law, and that the range $R(f)$ of f is segmentally open. Then, $f = f^u$ for some function $u : \Omega \to \mathbb{R}$ if and only if it satisfies (SA).*

Proof. . The 'only if' part is clear from theorem 3.2. To prove 'if' part, we need a lemma. This lemma is called the Shephard's lemma in economics.

Lemma 3.5. *Suppose that $f = f^{\succsim}$ is continuous and satisfies the Walras' law, where $\succsim$ is a preference relation. For each $x \in R(f)$, define*

$$E^x(q) = \inf\{q \cdot y | y \succsim x\}.$$

[4]See Mendelson [16].

Then, the function $E^x : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ *is concave and continuous. Moreover,* $DE^x(q) = f(q, E^x(q))$*, and if* $x = f(p, m)$*, then* $E^x(p) = m$*.*

Proof. Choose any $p_1, p_2 \in \mathbb{R}^n_{++}$ and $t \in [0, 1]$. Fix any $\varepsilon > 0$, and choose $y \in \Omega$ such that $y \succsim x$ and $p \cdot y \le E^x(p) + \varepsilon$, where $p = (1-t)p_1 + tp_2$. Then,

$$E^x(p) + \varepsilon \ge p \cdot y = (1-t)p_1 \cdot y + tp_2 \cdot y \ge (1-t)E^x(p_1) + tE^x(p_2).$$

Because $\varepsilon > 0$ is arbitrary, we have that E^x is concave. Because any concave function defined on an open set is continuous and the domain of E^x is open, we have E^x is concave and continuous.

Next, suppose that $x = f(p, m)$. If $y \succsim x$ and $y \ne x$, then $p \cdot y > m$. Meanwhile, $x \succsim x$ by completeness and $p \cdot x = m$ by Walras' law. Therefore, we have $E^x(p) = m$. Meanwhile, if $q \in \mathbb{R}^n_{++}$, then there exists $\varepsilon > 0$ such that $q \cdot y \le \varepsilon$ implies that $p \cdot y \le m$, which implies that $E^x(q) \ge \varepsilon > 0$. Therefore, $E^x(q)$ is always positive.

Define $x(q) = f(q, E^x(q))$. This function is continuous and $q \cdot x(q) = E^x(q)$. Fix any $\varepsilon > 0$ and define $x_\varepsilon(q) = f(q, E^x(q) + \varepsilon)$. By definition of $E^x(q)$, there exists $y \in \Omega$ such that $y \succsim x$ and $q \cdot y < E^x(q) + \varepsilon$. This implies that $x_\varepsilon(q) \succsim y$, and by transitivity, $x_\varepsilon(q) \succsim x$. Hence, for any $q, r \in \mathbb{R}^n_{++}$, we have $q \cdot x(q) = E^x(q) \le q \cdot x_\varepsilon(r)$. If $\varepsilon \downarrow 0$, then $x_\varepsilon(r) \to x(r)$ and thus $q \cdot x(q) \le q \cdot x(r)$.

Now, let e_i be the i-th unit vector and $q(t) = q + te_i$. Then,

$$\begin{aligned} E^x(q(t)) - E^x(q) &= (q + te_i) \cdot x(q + te_i) - q \cdot x(q) \\ &= q \cdot (x(q + te_i) - x(q)) + tx_i(q + te_i) \\ &\ge tf_i(q + te_i, E^x(q + te_i)). \end{aligned}$$

Therefore,

$$\lim_{t \downarrow 0} \frac{E^x(q(t)) - E^x(q)}{t} \ge f_i(q, E^x(q)) \ge \lim_{t \uparrow 0} \frac{E^x(q(t)) - E^x(q)}{t},$$

where both limits exist and $\lim_{t\downarrow 0} \frac{E^x(q(t)) - E^x(q)}{t} \le \lim_{t \uparrow 0} \frac{E^x(q(t)) - E^x(q)}{t}$ because E^x is concave. This means that $\frac{\partial E^x}{\partial q_i}(q) = f_i(q, E^x(q))$, and thus we have $DE^x(q) = f(q, E^x(q))$, as desired. ∎

We shall show 'if' part of theorem 3.4. Because f satisfies (SA), there exists a preference relation $\succsim$ such that $f = f^{\succsim}$. Now, for every $x \in R(f)$, define $E^x(p) = \inf\{p \cdot y | y \succsim x\}$. Fix $\bar{p} \in \mathbb{R}^n_{++}$ and define $u_{f,\bar{p}}(x) = E^x(\bar{p})$ if $x \in R(f)$ and $u_{f,\bar{p}}(x) = 0$ if $x \notin R(f)$. We will show that $f = f^{u_{f,\bar{p}}}$.

First, suppose that $x = f(p, m) = f^{\succsim}(p, m)$. If $y \in \Omega$ and $p \cdot y \le m$, then $x \succsim y$, and thus

$$\{z \in \Omega | z \succsim x\} \subset \{z \in \Omega | z \succsim y\}.$$

Therefore, clearly $u_{f,\bar{p}}(x) \ge u_{f,\bar{p}}(y)$.

Second, suppose that $p \cdot z < m$. We will show that $u_{f,\bar{p}}(z) < u_{f,\bar{p}}(x)$. Choose any $\varepsilon > 0$ such that $p \cdot z < m - \varepsilon$, and let $y = f(p, m - \varepsilon)$. Then, we have already shown that $u_{f,\bar{p}}(y) \ge u_{f,\bar{p}}(z)$. If $u_{f,\bar{p}}(z) = u_{f,\bar{p}}(x)$, then $u_{f,\bar{p}}(y) = u_{f,\bar{p}}(x)$. Define $c_1(t) = E^x((1-t)p + t\bar{p}), c_2(t) = E^y((1-t)p + t\bar{p})$. Then,

$$\dot{c}_i(t) = f((1-t)p + t\bar{p}, c_i(t)) \cdot (\bar{p} - p),\ c_i(1) = u_{f,\bar{p}}(x),$$

and thus, we have $c_1 \equiv c_2$ by Picard-Lindelöf's uniqueness theorem of the solution of an ordinary differential equation,[5] and

$$m = E^x(p) = c_1(0) = c_2(0) = E^y(p) = m - \varepsilon,$$

a contradiction. Therefore, we have $u_{f,\bar{p}}(z) < u_{f,\bar{p}}(x)$.

Last, choose any $z \in \Omega$ with $z \neq x$ and $p \cdot z \leq m$. If $z \notin R(f)$, then $u_{f,\bar{p}}(z) = 0 < u_{f,\bar{p}}(x)$. Suppose that $z \in R(f)$. Because $R(f)$ is segmentally open, there exists $t \in]0,1[$ and $y = (1-t)x + tz \in R(f)$. Suppose that $y = f(q,w)$. Then, by (WA), we have that $q \cdot x > w$, and thus $q \cdot z < w$, which implies that $u_{f,\bar{p}}(y) > u_{f,\bar{p}}(z)$. Therefore, $u_{f,\bar{p}}(x) > u_{f,\bar{p}}(z)$, and thus $x = f^{u_{f,\bar{p}}}(p,m)$. Hence, $f = f^{u_{f,\bar{p}}}$. This completes the proof. □

4. Second Result: Relationships between Axioms

4.1. **Rose's Theorem.** By theorem 3.2, we have that for a CoD f, f is a demand function if and only if f satisfies (SA). Meanwhile, (SA) implies (WA). Our next question is as follows: does (WA) imply (SA)? If so, then f is a demand function if and only if f satisfies (WA).

In the 1950's, many economists argued this topic frequently, and some group of economists claimed that (WA) is equivalent to (SA). Finally, it was found that an example of a CoD satisfies (WA) but violates (SA), and arguments closed. We will explain this example in the next subsection. However, in the example, the dimension n of the commodity space Ω is three. Is there such an example even in the case $n = 2$? The answer is almost negative, and probably this is the main reason why this argument was prolonged.

This result was obtained by Rose [19]. In this subsection, we will present the proof of Rose's result by using the notion of p-transitivity. A binary relation $\succsim$ on Ω is p-transitive if and only if for every $x, y, z \in \Omega$, if $x \succsim y$, $y \succsim z$ and $\dim(\mathrm{span}\{x,y,z\}) \leq 2$, then $x \succsim z$. Clearly, if $n = 2$, then p-transitivity is equivalent to the transitivity.

As we defined the transitive closure, we can define the p-transitive closure. Let $\rhd$ be a binary relation on Ω. If $(\rhd_i)$ is a collection of p-transitive binary relations on Ω, then $\cap_i \rhd_i$ is also p-transitive. Therefore, the intersection $\rhd^*$ of all p-transitive binary relations including $\rhd$ is the least p-transitive binary relation including $\rhd$. This binary relation $\rhd^*$ is called the p-transitive closure of $\rhd$. As in the arguments on transitive closure, we can show that $x \rhd^* y$ if and only if there exists $z_1, ..., z_k$ such that $\dim(\mathrm{span}\{z_1, ..., z_k\}) \leq 2$, $z_1 = x, z_k = y$ and $z_i \rhd z_{i+1}$ for $i \in \{1, ..., k-1\}$.

Recall the definition of $\succ_r$. $x \succ_r y$ if and only if $x \neq y$ and there exists (p,m) such that $x = f(p,m)$, $p \cdot y \leq m$. Let $\succ_{irp}$ be the p-transitive closure of $\succ_r$. Then, the following result holds.

Theorem 4.1. *Suppose that f is a CoD that satisfies Walras' law.*[6] *Define $\succ_r$, $\succ_{irp}$ as above. Then, $\succ_r$ is asymmetric if and only if $\succ_{irp}$ is asymmetric.*

Proof. The 'if' part is trivial, because $\succ_r \subset \succ_{irp}$. Thus, it suffices to show 'only if' part. Suppose that $x \succ_{irp} y$ and $y \succ_{irp} x$. Because $\dim(\mathrm{span}\{x,y\}) \leq 2$, we have

[5] The income-Lipschitzian property is used for assuring the applicability of this theorem.

[6] Recall that f satisfies Walras' law iff $p \cdot f(p,m) = m$ for all (p,m).

$x \succ_{irp} x$, and thus there exists z such that $x \succ_{irp} z$ and $z \succ_r x$. Therefore, to prove the asymmetry of $\succ_{irp}$, it suffices to show that if $x \succ_{irp} y$, then $y \not\succ_r x$.

Hence, suppose that $x \succ_{irp} y$. Then, there exists $z_1, ..., z_k$ such that $\dim(\mathrm{span}\{z_1, ..., z_k\}) \le 2$, $z_1 = x$, $z_k = y$ and $z_i \succ_r z_{i+1}$ for $i \in \{1, ..., k-1\}$. Therefore, for $i \in \{1, ..., k-1\}$, there exists (p_i, m_i) such that $z_i = f(p_i, m_i)$ and $p_i \cdot z_{i+1} \le m_i$. Because of Walras' law, we have $p_i \cdot z_i = m_i$, and thus

$$p_i \cdot z_{i+1} \le p_i \cdot z_i.$$

It suffices to show that $z_k \not\succ_r z_1$. We use mathematical induction on k. If $k = 2$, then $z_1 \succ_r z_k$, and thus clearly $z_k \not\succ_r z_1$ by the asymmetry of $\succ_r$.

Next, suppose that our claim holds if $k \le k^*$ for $k^* \ge 2$, and consider the case $k = k^* + 1$. Suppose that $z_k \succ_r z_1$. Then, $z_k \ne z_1$ and there exists (p_k, m_k) such that $z_k = f(p_k, m_k)$ and $p_k \cdot z_1 \le m_k$. Define $V = \mathrm{span}\{z_1, ..., z_k\}$. If $\dim V = 1$, then $z_i = c_i z_1$ for $c_i \in]0, 1]$, and $1 = c_1 > c_2 > ... > c_k$. Therefore,

$$p_k \cdot z_1 > p_k \cdot c_k z_1 = p_k \cdot z_k = m_k,$$

a contradiction. Thus, we have $\dim V = 2$. Let P_V be the orthogonal projection from $\mathbb{R}^n$ into V. By definition of the orthogonal projection, we have $P_V p_i \cdot z_j = p_i \cdot z_j$ for $i, j \in \{1, ..., k\}$. Define $q_i = \frac{1}{p_i \cdot x_1} P_V p_i$. Then, we have $q_i \cdot x_1 = 1$ for all $i \in \{1, ..., k\}$, and thus all q_i are included in the line $\{q \in V | q \cdot x_1 = 1\}$.

We separate our proof into three cases.

Case 1. $q_1 \in [q_{k-1}, q_k]$. In this case, $q_1 = (1-t)q_{k-1} + tq_k$ for some $t \in [0, 1]$. Therefore,

$$\begin{aligned} q_1 \cdot (z_1 - z_k) &= q_1 \cdot (z_1 - z_2) + q_1 \cdot (z_2 - z_k) \\ &= q_1 \cdot (z_1 - z_2) + (1-t)q_{k-1} \cdot (z_2 - z_k) + tq_k \cdot (z_2 - z_k) \\ &= q_1 \cdot (z_1 - z_2) + (1-t)q_{k-1} \cdot (z_2 - z_{k-1}) \\ &\quad + (1-t)q_{k-1}(z_{k-1} - z_k) + tq_k(z_2 - z_k) \ge 0, \end{aligned}$$

where the last inequality follows from the induction hypothesis. Therefore, we have

$$m_1 = p_1 \cdot z_1 \ge p_1 \cdot z_k,$$

which implies that $z_1 \succ_r z_k$. However, this contradicts the asymmetry of $\succ_r$.

Case 2. $q_{k-1} \in [q_1, q_k]$. In this case, $q_{k-1} = (1-t)q_1 + tq_k$ for some $t \in [0, 1]$. Therefore,

$$\begin{aligned} &(1-t)q_1 \cdot (z_1 - z_k) + tq_k \cdot (z_1 - z_k) \\ &= q_{k-1} \cdot (z_1 - z_k) \\ &= q_{k-1} \cdot (z_1 - z_{k-1}) + q_{k-1} \cdot (z_{k-1} - z_k) > 0. \end{aligned}$$

Because $q_k \cdot (z_1 - z_k) \le 0$, we have $(1-t)q_1 \cdot (z_1 - z_k) > 0$. Therefore, we have $t < 1$ and $z_1 \succ_r z_k$, which is absurd.

Case 3. The other case. Define $v = q_1 - q_k$. The case that $v = 0$ is included in our case 1, and thus we have $v \ne 0$ and $q_i = q_k + t_i v$ for some $t_i \in \mathbb{R}$. By definition, $t_1 = 1$. The case $t_{k-1} \ge 0$ is included in either case 1 or case 2, and thus we have

$t_{k-1} < 0$ for some $k-1$. Therefore, we must have that there exists $i \in \{1, ..., k-2\}$ such that $t_i \ge 0$ and $t_{i+1} \le 0$, and thus, $q_k \in [q_i, q_{i+1}]$. Then, $q_k = (1-t)q_i + tq_{i+1}$ for some $t \in [0,1]$, and

$$\begin{aligned} 0 \ge q_k \cdot (z_1 - z_k) &= q_k \cdot (z_1 - z_{i+1}) + q_k \cdot (z_{i+1} - z_k) \\ &= (1-t)q_i \cdot (z_1 - z_{i+1}) + tq_{i+1} \cdot (z_1 - z_{i+1}) + q_k \cdot (z_{i+1} - z_k) \\ &= (1-t)q_i \cdot (z_1 - z_i) + (1-t)q_i \cdot (z_i - z_{i+1}) \\ &\quad + tq_{i+1} \cdot (z_1 - z_{i+1}) + q_k \cdot (z_{i+1} - z_k) \ge 0, \end{aligned}$$

by the induction hypothesis. Therefore, all terms of both sides are zero. If $i \ge 2$ or $t > 0$, then the right-hand side is positive, which is absurd. Therefore, $i = 1$ and $t = 0$, which implies that $q_k = q_1$ and thus $v = 0$, a contradiction.

Therefore, in all cases there is a contradiction. Thus, we conclude that $z_k \not\succ_r z_1$, and hence our claim is correct. This completes the proof. □

As its direct corollary, we have the following Rose's theorem.

Corollary 4.2. *Suppose that $n = 2$, and let f be a CoD that satisfies Walras' law. Then, f satisfies (WA) if and only if f satisifies (SA).*

Proof. If $n = 2$, then $\succ_{irp} = \succ_{ir}$. □

4.2. **Gale's Example.** The following example was obtained by Gale [5]. Let

$$A = \begin{pmatrix} -3 & 4 & 0 \\ 0 & -3 & 4 \\ 4 & 0 & -3 \end{pmatrix},$$

and define a function $h_A(p)$ as

$$h_A(p) = \frac{1}{p^T Ap} Ap,$$

where p^T denotes the transpose of p. Let $C = \{p \in \mathbb{R}^n_{++} | Ap \ge 0\}$. We can easily show that if $p \in C$, then $p^T Ap \ge 0$. For every $p \in \mathbb{R}^n_{++}$, define $\bar{p}$ as follows.[7]

Case 1. If $p \in C$, then $\bar{p} = p$.

Case 2. Suppose that (i, j, k) is $(1,2,3), (2,3,1)$ or $(3,1,2)$. If $-3p_i + 4p_j \le 0$ and $-3p_j + 4p_k \le 0$, then $\bar{p}_i = \frac{16}{9}p_k, \bar{p}_j = \frac{4}{3}p_k$, and $\bar{p}_k = p_k$. Note that $\bar{p} \in C,\ \bar{p} \le p$.

Case 3. Again suppose that (i, j, k) is $(1,2,3), (2,3,1)$ or $(3,1,2)$. If $-3p_i + 4p_j \le 0,\ -3p_j + 4p_k > 0$ and $-3p_k + 4p_i > 0$, then separate this case into two subcases.

Subcase 3-1. If $16p_j - 9p_k \ge 0$, then define $\bar{p}_i = \frac{4}{3}p_j, \bar{p}_j = p_j, \bar{p}_k = p_k$, and $f(p, m) = h_A(\bar{p})m$. We can easily check that $\bar{p} \in C,\ \bar{p} \le p$.

[7]Note that if $p \ge 0$ and $-3p_1 + 4p_2 \le 0, -3p_2 + 4p_3 \le 0$, and $-3p_3 + 4p_1 \le 0$, then $p = 0$.

Subcase 3-2. If $16p_j - 9p_k < 0$, then define $\bar{p}_i = \frac{4}{3}p_j, \bar{p}_j = p_j, \bar{p}_k = \frac{16}{9}p_j$ We can easily check that $\bar{p} \in C,\ \bar{p} \le p$.

Define $f(p,m) = h_A(\bar{p})m$. We can easily check that f is a well-defined function from $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$ into Ω, and satisfies Walras' law.

Suppose that f violates (WA). Then, there exist $x = f(p,m), y = f(q,w)$ such that $x \ne y$, $q \cdot x \le w$ and $p \cdot y \le m$. Because $\bar{q} \le q$ and $\bar{p} \le p$, we have $x = f(\bar{p},m)$, $y = f(\bar{q},w)$ and $\bar{p} \cdot y \le m$, $\bar{q} \cdot x \le w$. Therefore, we can assume that $\bar{p} = p, \bar{q} = q$. Moreover, by changing (p,m) into $(\frac{1}{m}p, 1)$ and (q,w) into $(\frac{1}{w}q, 1)$, we can assume that $m = w = 1$. Then, we have

$$q^T Ap \le p^T Ap,\ p^T Aq \le q^T Aq.$$

Define

$$\lambda = \frac{p^T Ap}{q^T Ap}.$$

Then, $\lambda \ge 1$. Let $r = \lambda q$. Then, we have $(r-p)^T Ap = 0$. Because $\lambda \ge 1$, we have

$$p^T Ar = \lambda p^T Aq \le \lambda^2 p^T Aq \le \lambda^2 q^T Aq = r^T Ar,$$

and thus, we have

$$0 \le (r-p)^T Ar = (r-p)^T A(r-p).$$

Define $z = r - p$. Then, by the above inequality, we have

$$z^T Az \ge 0, z^T Ap = 0.$$

Suppose that $z \ne 0$. Because of the second equality, without loss of generality we can assume that $z_1 \ge 0, z_2 \ge 0, z_3 \le 0$ and $z_1 - z_3 > 0$. Therefore,

$$\begin{aligned} 0 \le z^T Az = &\ -3z_1^2 - 3z_2^2 - 3z_3^2 + 4z_1z_2 + 4z_2z_3 + 4z_3z_1 \\ = &\ -(3z_1^2 - 4z_1z_2 + 3z_2^2) - 3z_3^2 + 4z_3(z_1 + z_2) < 0, \end{aligned}$$

a contradiction. Therefore, we must have $z = 0$, and thus q is proportional to p. But this means $x = y$, a contradiction. Therefore, f must satisfy (WA).

Next, suppose that f satisfies (SA). Then, $f = f^{\succsim}$ for some preference relation $\succsim$. For $x = (1,1,1) = f(1,1,1,3)$, define

$$E^x(q) = \inf\{q \cdot y | y \succsim x\}.$$

Then, by lemma 3.5, we have for $p = (1,1,1)$ and $m = 3$,

$$DE^x(q) = f(q, E^x(q)),\ E^x(p) = m.$$

Because f is continuously differentiable around (p,m), we have that E^x is twice continuously differentiable at p, and

$$D^2E^x(p) = S_f(p,m),$$

and thus $S_f(p,m)$ must be symmetric. However, to calculate $S_f(p,m)$ directly, we have

$$s_{12}(p,m) = \frac{11}{3} \ne -\frac{1}{3} = s_{21}(p,m),$$

a contradiction. Therefore, f violates (SA).

Hence, corollary 4.2 can hold only in the case $n = 2$.

Remark 4.3. Gale [5] did not use lemma 3.5 to show f violates (SA). Instead, he showed that the corresponding inverse demand function (later, we will define this term rigorously) violates Jacobi's integrability condition.

The author guesses that lemma 3.5 was not known in the 1950's, and thus Gale could not use this result. Meanwhile, Samuelson [21] showed that if $f = f^u$ for some twice continuously differentiable function u, then the **Antonelli matrix** is symmetric, and to use this result, he derived the symmetry of the Slutsky matrix. Hosoya [6] showed that the symmetry of the Antonelli matrix is equivalent to Jacobi's integrability condition of the inverse demand function. Therefore, Gale's argument is not so misdirected. At least in the 1950's, however, this result was not known by economists, and thus the author thinks that Gale's proof itself is incomplete.

Meanwhile, by using theorem 1 of Hosoya [7], we can show that the restriction of f to $f^{-1}(\mathbb{R}^n_{++})$ corresponds with some complete, continuous, and p-transitive binary relation on $\mathbb{R}^n_{++}$. The author guesses that this relation can be naturally extended to some complete, continuous, and p-transitive binary relation on Ω, and $f = f^{\succsim}$. However, such a result is not related to the main concern of this paper, and thus we omit this argument.

5. Third Result: Under Differentiability

5.1. **(NSD) and (S) as Axioms.** In this section, we treat only CoDs that satisfy Walras' law and are continuously differentiable. Therefore, the Slutsky matrix can be always defined, and conditions (NSD) and (S) are meaningful. The following theorem was obtained by Hosoya [8].

Theorem 5.1. *Suppose that f is a CoD that is continuously differentiable and satisfies Walras' law. Then, $f = f^{\succsim}$ for some preference relation $\succsim$ if and only if both (NSD) and (S) hold. Moreover, in this case $f = f^u$ for some $u : \Omega \to \mathbb{R}$.*

We omit the full proof of this theorem because it is too long. Instead, we give a sketch of the proof of this theorem. To prove 'only if' part, we use lemma 3.5. If $x = f(p,m)$, then

$$DE^x(q) = f(q, E^x(q)),\ E^x(p) = m.$$

Because f is continuously differentiable, we have E^x is twice continuously differentiable at p, and

$$D^2E^x(p) = S_f(p,m).$$

Because E^x is concave, $S_f(p,m)$ is negative semi-definite and symmetric. Hence, (NSD) and (S) must be satisfied.

To prove 'if' part is not so easy. First, consider the following partial differential equation:

$$DE(q) = f(q, E(q)),\ E(p) = m. \tag{5.1}$$

If $f = f^{\succsim}$ for some $\succsim$, then $E = E^x$ solves the above equation, where $x = f(p,m)$. Actually, the local existence of the solution of the above equation is equivalent to (S).[8] By using (NSD), we can extend this local existence result to the global existence result. Moreover, this solution $E : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ must be concave.

[8]See theorem 10.9.4 of Dieudonne [4].

Next, we will show the following result: suppose that $x \neq y$, $x = f(p,m)$, $y = f(q,w)$ and for a solution $E : \mathbb{R}^n_{++} \to \mathbb{R}$ of (5.1), $w \geq E(q)$. Then, $p \cdot y > m$. The meaning of this result is as follows. Because E is a solution of (5.1) and the solution of this equation is unique in this setup,[9] probably $E = E^x$. It is known that if $f = f^u$ for some continuous function u, then for every $\bar{p} \in \mathbb{R}^n_{++}$,

$$u(x) \geq u(y) \Leftrightarrow E^x(\bar{p}) \geq E^y(\bar{p}).$$

Therefore, $w = E^y(q) \geq E^x(q)$ means that $u(y) \geq u(x)$, and thus $y \not\succ_r x$, which implies that $p \cdot y > m$.[10]

(WA) can be shown by the above result. Using (WA) and Picard-Lindelöf's theorem, we can show that if $x = f(p_1, m_1) = f(p_2, m_2)$ and $E_i : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is a solution of (5.1) with $p = p_i, m = m_i$, then $E_1 \equiv E_2$. Thus, if we define $u_{f,\bar{p}}(x) = 0$ if x is not in the range of f, and for $x = f(p,m)$, $u_{f,\bar{p}}(x) = E(\bar{p})$, where $E : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ is a solution of (5.1), then $u_{f,\bar{p}}$ is a well-defined function on Ω. The proof of $f = f^{u_{f,\bar{p}}}$ is not so difficult.

5.2. **Characterization of (NSD).** We showed in subsection 2.3 that under (WA), if $\Delta(p,m) = \Delta(q,w)$, then $f(p,m) = f(q,w)$. Particularly, because $\Delta(p,m) = \Delta(ap,am)$ for every $a > 0$, we have $f(p,m) = f(ap,am)$. This property is called the **homogeneity of degree zero**. As above, the homogeneity of degree zero is weaker than (WA).

Kihlstrom, Mas-Colell, and Sonnenschein [13] presented two results. First is the following: if f satisfies (ND), then f satisfies (WA). Second is the following: f satisfies (NSD) if and only if f satisfies some condition, named (WWA). The formal statement of (WWA) is as follows: if $x = f(p,m)$, $y = f(q,w)$ and $p \cdot y < m$, then $q \cdot x > w$. Under Walras' law, this is trivially weaker than (WA).[11]

To prove their results, we need a lemma.

Lemma 5.2. *Suppose that f is a continuously differentiable CoD that satisfies Walras' law. If $x = f(p,m)$ and $v \in \mathbb{R}^n$, define $p(t) = p + tv$. Then,*

$$\lim_{t \to 0} \frac{1}{t^2}(p - p(t)) \cdot (f(p,m) - f(p(t), p(t) \cdot x)) = v^T S_f(p,m) v. \tag{5.2}$$

Proof. By definition of $p(t)$, we have $p - p(t) = -tv$. Therefore, it suffices to show that

$$\lim_{t \to 0} \frac{1}{t} \left| f_i(p(t), p(t) \cdot x) - f_i(p,m) - \sum_{j=1}^n s_{ij}(p,m) v_j \right| = 0.$$

However, this is easy to prove by a simple calculation. □

[9]This uniqueness holds when f is income-Lipschitzian. In this section, f is assumed to be continuously differentiable, and thus it is income-Lipschitzian. Meanwhile, if f is not income-Lipschitzian, then this uniqueness is broken. See Mas-Colell [14].

[10]Although the interpretation of this requirement is not so vague, the actual proof of this fact is not so easy.

[11](WWA) is an abbreviation of the weak weak axiom of revealed preference.

Theorem 5.3. *Suppose that f is a continuously differentiable CoD that satisfies Walras' law. If f satisfies (ND), then f satisfies (WA).*[12]

Proof. We prove the contraposition of the claim. Suppose that f violates (WA). We separate the proof into four cases.

Case 0. f is not homogeneous of degree zero. In this case, there exists (p,m) such that

$$0\neq \frac{d}{da}f(ap,am)\bigg|_{a=1}=S_f(p,m)p.$$

By Walras' law, we have $p^TS_f(p,m)=0^T$,[13] and thus the rank of $S_f(p,m)$ is less than n. Therefore, there exists $v\in\mathbb{R}^n$ such that v is not proportional to p and $S_f(p,m)v=0$, which implies that $v^TS_f(p,m)v=0$, and f violates (ND).

Hereafter, we assume that f is homogeneous of degree zero.

Case 1. There exist (p,m) and (q,w) such that $p\cdot f(q,w)=m$ and $q\cdot f(p,m)<w$. Then, define $x=f(p,m), y=f(q,w)$ and $p(s)=p+s(q-p)$, and let s_0 be the smallest positive s such that $q\cdot f(p(s),p(s)\cdot y)=w$. Because $f(p(1),p(1)\cdot y)=y$ and $f(p(0),p(0)\cdot y)=x$, we have that $s_0>0$ is well-defined. Define $r=p(s_0)$ and $z=f(p(s_0),p(s_0)\cdot y)$. Then,

$$(1-s_0)m+s_0w=p(s_0)\cdot y=r\cdot z=(1-s_0)p\cdot z+s_0w,$$

and thus we have either $s_0=1$ or $p\cdot z=m$. In both cases, we have $p\cdot z=m$, and hence $p(s)\cdot y=p(s)\cdot z=(1-s)m+sw$ for every $s\in[0,1]$. If $r\cdot f(p(s),p(s)\cdot z)=r\cdot z$ for $s\in]0,s_0[$, then

$$p(s_0)\cdot f(p(s),p(s)\cdot z)=(1-s_0)m+s_0w,$$
$$p(s)\cdot f(p(s),p(s)\cdot z)=(1-s)m+sw,$$

and thus,

$$(q-p)\cdot f(p(s),p(s)\cdot z)=w-m.$$

This implies that

$$0<w-q\cdot f(p(s),p(s)\cdot z)=m-p\cdot f(p(s),p(s)\cdot z),$$

and thus

$$p(s)\cdot f(p(s),p(s)\cdot z)<p(s)\cdot z,$$

a contradiction. Therefore, for every $s\in]0,s_0[$, we have

$$r\cdot f(p(s),p(s)\cdot z)<r\cdot z.$$

Define v as the orthogonal projection of $p-r$ on $T_r=\{p'\in\mathbb{R}^n|p'\cdot r=0\}$. Note that, because of the homogeneity of degree zero, we must have that p is not proportional to r, and thus $v\neq 0$. Define $r(t)=r+tv$. By the above calculation, we have for every $r'\in[p,r[$,

$$r\cdot f(r',r'\cdot z)<r\cdot z.$$

[12] Check that the proof of this theorem only requires that the domain of f is an open and convex cone included in $\mathbb{R}^n_{++}\times\mathbb{R}_{++}$. Later we will use this fact.

[13] See the proof of lemma 5.9.

By definition of v, we have

$$v = p - r - \frac{(p-r)\cdot r}{\|r\|^2} r \equiv p - dr,$$

for some $d > 0$, and thus for every sufficiently small $t > 0$, $r(t) = (1-dt)r + tp$ is proportional to $(1 - c(t)t)r + c(t)tp \equiv r' \in [p, r[$, where $c(t) = \frac{1}{1-dt+t}$. Then,

$$f(r(t), r(t)\cdot z) = f(r', r'\cdot z),$$

which implies that

$$r \cdot f(r(t), r(t)\cdot z) < r \cdot z$$

for sufficiently small $t > 0$. Therefore,

$$(r - r(t))\cdot [f(r, r\cdot z) - f(r(t), r(t)\cdot z)] > 0,$$

and thus, by using (5.2), we have

$$v^T S_f(r, r\cdot z) v = \lim_{t\downarrow 0} \frac{1}{t^2}(r - r(t))\cdot [f(r, r\cdot z) - f(r(t), r(t)\cdot z)] \geq 0,$$

which implies that f violates (ND).

Case 2. There exist (p, m) and (q, w) such that $p \cdot f(q, w) < m$ and $q \cdot f(p, m) < w$. Define $p(t) = (1-t)p + tq$, $m(t) = (1-t)m + tw$, and $x(t) = f(p(t), m(t))$. If $t > 0$ is sufficiently near to 0, then $q \cdot x(t) < w$. If $t < 1$ is sufficiently near to 1, then $p \cdot x(t) < m$. Because $p(t)\cdot x(t) = m(t)$, this implies that $q \cdot x(t) > w$. Therefore, by intermediate value theorem, there exists $t^* \in]0, 1[$ such that $q \cdot x(t^*) = w$. Because $p(t^*)\cdot x(t^*) = m(t^*)$, we must have $p \cdot x(t^*) = m$. Define $r = p(t^*)$ and $c = m(t^*)$. Then, $r \cdot f(q, w) < c$ and $q \cdot f(r, c) = q \cdot x(t^*) = w$, and thus (r, c) and (q, w) satisfies the requirement of case 1. Therefore, again f violates (ND).

Case 3. There exist (p, m) and (q, w) such that $f(p, m) \neq f(q, w)$, $q \cdot f(p, m) = w$ and $p \cdot f(q, w) = m$. Define $p(t) = (1-t)p + tq$, $m(t) = (1-t)m + tw$ and $x(t) = f(p(t), m(t))$. Then, the sign of $m - p\cdot x(t)$ is the same as that of $q \cdot x(t) - w$ for all $t \in [0, 1]$. If $p \cdot x(t) < m$ (resp. $q \cdot x(t) < w$) for some $t \in]0, 1[$, then (p, m) and $(p(t), m(t))$ (resp. (q, w) and $(p(t), m(t))$) satisfy all requirements of case 1, and thus f violates (ND). Hence, we can assume that $p \cdot x(t) = m$ and $q \cdot x(t) = w$ for all $t \in [0, 1]$. Let v be the orthogonal projection of $q - p$ to $T_p = \{p' \in \mathbb{R}^n | p' \cdot p = 0\}$. Then, $v \neq 0$ by the homogeneity of degree zero, and by almost the same arguments as in case 1, we can show that for sufficiently small $t > 0$,

$$p \cdot f(p + tv, (p + tv)\cdot f(p, m)) = m.$$

Therefore, we have

$$(p - (p + tv))\cdot [f(p, m) - f(p + tv, (p + tv)\cdot f(p, m))] = 0$$

for all sufficiently small $t > 0$, and thus

$$v^T S_f(p, m) v \geq 0,$$

which implies that f violates (ND). This completes the proof. □

Theorem 5.4. *Suppose that f is a continuously differentiable CoD that is homogeneous of degree zero and satisfies Walras' law. Then, f satisfies (WWA) if and only if f satisfies (NSD).*

Proof. Suppose that f satisfies (WWA). Choose any $(p, m) \in \mathbb{R}^n_{++} \times \mathbb{R}_{++}$ and define $x = f(p, m)$. For every $v \in \mathbb{R}^n$, define $p(t) = p + tv$. Then, $p(t) \in \mathbb{R}^n_{++}$ for every t sufficiently near to 0. For such t, we have

$$p(t) \cdot f(p, m) = p(t) \cdot x < p(t) \cdot f(p(t), p(t) \cdot x + \varepsilon),$$

and thus, for (WWA),

$$p \cdot f(p(t), p(t) \cdot x + \varepsilon) > m = p \cdot f(p, m),$$

for every $\varepsilon > 0$. Therefore, we have

$$p \cdot f(p(t), p(t) \cdot x) \geq m,$$

and thus,

$$(p - p(t)) \cdot (f(p, m) - f(p(t), p(t) \cdot x)) \leq 0.$$

Hence, we have

$$v^T S_f(p, m) v \leq 0,$$

which implies (NSD).

To prove the converse, suppose on the contrary that f satisfies (NSD) but violates (WWA). Then, for some (p, m) and (q, w), $x = f(p, m) \neq f(q, w) = y$, $p \cdot y < m$ and $q \cdot x \leq w$. Because f is continuous, we can assume without loss of generality that $q \cdot x < w$. Define $h(r) = -\|r\|$, and let V be some open and convex cone in $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$ such that it includes (p, m) and (q, w) and every limit point of V different from 0 is included in $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$. Then, for sufficiently small $s > 0$, $f_s(r, c) = f(r, c - sh(r)) + sDh(r)$ can be defined for every $(r, c) \in V$. We can easily show that f_s satisfies Walras' law and homogeneity of degree zero, and f_s satisfies (ND) on V. By repeating the proof of theorem 5.3, we can show that f_s satisfies (WA) on V. However, if $s > 0$ is sufficiently small, then we have $q \cdot f_s(p, m) < w$ and $p \cdot f_s(q, w) < m$, a contradiction. □

Remark 5.5. Samuelson [20] claimed that (WA) implies (NSD), and this claim is true because (WA) implies (WWA). Does (NSD) imply (WA)? Theorem 5.4 only says that (NSD) is equivalent to (WWA). Kihlstrom, Mas-Colell, and Sonnenschein [13] presented an example that satisfies (NSD) but violates (WA). Their example is,

$$f(p, m) = \left(\frac{p_2}{p_3}, -\frac{p_1}{p_3}, \frac{m}{p_3} \right).$$

However, the range of this function is not Ω. They said that whether there is such an example whose range is included in Ω is unknown, and this problem remains open.

We stress that the range of f is very important in this theory. For example, the actual proof of theorem 5.1 heavily depends on the boundedness of $\Delta(p, m)$, and if Ω is not included in $\mathbb{R}^n_+$, then this requirement is violated.

5.3. **Ville's Axiom and (S).** By theorem 5.4, we obtain a characterization of (NSD) in revealed preference theory. The remaining condition that needs to be characterized is (S). We use the notion of the budgeter to characterize (S).

Let $x \in \mathbb{R}^n_{++}$ and U be an open neighborhood of x. A function $g : U \to \mathbb{R}^n_{++}$ is called a **local budgeter** around x, and if $U = \mathbb{R}^n_{++}$, then it is called a budgeter. Suppose that f is a CoD that satisfies the homogeneity of degree zero and Walras' law. For a local budgeter $g : U \to \mathbb{R}^n_{++}$ around x, if

$$y = f(g(y), g(y) \cdot y)$$

for all $y \in U$, then g is called a **local inverse demand function** of f around x. Again, if $U = \mathbb{R}^n_{++}$, then g is called an **inverse demand function** of f.

Next, suppose that $g : U \to \mathbb{R}^n_{++}$ is a continuously differentiable local budgeter around x. We say that g satisfies (SBA) if and only if the following Jacobi's integrability condition holds: for all $i, j, k \in \{1, ..., n\}$,

$$\begin{aligned} & g_i(y)\left(\frac{\partial g_j}{\partial y_k}(y) - \frac{\partial g_k}{\partial y_j}(y)\right) \\ & + g_j(y)\left(\frac{\partial g_k}{\partial y_i}(y) - \frac{\partial g_i}{\partial y_k}(y)\right) \\ & + g_k(y)\left(\frac{\partial g_i}{\partial y_j}(y) - \frac{\partial g_j}{\partial y_i}(y)\right) = 0. \end{aligned} \tag{5.3}$$

We also say that g satisfies (NSDBA) if and only if for every $v \in \mathbb{R}^n$, $g(y) \cdot v = 0$ implies $v^T Dg(x) v \le 0$.

If g is a continuously differentiable inverse demand function of a continuously differentiable CoD f, then it is known that f satisfies (NSD) if and only if g satisfies (NSDBA), and f satisfies (S) if and only if g satisfies (SBA).[14] Therefore, to characterize (S), we can consider (SBA).

If $g : U \to \mathbb{R}^n_{++}$ is a continuously differentiable local budgeter around x, then a piecewise C^1 function[15] $z : [0, T] \to U$ is called a **Ville cycle** of g if and only if $T > 0$, $z(0) = z(T)$ and

$$g(z(t)) \cdot \dot{z}(t) > 0$$

for every $t \in]0, T[$ except non-differentiable points of z. Then, g satisfies the **Ville's axiom of revealed preference** (abbreviated as (VA)) if and only if for every $y \in U$, there exists an open neighborhood $V \subset U$ of y such that there is no Ville cycle of the restriction g_V of g to V.

The interpretation of (VA) is as follows. First, suppose that $u : \Omega \to \mathbb{R}$ is a twice continuously differentiable and increasing function that satisfies the **strict bordered Hessian condition**, that is,

$$v \neq 0,\ v \cdot Du(x) = 0 \Rightarrow v^T D^2 u(x) v < 0.$$

Then, it is known that u is strictly quasi-concave, and if $f^u(p, m) \in \mathbb{R}^n_{++}$, then f^u is continuously differentiable at (p, m). (See Debreu [1, 3].) By Lagrange's first-order

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[14] See the mathematical appendix of Samuelson [21] or Hosoya [7].

[15] A function $h : [0, T] \to \mathbb{R}^n$ is piecewise C^1 if and only if it is continuous and there exists a finite set $\{x_0, ..., x_k\} \subset [0, T]$ such that $0 = x_0 < x_1 < ... < x_k = T$ and for every $i \in \{1, ..., k\}$, the restriction of h into $[x_{i-1}, x_i]$ is continuously differentiable.

condition, we must have $g(y) \equiv Du(y)$ is an inverse demand function. If z is a Ville cycle of g, then

$$\frac{d}{dt}u(z(t)) > 0$$

for almost every $t \in [0, T]$, and thus $u(z(T)) > u(z(0))$ and $z(T) = z(0)$, a contradiction. Therefore, there must be no Ville cycle of g.

Theorem 5.6. *Suppose that g is a continuously differentiable local budgeter around x. Then, g satisfies (VA) if and only if g satisfies (SBA).*

Proof. [16] Suppose that g satisfies (SBA), and choose any $y \in U$. Then, by Frobenius theorem (see Hosoya [6]), there exists an open neighborhood V of y, a continuous function $\lambda : V \to \mathbb{R}_{++}$, and a continuously differentiable function $u : V \to \mathbb{R}$ such that

$$Du(z) = \lambda(z)g(z)$$

for every $z \in V$. If $z : [0, T] \to V$ is a Ville cycle of g_V, then

$$\frac{d}{dt}(u(z(t))) > 0$$

for almost every $t \in]0, T[$, and thus

$$u(z(0)) = u(z(T)) > u(z(0)),$$

a contradiction. Therefore, we must have that there is no Ville cycle of g_V, and thus g satisfies (VA).

To prove the converse, we first mention the following fact. Suppose that $g : U \to \mathbb{R}^n_{++}$ is a continuously differentiable local budgeter and $c : U \to \mathbb{R}_{++}$ is a continuously differentiable function. Define $h(y) = c(y)g(y)$. Then, g satisfies (VA) if and only if h satisfies (VA), and g satisfies (SBA) if and only if h satisfies (SBA). The former is obvious. The proof of the latter is a simple calculation:

$$\begin{aligned}
&h_i(y)\left(\frac{\partial h_j}{\partial y_k}(y) - \frac{\partial h_k}{\partial y_j}(y)\right) \\
&+ h_j(y)\left(\frac{\partial h_k}{\partial y_i}(y) - \frac{\partial h_i}{\partial y_k}(y)\right) \\
&+ h_k(y)\left(\frac{\partial h_i}{\partial y_j}(y) - \frac{\partial h_j}{\partial y_i}(y)\right) \\
&= (c(y))^2\left[g_i(y)\left(\frac{\partial g_j}{\partial y_k}(y) - \frac{\partial g_k}{\partial y_j}(y)\right)\right. \\
&+ g_j(y)\left(\frac{\partial g_k}{\partial y_i}(y) - \frac{\partial g_i}{\partial y_k}(y)\right) \\
&+ \left. g_k(y)\left(\frac{\partial g_i}{\partial y_j}(y) - \frac{\partial g_j}{\partial y_i}(y)\right)\right].
\end{aligned}$$

[16] This proof requires knowledge on theory of ordinary differential equations. We mention that all results needed to understand this proof are standard, and many textbooks include these results. For example, see Pontryagin [17]. These results are also in appendix of Hosoya [9].

Particularly, we can choose $c(y) = 1/(g_1(y))$, and thus to prove the converse, we can assume that $g_1(y) \equiv 1$.

Thus, suppose that $g : U \to \mathbb{R}^n_{++}$ satisfies (VA), and $g_1(y) \equiv 1$ on U. Choose any $y \in U$. We can assume without loss of generality that there is no Ville cycle of g itself. Because g is continuous, we can assume that there exists $M \geq 1$ such that

$$g_i(z) \leq M$$

for every $i \in \{2, ..., n\}$ and $z \in U$, and $U = \{z \in \mathbb{R}^n || z_i - y_i| < \varepsilon \text{ for all } i\}$ for some $\varepsilon > 0$. Define $\delta = \frac{\varepsilon}{M}$ and $V = \{z \in \mathbb{R}^n | \sum_{i=1}^n |z_i - y_i| < \delta\}$.

Choose any continuously differentiable function $\alpha : [0, T] \to \{z \in V | z_1 = y_1\}$, and consider the following parametrized ordinary differential equation:

$$\dot{c}(t) = -\sum_{i=2}^n g_i(c(t), \alpha_2(t), ..., \alpha_n(t))\dot{\alpha}_i(t) + \mu. \tag{5.4}$$

Choose any $z \in V$ and define $\alpha(t) = (y_1, ty_2 + (1-t)z_2, ..., ty_n + (1-t)z_n)$. Then, we can easily show that equation (5.4) has a unique solution $c_z^1(t; \mu)$ with $c_z^1(0; \mu) = z_1$ and $(c_z^1(t; \mu), \alpha_2(t), ..., \alpha_n(t)) \in U$ for all $t \in [0, 1]$ if $\mu = 0$, and thus the same result holds for all μ such that $|\mu|$ is sufficiently small. Define $u(z) = c_z^1(1; 0)$.

We will show that for every continuously differentiable function $\beta : [t_1, t_2] \to V$ such that $g(\beta(t)) \cdot \dot{\beta}(t) = 0$ for all $t \in [t_1, t_2]$, $u(\beta(t_1)) = u(\beta(t_2))$. Suppose on the contrary that there exists a continuously differentiable function $\beta : [t_1, t_2] \to V$ such that $g(\beta(t)) \cdot \dot{\beta}(t) = 0$ for all $t \in [t_1, t_2]$ and $u(\beta(t_1)) \neq u(\beta(t_2))$. We can assume without loss of generality that $t_1 = 1, t_2 = 2$ and $u(\beta(1)) > u(\beta(2))$. Define $\beta_1(t; \mu)$ as the unique solution of (5.4) such that $\alpha = \beta$ and $c(1; \mu) = \beta_1(1)$. Note that by definition, we have $\beta_1(t; 0) = \beta_1(t)$ for all $t \in [1, 2]$, and thus for every sufficiently small $\mu > 0$, $\beta_1(t; \mu)$ can be defined on $[1, 2]$. Define $\beta_i(t; \mu) = \beta_i(t)$ for all $t \in [1, 2]$, $i \in \{2, ..., n\}$ and $\mu > 0$. Then, for sufficiently small $\mu > 0$, we have $\beta(t; \mu) \in V$ for every $t \in [1, 2]$. Next, define $\alpha_i(t) = (1-t)y_i + t\beta_i(1)$ for all $i \in \{2, ..., n\}$, and define $q_{\beta(1)}(t; \mu) = c^1_{\beta(1)}(1 - t; -\mu)$. Again, $q_{\beta(1)}(\cdot; 0)$ can be defined on $[0, 1]$ and $(q_{\beta(1)}(t; 0), \alpha_2(t), ..., \alpha_n(t)) \in U$ for all $t \in [0, 1]$, and thus $q_{\beta(1)}(\cdot; \mu)$ also have the same properties for every sufficiently small $\mu > 0$. Define $\gamma(t; \mu) = (t-2)y + (3-t)\beta(2; \mu)$ and

$$\eta(t; \mu) = \begin{cases} (q_{\beta(1)}(t; \mu), \alpha_2(t), ..., \alpha_n(t)) & \text{if } t \in [0, 1], \\ \beta((t-1)2 + (2-t)1; \mu) & \text{if } t \in [1, 2], \\ (c^1_{\beta(2;\mu)}(t-2; \mu), \gamma_2(t), ..., \gamma_n(t)) & \text{if } t \in [2, 3], \\ (4-t)\eta(3; \mu) + (t-3)\eta(0; \mu) & \text{if } t \in [3, 4]. \end{cases}$$

Then, $t \mapsto \eta(t; \mu)$ is a piecewise C^1 function from $[0, 4]$ into U for all sufficiently small $\mu > 0$. Because $\eta_1(0; 0) = u(\beta(1)) > u(\beta(2)) = \eta_1(3; 0)$, we have $\eta_1(0; \mu) > \eta_1(3; \mu)$ for sufficiently small $\mu > 0$. Then,

$$g(\eta(t; \mu)) \cdot \dot{\eta}(t; \mu) = \mu > 0$$

for all $t \in [0, 3]$ except $t = 0, 1, 2, 3$ and

$$g(\eta(t; \mu)) \cdot \dot{\eta}(t; \mu) = (\eta_1(0; \mu) - \eta_1(3, \mu)) > 0,$$

for all $t \in]3, 4[$. Thus, $\eta(\cdot; \mu)$ is a Ville cycle on U, a contradiction.

Now, by definition and known results on ordinary differential equations, we must have that u is continuously differentiable. Choose any $z \in V$ such that $z_2 = y_2, ..., z_n = y_n$. Then, $u(z) = z_1$, and thus $\frac{\partial u}{\partial z_1}(z) = 1$. Hence, there is an open neighborhood $W \subset V$ of y such that $\frac{\partial u}{\partial z_1}(z) > 0$ for all $z \in W$. Choose any $z \in W$ and $i \in \{2, ..., n\}$ and for $j \in \{1, ..., n\}$, define

$$\beta_j(t) = \begin{cases} z_j & \text{if } j \neq i, \\ z_j + t & \text{if } j = i. \end{cases}$$

Let $d_1(t)$ be the solution of (5.4) for $\alpha = \beta$ and $\mu = 0$. Because

$$g(d_1(t), \beta_2(t), ..., \beta_n(t)) \cdot \frac{d}{dt}(d_1(t), \beta_2(t), ..., \beta_n(t)) = 0,$$

we have

$$\left.\frac{d}{dt}u(d_1(t), \beta_2(t), ..., \beta_n(t))\right|_{t=0} = 0,$$

and thus,

$$-\frac{\partial u}{\partial z_1}(z)g_i(z) + \frac{\partial u}{\partial z_i}(z) = 0.$$

Hence, if we define $\lambda(z) = \frac{\partial u}{\partial z_1}(z)$, then

$$Du(z) = \lambda(z)g(z)$$

for all $z \in W$.

By implicit function theorem, there exists a continuously differentiable function $w(z_2, ..., z_n)$ such that

$$w(y_2, ..., y_n) = y_1,\ u(w(z_2, ..., z_n), z_2, ..., z_n) = u(y),$$

for every $(z_2, ..., z_n)$ that is sufficiently near to $(y_2, ..., y_n)$. By the above result, w satisfies the following partial differential equation (where $\tilde{z} = (z_2, ..., z_n)$):

$$Dw(\tilde{z}) = -(g_2(w(\tilde{z}), \tilde{z}), ..., g_n(w(\tilde{z}), \tilde{z})).$$

Because g is continuously differentiable, we have w is twice continuously differentiable, and thus $D^2w(\tilde{y})$ is symmetric. Therefore, for all $i, j \in \{2, ..., n\}$,

$$\frac{\partial g_i}{\partial z_j}(y) - \frac{\partial g_i}{\partial z_1}(y)g_j(y) = \frac{\partial g_j}{\partial z_i}(y) - \frac{\partial g_j}{\partial z_1}(y)g_i(y). \tag{5.5}$$

We will show that (5.5) implies Jacobi's integrability condition (5.3). Choose any $i, j, k \in \{1, ..., n\}$. If $k = 1$, then (5.5) is equivalent to (5.3). Thus, suppose that $i, j, k \in \{2, ..., n\}$. If two of i, j, k are the same, then (5.3) is trivial, and thus we can assume that $i \neq j \neq k \neq i$. By (5.5), we have

$$\frac{\partial g_i}{\partial z_j}(y) - \frac{\partial g_i}{\partial z_1}(y)g_j(y) = \frac{\partial g_j}{\partial z_i}(y) - \frac{\partial g_j}{\partial z_1}(y)g_i(y),$$
$$\frac{\partial g_j}{\partial z_k}(y) - \frac{\partial g_j}{\partial z_1}(y)g_k(y) = \frac{\partial g_k}{\partial z_j}(y) - \frac{\partial g_k}{\partial z_1}(y)g_j(y),$$
$$\frac{\partial g_k}{\partial z_i}(y) - \frac{\partial g_k}{\partial z_1}(y)g_i(y) = \frac{\partial g_i}{\partial z_k}(y) - \frac{\partial g_i}{\partial z_1}(y)g_k(y).$$

Therefore,

$$g_i(y)\left(\frac{\partial g_j}{\partial z_k}(y)-\frac{\partial g_k}{\partial z_j}(y)\right)=g_i(y)g_k(y)\frac{\partial g_j}{\partial z_1}(y)-g_i(y)g_j(y)\frac{\partial g_k}{\partial z_1}(y),$$
$$g_j(y)\left(\frac{\partial g_k}{\partial z_i}(y)-\frac{\partial g_i}{\partial z_k}(y)\right)=g_j(y)g_i(y)\frac{\partial g_k}{\partial z_1}(y)-g_j(y)g_k(y)\frac{\partial g_i}{\partial z_1}(y),$$
$$g_k(y)\left(\frac{\partial g_i}{\partial z_j}(y)-\frac{\partial g_j}{\partial z_i}(y)\right)=g_k(y)g_j(y)\frac{\partial g_i}{\partial z_1}(y)-g_k(y)g_i(y)\frac{\partial g_j}{\partial z_1}(y).$$

Summing up these equations, we have that (5.3) holds. Therefore, (SBA) holds. This completes the proof. □

As its corollary, we can obtain the following result.

Theorem 5.7. *Suppose that f is a continuously differentiable CoD that satisfies homogeneity of degree zero, Walras' law, and (R). Let $f(p,m)=x\in\mathbb{R}^n_{++}$. Then, S_f is symmetric around (p,m) if and only if for every continuously differentiable local inverse demand function g around x, there exists an open neighborhood U of x such that the restriction g_U of g to U satisfies (VA). Particularly, if the range of f is inclced in $\mathbb{R}^n_{++}$, then (S) is equivalent to the absence of the local inverse demand function that violates (VA).*

Proof. . We need three lemmas.

Lemma 5.8. *If f is a continuously differentiable CoD that is homogeneous of degree zero, then $s_{ij}(p,m)$ is homogeneous of degree -1. That is, for every $a>0$,*

$$s_{ij}(ap,am)=a^{-1}s_{ij}(p,m).$$

Proof. It suffices to show that $\frac{\partial f_i}{\partial p_j}$ and $\frac{\partial f_i}{\partial m}$ are homogeneous of degree -1. Let e_j be the j-th unit vector in $\mathbb{R}^n$. Then, for $a>0$,

$$\begin{aligned}\frac{\partial f_i}{\partial p_j}(ap,am)&=\lim_{t\to0}\frac{f_i(ap+te_j,am)-f_i(ap,am)}{t}\\&=\lim_{t\to0}\frac{f_i(p+a^{-1}te_j)-f_i(p,m)}{t}=a^{-1}\frac{\partial f_i}{\partial p_j}(p,m),\\\frac{\partial f_i}{\partial m}(ap,am)&=\lim_{t\to0}\frac{f_i(ap,am+t)-f_i(ap,am)}{t}\\&=\lim_{t\to0}\frac{f_i(p,m+a^{-1}t)-f_i(p,m)}{t}=a^{-1}\frac{\partial f_i}{\partial m}(p,m),\end{aligned}$$

as desired. □

Lemma 5.9. *If f is a continuously differentiable CoD that is homogeneous of degree zero and satisfies Walras' law, then*

$$p^TS_f(p,m)=0^T,\ \ S_f(p,m)p=0.$$

Proof. By Walras' law,

$$p^TD_pf(p,m)+f^T(p,m)=0^T,\ \ p^TD_mf(p,m)=1.$$

Therefore,

$$p^T S_f(p,m) = p^T D_p f(p,m) + p^T D_m f(p,m) f^T(p,m) = 0^T.$$

By homogeneity of degree zero,

$$\frac{d}{da} f(ap, am)\Big|_{a=1} = D_p f(p,m)p + D_m f(p,m)m = 0,$$

and by Walras' law,

$$S_f(p,m)p = D_p f(p,m)p + D_m f(p,m)m = 0,$$

as desired. □

Lemma 5.10. *Suppose that f is a continuously differentiable CoD that is homogeneous of degree zero and satisfies Walras' law, and $x = f(p,m) \in \mathbb{R}^n_{++}$. Then, the rank of $S_f(q,w)$ is $n-1$ for every (q,w) in some neighborhood of (p,m) if and only if there exists a continuously differentiable local inverse demand function $g : U \to \mathbb{R}^n_{++}$ of f around x and an open neighborhood V of (p,m) such that $g_1(y) \equiv 1$, and if $(q,w) \in V$, then $y = f(q,w) \in U$ and $(g(y), g(y) \cdot y)$ is proportional to (q,w).*

Proof. First, suppose that such g exists. For every $y \in U$, define

$$a_{ij}(y) = \frac{\partial g_{i+1}}{\partial z_{j+1}}(y) - \frac{\partial g_{i+1}}{\partial z_1}(y) g_{j+1}(y),$$

for all $i, j \in \{1, ..., n-1\}$, and let $A_g(y)$ be the $(n-1) \times (n-1)$ matrix whose (i,j)-th element is $a_{ij}(y)$.

Choose any $(q,w) \in V$ and let $y = f(q,w)$ and $\hat{S}$ be a matrix whose first column is $\frac{\partial f}{\partial m}(g(y), g(y) \cdot y)$ and whose i-th column is the same as the i-th column of $S_f(g(y), g(y) \cdot y)$ for $i \in \{2, ..., n\}$. Also, let F be a matrix whose first column is $\frac{\partial f}{\partial m}(g(y), g(y) \cdot y)$ and whose i-th column is the same as the i-th column of $D_p f(g(y), g(y) \cdot y)$ for $i \in \{2, ..., n\}$. Moreover, define[17]

$$H = \begin{pmatrix} \frac{\partial g_2}{\partial z_2} & \cdots & \frac{\partial g_2}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial z_2} & \cdots & \frac{\partial g_n}{\partial z_n} \end{pmatrix},$$

$$b = \left(\frac{\partial g_2}{\partial z_1}, ..., \frac{\partial g_n}{\partial z_1}\right)^T,$$

$$\hat{g} = (g_2, ..., g_n)^T,$$

$$c = \left(\frac{\partial}{\partial z_2}[g(z) \cdot z], ..., \frac{\partial}{\partial z_n}[g(z) \cdot z]\right)\Big|_{z=y},$$

$$\hat{f} = (f_2, ..., f_n)^T.$$

[17]Hereafter, we frequently abbreviate variables of functions.

To differentiate $z = f(g(z), g(z)\cdot z)$ by z at $z = y$, we have

$$\begin{aligned} I_n &= D_p f(g(y), g(y)\cdot y)Dg(y) + D_m f(g(y), g(y)\cdot y)D[g(z)\cdot z]|_{z=y} \\ &= F \times \begin{pmatrix} \frac{\partial}{\partial y_1}[g(z)\cdot z]\Big|_{z=y} & c \\ b & H \end{pmatrix}. \end{aligned}$$

Therefore, F is regular, and

$$\begin{aligned} F^{-1} &= \begin{pmatrix} \frac{\partial}{\partial x_1}[g(z)\cdot z]\Big|_{z=y} & c \\ b & H \end{pmatrix} \\ &= \begin{pmatrix} 1 & \hat{f}^T \\ 0 & I_{n-1} \end{pmatrix} \times \begin{pmatrix} 1 & \hat{g}^T \\ b & H \end{pmatrix} \\ &= \begin{pmatrix} 1 & \hat{f}^T \\ 0 & I_{n-1} \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ b & A_g \end{pmatrix} \times \begin{pmatrix} 1 & \hat{g}^T \\ 0 & I_{n-1} \end{pmatrix}. \end{aligned}$$

Hence,

$$\begin{aligned} \hat{S} &= F \times \begin{pmatrix} 1 & \hat{f}^T \\ 0 & I_{n-1} \end{pmatrix} \\ &= \begin{pmatrix} 1 & \hat{g}^T \\ 0 & I_{n-1} \end{pmatrix}^{-1} \times \begin{pmatrix} 1 & 0 \\ b & A_g \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 & -\hat{g}^T \\ 0 & I_{n-1} \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -(A_g)^{-1}b & (A_g)^{-1} \end{pmatrix}. \end{aligned}$$

Therefore, the last $n-1$ columns of $S_f(g(y), g(y)\cdot y)$ are linearly independent, and thus the rank of $S_f(g(y), g(y)\cdot y)$ is $n-1$ by lemma 5.9. Because of lemma 5.8, the rank of $S_f(q, w)$ is also $n-1$. Hence, the rank of S_f is $n-1$ on V, which completes the proof of 'if' part.

To prove 'only if' part, note that because the rank of $S_f(p, m)$ is $n-1$, there exists i such that $s_1(p, m), ..., s_{i-1}(p, m), s_{i+1}(p, m), ..., s_n(p, m)$ are linearly independent, where $s_j(p, m)$ is the j-th column of $S_f(p, m)$. In this case, we prove that there exists a continuously differentiable local inverse demand function $h : U \to \mathbb{R}^n_{++}$ and an open neighborhood $V \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++}$ of (p, m) such that $h_i(y) \equiv 1$ and if $(q, w) \in V$, then $y = f(q, w) \in U$ and $(h(y), h(y)\cdot y)$ is proportional to (q, w). Then, $g(y) = \frac{1}{h_1(y)}h(y)$ and V satisfies the claim of this lemma.

We treat the case in which $i = 1$, because the remaining cases can be treated symmetrically. Define for $\tilde{q} = (q_2, ..., q_n) \in \mathbb{R}^{n-1}_{++}$ and $w > 0$,

$$\hat{f}(w, \tilde{q}) = f(1, \tilde{q}, w).$$

Then, $D\hat{f}(w, \tilde{q}) = F$, where F is defined as above. Because

$$\hat{S} = F \times \begin{pmatrix} 1 & \hat{f}^T \\ 0 & I_{n-1} \end{pmatrix},$$

we have that $D\hat{f}$ is regular at $(\bar{m}, \bar{p}) = \frac{1}{p_1}(m, \tilde{p})$.[18] Therefore, by inverse function theorem, there exists an open neighborhood U of x and an open neighborhood W

[18]Note that by lemma 5.9, we can show that $\hat{S}$ is regular.

of $(\bar{m}, \bar{p})$ such that $\hat{f}$ is a bijection from W onto U. Because of the definition of W, there exists an open neighborhood V of (p, m) such that for every $(q, w) \in V$, $\frac{1}{q_1}(\tilde{q}, w) \in W$. Define $h_1(y) = 1$ and $h_j(y) = (\hat{f})_j^{-1}(y)$ for $j \in \{2, ..., n\}$. Then, these $h : U \to \mathbb{R}^n_{++}$ and V satisfy all requirements of our claim. This completes the proof. □

Now, we will prove theorem 5.7. Suppose that $x = f(p, m) \in \mathbb{R}^n_{++}$ and S_f is symmetric on some neighborhood of (p, m). Let $h : U' \to \mathbb{R}^n_{++}$ be a continuously differentiable local inverse demand function of f around x, and define $g(y) = \frac{1}{h_1(y)} h(y)$. Let $U \subset U'$ be an open neighborhood of x such that $S_f(g(y), g(y) \cdot y)$ is symmetric for every $y \in U$. By repeating the proof of lemma 5.10, we have that $A_g(y)$ is the inverse matrix of the matrix $(s_{ij}(g(y), g(y) \cdot y))_{i,j=2}^n$, and thus symmetric. This implies that the condition (5.5) in the proof of theorem 5.6 holds, and as in the proof of theorem 5.6, we have that the restriction $h_U : U \to \mathbb{R}^n_{++}$ of h to U satisfies (SBA), and therefore it satisfies (VA).

To prove the converse, suppose that $g : U \to \mathbb{R}^n_{++}$ is a continuously differentiable local inverse demand function of f around x and $V \subset \mathbb{R}^n_{++} \times \mathbb{R}_{++}$ is an open neighborhood of (p, m) such that $g_1(y) \equiv 1$, and if $(q, w) \in V$, then $y = f(q, w) \in U$ and $(g(y), g(y) \cdot y)$ is proportional to (q, w).[19] If g satisfies (VA), then it satisfies (SBA) by theorem 5.6, and thus $A_g(y)$ is symmetric. This implies that $(s_{ij}(q, w))_{i,j=2}^n$ is also symmetric on V. By lemma 5.9,

$$s_{1i}(q, w) = -\sum_{j=2}^{n} \frac{q_j}{q_1} s_{ji}(q, w) = -\sum_{j=2}^{n} \frac{q_j}{q_1} s_{ij}(q, w) = s_{i1}(q, w),$$

for all $i \in \{2, ..., n\}$, and thus $S_f(q, w)$ is symmetric on V.

Finally, suppose that the range of f is included in $\mathbb{R}^n_{++}$. If there is no continuously differentiable local inverse demand function of f that violates (VA), then for every (p, m), $S_f(p, m)$ is symmetric by the above arguments, which implies that f satisfies (S). Conversely, suppose that f satisfies (S), and $h : U \to \mathbb{R}^n_{++}$ is a continuously differentiable local inverse demand function of f. Define $g(y) = \frac{1}{h_1(y)} h(y)$ for every $y \in U$. Then, by repeating the above arguments, we can show that $A_g(y)$ is symmetric for every $y \in U$. Therefore, h satisfies (SBA), and thus it satisfies (VA). This completes the proof. □

Remark 5.11. Theorem 5.6 was first claimed by Ville [24], and then proved by Hurwicz and Richter [11, 12]. However, the existence of the local inverse demand function had not been researched until Hosoya [7]. Therefore, theorem 5.7 is a new result. Hosoya [7] also showed that if f satisfies homogeneity of degree zero, Walras' law, (R), and (WA), and the range of f is $\mathbb{R}^n_{++}$, then there exists the unique inverse demand function that satisfies $g_n(x) \equiv 1$.

Meanwhile, Hurwicz and Richter [11, 12] claimed that if there is no Ville cycle that is C^∞, then (SBA) holds. However, the author could not verify that this claim is correct. They used the Weierstrass's approximation theorem for η. However, to the best understanding of the author, Weierstrass's theorem can be used for only

[19] The existence of such g and V is assured by lemma 5.10.

the uniform topology, and thus the approximated cycle need not to be a Ville cycle. Thus, whether their claim is correct remains open.

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