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SOLUTION OF A ZERO-SUM LINEAR-QUADRATIC DIFFERENTIAL GAME WITH STATE-DEPENDENT IMPULSE IN DYNAMICS

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Abstract. A zero-sum linear-quadratic differential game is considered. The dynamics of this game is impulsive, and the value of the impulse depends on its momentum and the value of the game's state variable at this momentum. The momentum of the impulse is not given, and it is considered as an additional control of the minimizing player (the minimizer). A minimum guaranteed outcome of the game is derived and a minimum guaranteeing minimizer's strategy is designed. These results are applied to solution of a planar pursuit-evasion game with pursuer's hybrid dynamics.

1. Introduction

Control problems with impulsive dynamics were studied extensively in the open literature. Mostly, the problems with one decision maker were analyzed (see e.g., [1, 3, 5, 9, 15, 18] and references therein). Differential games with impulsive dynamics were investigated much less. Thus, in [17], a zero-sum differential game with an impulsive control of the minimizing player (the minimizer) was studied. The right-hand side of the differential equation in this game is a sum of two terms. One of these terms is a function of the game's state variable, the non-impulsive control of the maximizing player and the independent variable (time). The second term is the control of the minimizer. Based on the approach of viscosity solutions for Hamilton-Jacobi equations, the existence of the game's value was established. In [2], some impulsive evolutionary games were solved by conditional viability approach. In [4], a two-person zero-sum game with separated impulsive time-invariant dynamics was considered. The game was analyzed applying the concepts of differential inclusion and viability. In [16], the differential game with a scalar impulsive dynamics, arising in the population biology, was considered. In this game, the impulses momentums and the values of impulses are given. Some periodicity and stability properties of the solution to this game were studied. In [7], bilevel qualitative games with impulsive dynamics, described by measure differential equations, were considered. The case of fixed impulses and the case of impulsive control of the lower level player were analyzed. Some invariance properties of the games' solutions were established.

In the present paper, a zero-sum linear-quadratic differential game with impulsive dynamics is considered. The impulsive nature of the dynamics is due to the presence of the Dirac delta-function in the right-hand side of the game's differential equation.

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This delta-function multiplies the state variable of the game with a continuous coefficient, meaning that the impulse depends on the game's state variable. Moreover, the momentum of the delta-function's impulse is not specified. Thus, the impulse in the dynamics of the game also depends on this momentum. In this paper, we consider the impulse momentum as an additional control of the minimizing player (the minimizer). Minimum guaranteed outcome of the considered game is derived. Minimum guaranteeing minimizer's strategy is designed. To obtain these results, a matrix differential Riccati equation with a state-dependent impulse is considered. We solve this equation, based on the definition of the solution to a nonlinear differential equation with a state-dependent impulse, proposed in [6]. Using the solution of the Riccati equation, the above mentioned minimum guaranteed game's outcome and minimum guaranteeing minimizer's strategy are obtained.

The theoretical results of this paper are applied to solution of a planar pursuit-evasion differential game with hybrid pursuer's dynamics. In [10, 11, 12, 14], various given duration planar pursuit-evasion games with bounded controls of the players and hybrid dynamics of one/both players were studied. In these works, the cost functional, to be minimized by the pursuer and maximized by the evader, is the miss distance (the final distance between the players). The solutions of these games yield the optimal order of the possible dynamics, the optimal switch time momentum from one dynamics to another and the optimal feedback bang-bang control for each player. However, such a type of the feedback control produces as a rule sliding modes, resulting in a control chattering which is extremely undesirable in practical implementations. In the present paper, we consider the planar pursuit-evasion in the frame of a given duration linear-quadratic differential game. The cost functional in this game is a sum of two addends. The first one is the square of the miss distance. The second addend is the integral of the linear combination of the squares of the players' controls with a positive coefficient for the pursuer and a negative coefficient for the evader. There are no geometric constraints imposed on the players' controls. The pursuer's dynamics is hybrid. Namely, the pursuer can change a parameter of its dynamics (the time constant) once during the game using a given set of two possible values of this parameter. Time momentum of such a changing is an additional pursuer's control. The objective of the pursuer is to minimize the cost functional, while the evader tries to maximize it. This game is reduced to a game of a lower dimension dynamics with a state-dependent impulse. Momentum of this impulse is the switch momentum from one value of the time constant to the other in the pursuer dynamics. For this game, the minimum guaranteed outcome and the minimum guaranteeing pursuer's strategy are obtained. This strategy consists of the switch momentum and the state-feedback pursuer's control. The latter is a linear function of the state variables. Therefore, it does not yield a sliding mode and is chattering-free. In a real-life situation, such a control allows to decrease the pursuer's control expenditure and to avoid a damage to the pursuer during the game.

2. Problem statement

The dynamics of the game is described by the following system

$$\frac{dx(t)}{dt} = A_1(t)x(t) + \big[A_{21}(t) + A_{22}(t)\delta(t - t_{\rm im})\big]y(t)$$
$$+B_{\rm u,1}(t)u(t) + B_{\rm v,1}(t)v(t), \quad x(0) = x_0, \tag{2.1}$$

$$\frac{dy(t)}{dt} = A_3(t)y(t) + B_{\rm u,2}(t)u(t), \quad y(0) = y_0, \tag{2.2}$$

where $t \in [0, t_f]$, ($t_f > 0$ is a given time instant); $x(t) \in E^n$ and $y(t) \in E^m$ are state variables; $u(t) \in E^r$ and $v(t) \in E^s$ are controls of the players; $\delta(t - t_{\rm im})$ is the Dirac delta-function with the impulse momentum at $t = t_{\rm im}$, ($t_{\rm im} \in (0, t_f)$), which is not given in advance; $A_1(t)$, $A_{21}(t)$, $A_{22}(t)$, $A_3(t)$, $B_{\rm u,1}(t)$, $B_{\rm v,1}(t)$, $B_{\rm u,2}(t)$ are given matrices of corresponding dimensions; $x_0 \in E^n$ and $y_0 \in E^m$ are given vectors; E^q is the q-dimensional real Euclidean space.

The cost functional of the game is

$$J\big(u, v, t_{\rm im}\big) = z^T(t_f)Fz(t_f) + \int_0^{t_f} \big[z^T(t)D(t)z(t)$$
$$+u^T(t)R_{\rm u}(t)u(t) - v^T(t)R_{\rm v}(t)v(t)\big]dt, \tag{2.3}$$

where $z(t) = \mathrm{col}\big(x(t), y(t)\big)$; F, $D(t)$, $R_{\rm u}(t)$, $R_{\rm v}(t)$ are given symmetric matrices of corresponding dimensions, and $F \geq 0$, $D(t) \geq 0$, $R_{\rm u}(t) > 0$, $R_{\rm v}(t) > 0$, $t \in [0, t_f]$.

The cost functional (2.3) is minimized by a proper choice of the pair $\big(t_{\rm im}, u\big)$ and it is maximized by a proper choice of v.

We solve the game (2.1)-(2.2),(2.3) from the minimizing player (the minimizer) viewpoint. Namely, we look for the minimum guaranteed game outcome in the form:

$$J_{\rm u} \overset{\triangle}{=} \min_{t_{\rm im} \in (0, t_f)} \min_u \max_v J\big(u, v, t_{\rm im}\big), \tag{2.4}$$

where the players' controls are of state-feedback forms $u = u(x, y, t)$, $v = v(x, y, t)$. The classes of the controls $v(x, y, t)$ and $u(x, y, t)$, in which the corresponding maximization and minimization are carried out, are defined in the next section.

3. Main definitions

In what follows, we assume:

A1. The matrix $A_{22}(t)$ is continuous in the interval $[0, t_f]$, while the other time-dependent matrices of the coefficients in the game (2.1)-(2.2),(2.3) are piecewise continuous in the interval $[0, t_f]$.

Let us denote $z \overset{\triangle}{=} \mathrm{col}(x, y)$, $x \in E^n$, $y \in E^m$; $z_0 \overset{\triangle}{=} \mathrm{col}(x_0, y_0)$, $x_0 \in E^n$, $y_0 \in E^m$.

Consider the set $\mathcal{U}$ of all functions $u = u(z, t) : E^{n+m} \times [0, t_f] \to E^r$, which are measurable w.r.t. $t \in [0, t_f]$ for any fixed $z \in E^{n+m}$ and satisfy the local Lipschitz condition w.r.t. $z \in E^{n+m}$ uniformly in $t \in [0, t_f]$. Also, consider the set $\mathcal{V}$ of all functions $v = v(z, t) : E^{n+m} \times [0, t_f] \to E^s$ with the same properties.

For any given $t_{\rm im} \in (0, t_f)$, $u(z,t) \in \mathcal{U}$ and $v(z,t) \in \mathcal{V}$, consider the following two initial-value problems:

$$\begin{aligned} \frac{dx_1(t)}{dt} &= A_1(t)x_1(t) + A_{21}(t)y_1(t) \\ &\quad + B_{\rm u,1}(t)u(z_1(t),t) + B_{\rm v,1}(t)v(z_1(t),t), \quad x_1(0) = x_0, \end{aligned} \tag{3.1}$$

$$\frac{dy_1(t)}{dt} = A_3(t)y_1(t) + B_{\rm u,2}(t)u(z_1(t),t), \quad y_1(0) = y_0, \tag{3.2}$$

where $t \in [0, t_{\rm im}]$, $z_1(t) = {\rm col}\big(x_1(t), y_1(t)\big)$,
and

$$\begin{aligned} \frac{dx_2(t)}{dt} &= A_1(t)x_2(t) + A_{21}(t)y_2(t) \\ &\quad + B_{\rm u,1}(t)u(z_2(t),t) + B_{\rm v,1}(t)v(z_2(t),t), \\ x_2(t_{\rm im}) &= x_1(t_{\rm im} - 0) + A_{22}(t_{\rm im})y_1(t_{\rm im} - 0), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{dy_2(t)}{dt} &= A_3(t)y_2(t) + B_{\rm u,2}(t)u(z_2(t),t), \\ y_2(t_{\rm im}) &= y_1(t_{\rm im} - 0), \end{aligned} \tag{3.4}$$

where $t \in [t_{\rm im}, t_f]$, $z_2(t) = {\rm col}\big(x_2(t), y_2(t)\big)$.

Definition 3.1. For any given $t_{\rm im} \in (0, t_f)$, $u(z,t) \in \mathcal{U}$ and $v(z,t) \in \mathcal{V}$, the following pair of function $\big(x(t; t_{\rm im}), y(t; t_{\rm im})\big)$, $t \in [0, t_f]$ is called the solution of the initial-value problem (2.1)-(2.2) with $u(t) = u(z,t)$, $v(t) = v(z,t)$:

$$\big(x(t; t_{\rm im}), y(t; t_{\rm im})\big) = \begin{cases} \big(x_1(t; t_{\rm im}), y_1(t; t_{\rm im})\big), & t \in [0, t_{\rm im}), \\ \big(x_2(t; t_{\rm im}), y_2(t; t_{\rm im})\big), & t \in [t_{\rm im}, t_f], \end{cases} \tag{3.5}$$

where $\big(x_1(t; t_{\rm im}), y_1(t; t_{\rm im})\big)$, $t \in [0, t_{\rm im}]$ and $\big(x_2(t; t_{\rm im}), y_2(t; t_{\rm im})\big)$, $t \in [t_{\rm im}, t_f]$ are the solutions of the initial-value problems (3.1)-(3.2) and (3.3)-(3.4), respectively.

Definition 3.2. By UV, we denote the set of all pairs $\big(u(z,t), v(z,t)\big)$ such that the following conditions are valid:

(i) $u(z,t) \in \mathcal{U}$, $v(z,t) \in \mathcal{V}$;
(ii) the initial-value problem (2.1)-(2.2) for $u(t) = u(z,t)$, $v(t) = v(z,t)$ and any $z_0 \in E^{n+m}$, $t_{\rm im} \in (0, t_f)$ has the unique piecewise continuous solution $z_{\rm uv}(t; z_0, t_{\rm im}) = {\rm col}\big(x_{\rm uv}(t; z_0, t_{\rm im}), y_{\rm uv}(t; z_0, t_{\rm im})\big)$ in the interval $[0, t_f]$;
(iii) $u\big(z_{\rm uv}(t; z_0, t_{\rm im}), t\big) \in L^2[0, t_f; E^r]$;
(iv) $v\big(z_{\rm uv}(t; z_0, t_{\rm im}), t\big) \in L^2[0, t_f; E^s]$.

The set UV is called a set of all admissible pairs of the players' state-feedback controls in the game (2.1)-(2.2),(2.3).

For a given $u(z,t) \in \mathcal{U}$, consider the sets $\mathcal{F}_{\rm v}\big(u(z,t)\big) \stackrel{\triangle}{=} \{v(z,t) \in \mathcal{V} : \big(u(z,t), v(z,t)\big) \in UV\}$ and $\mathcal{H}_{\rm u} \stackrel{\triangle}{=} \{u(z,t) \in \mathcal{U} : \mathcal{F}_{\rm v}\big(u(z,t)\big) \neq \emptyset\}$.

Now, we can rewrite the equation (2.4) in the following more precise form.

Definition 3.3. The value

$$J_{\mathrm{u}}^* \stackrel{\triangle}{=} \inf_{t_{\mathrm{im}}\in(0,t_f)} \inf_{u\in\mathcal{H}_{\mathrm{u}}} \sup_{v\in\mathcal{F}_{\mathrm{v}}(u)} J\big(u, v, t_{\mathrm{im}}\big) \tag{3.6}$$

is called the minimum guaranteed outcome in the game (2.1)-(2.2),(2.3). If there exist a value $t_{\mathrm{im}}^* \in (0, t_f)$ and a function $u^*(z,t) \in \mathcal{H}_{\mathrm{u}}$ such that

$$\sup_{v\in\mathcal{F}_{\mathrm{v}}(u^*)} J\big(u^*(z,t), v, t_{\mathrm{im}}^*\big) = J_{\mathrm{u}}^*, \tag{3.7}$$

then the pair $\big(t_{\mathrm{im}}^*, u^*(z,t)\big)$ is called a minimum guaranteeing minimizer's strategy.

In what follows, the game (2.1)-(2.2),(2.3),(3.6)-(3.7) is called the Original Differential Game (ODG).

4. Solution of the ODG

4.1. **Saddle point of the game (2.1)-(2.2),(2.3) with a given t_{im}.** Let $t_{\mathrm{im}} \in (0.t_f)$ be any given. Also, for a given $v(z,t) \in \mathcal{V}$, consider the sets $\mathcal{F}_u\big(v(z,t)\big) \stackrel{\triangle}{=} \{u(z,t) \in \mathcal{U} : \big(u(z,t), v(z,t)\big) \in UV\}$ and $\mathcal{H}_v \stackrel{\triangle}{=} \{v(z,t) \in \mathcal{V} : \mathcal{F}_u\big(v(z,t)\big) \neq \emptyset\}$.

If the following equality is fulfilled:

$$\sup_{v\in\mathcal{H}_{\mathrm{v}}} \inf_{u\in\mathcal{F}_{\mathrm{u}}(v)} J(u, v, t_{\mathrm{im}}) = \inf_{u\in\mathcal{H}_{\mathrm{u}}} \sup_{v\in\mathcal{F}_{\mathrm{v}}(u)} J(u, v, t_{\mathrm{im}}) \tag{4.1}$$

in the game (2.1)-(2.2),(2.3) with any given t_{im}, then

$$J^0(t_{\mathrm{im}}) \stackrel{\triangle}{=} \sup_{v\in\mathcal{H}_{\mathrm{v}}} \inf_{u\in\mathcal{F}_{\mathrm{u}}(v)} J(u, v, t_{\mathrm{im}}) = \inf_{u\in\mathcal{H}_{\mathrm{u}}} \sup_{v\in\mathcal{F}_{\mathrm{v}}(u)} J(u, v, t_{\mathrm{im}}) \tag{4.2}$$

is called the value of this game, while a pair $\big(u^0(z,t), v^0(z,t)\big)$, for which

$$J\big(u^0(z,t), v^0(z,t), t_{\mathrm{im}}\big) = J^0(t_{\mathrm{im}}), \tag{4.3}$$

is called a state-feedback saddle point in this game.

4.1.1. *Matrix differential Riccati equation with impulse.* Consider the block matrices $A(t, t_{\mathrm{im}})$, $B_{\mathrm{u}}(t)$ and $B_{\mathrm{v}}(t)$ of the dimensions $(n+m)\times(n+m)$, $(n+m)\times r$ and $(n+m)\times s$, respectively:

$$A(t, t_{\mathrm{im}}) = \begin{pmatrix} A_1(t) & A_{21}(t) + A_{22}(t)\delta(t - t_{\mathrm{im}}) \\ 0 & A_3(t) \end{pmatrix}, \tag{4.4}$$

$$B_{\mathrm{u}}(t) = \begin{pmatrix} B_{\mathrm{u},1}(t) \\ B_{\mathrm{u},2}(t) \end{pmatrix}, \quad B_{\mathrm{v}}(t) = \begin{pmatrix} B_{\mathrm{v},1}(t) \\ 0 \end{pmatrix}. \tag{4.5}$$

Using these matrices, consider the terminal-value problem for the matrix differential Riccati equation with respect to a symmetric matrix-valued function $P(t)$

$$\begin{aligned} \frac{dP(t)}{dt} &= -P(t)A(t, t_{\mathrm{im}}) - A^T(t, t_{\mathrm{im}})P(t) \\ &\quad + P(t)\big(S_{\mathrm{u}}(t) - S_{\mathrm{v}}(t)\big)P(t) - D(t), \quad P(t_f) = F, \end{aligned} \tag{4.6}$$

where $t \in [0, t_f]$,

$$S_{\rm u}(t) = B_{\rm u}(t)R_{\rm u}^{-1}(t)B_{\rm u}^T(t) = \begin{pmatrix} S_{\rm u,1}(t) & S_{\rm u,2}(t) \\ S_{\rm u,2}^T(t) & S_{\rm u,3}(t) \end{pmatrix}, \tag{4.7}$$

$$S_{\rm v}(t) = B_{\rm v}(t)R_{\rm v}^{-1}(t)B_{\rm v}^T(t) = \begin{pmatrix} S_{\rm v,1}(t) & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.8}$$

and

$$S_{\rm u,1}(t) = B_{\rm u,1}(t)R_{\rm u}^{-1}B_{\rm u,1}^T(t), \quad S_{\rm u,2}(t) = B_{\rm u,1}(t)R_{\rm u}^{-1}B_{\rm u,2}^T(t),$$
$$S_{\rm u,3}(t) = B_{\rm u,2}(t)R_{\rm u}^{-1}B_{\rm u,2}^T(t), \quad S_{\rm v,1}(t) = B_{\rm v,1}(t)R_{\rm v}^{-1}B_{\rm v,1}^T(t).$$

Let us partition the matrices $D(t)$, F and $P(t)$ into blocks as:

$$D(t) = \begin{pmatrix} D_1(t) & D_2(t) \\ D_2^T(t) & D_3(t) \end{pmatrix}, \quad F = \begin{pmatrix} F_1 & F_2 \\ F_2^T & F_3 \end{pmatrix}, \tag{4.9}$$

$$P(t) = \begin{pmatrix} P_1(t) & P_2(t) \\ P_2^T(t) & P_3(t) \end{pmatrix}, \tag{4.10}$$

where the blocks $D_1(t)$, F_1 and $P_1(t)$ are of the dimension $n \times n$, the blocks $D_2(t)$, F_2 and $P_2(t)$ are of the dimension $n \times m$, the blocks $D_3(t)$, F_3 and $P_3(t)$ are of the dimension $m \times m$.

Substitution of the block representations for the matrices $A(t, t_{\rm im})$, $S_{\rm u}(t)$, $S_{\rm v}(t)$, $D(t)$, F and $P(t)$ converts the terminal-value problem (4.6) to the following equivalent terminal-value problem for the set of three Riccati-type matrix differential equations:

$$\begin{aligned} \frac{dP_1(t)}{dt} &= -P_1(t)A_1(t) - A_1^T(t)P_1(t) \\ &+P_1(t)\big(S_{\rm u,1}(t) - S_{\rm v,1}(t)\big)P_1(t) + P_2(t)S_{\rm u,2}^T(t)P_1(t) \\ &+P_1(t)S_{\rm u,2}(t)P_2^T(t) + P_2(t)S_{\rm u,3}(t)P_2^T(t) - D_1(t), \end{aligned} \tag{4.11}$$

$$\begin{aligned} \frac{dP_2(t)}{dt} &= -P_1(t)\big[A_{21}(t) + A_{22}(t)\delta(t - t_{\rm im})\big] - P_2(t)A_3(t) \\ &-A_1^T(t)P_2(t) + P_1(t)\big(S_{\rm u,1}(t) - S_{\rm v,1}(t)\big)P_2(t) \\ &+P_2(t)S_{\rm u,2}^T(t)P_2(t) + P_1(t)S_{\rm u,2}(t)P_3(t) \\ &+P_2(t)S_{\rm u,3}(t)P_3(t) - D_2(t), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \frac{dP_3(t)}{dt} &= -P_2^T(t)\big[A_{21}(t) + A_{22}(t)\delta(t - t_{\rm im})\big] - P_3(t)A_3(t) \\ &-\big[A_{21}^T(t) + A_{22}^T(t)\delta(t - t_{\rm im})\big]P_2(t) - A_3^T(t)P_3(t) \\ &+P_2^T(t)\big(S_{\rm u,1}(t) - S_{\rm v,1}(t)\big)P_2(t) + P_3(t)S_{\rm u,2}^T(t)P_2(t) \\ &+P_2^T(t)S_{\rm u,2}(t)P_3(t) + P_3(t)S_{\rm u,3}(t)P_3(t) - D_3(t), \end{aligned} \tag{4.13}$$

$$P_1(t_f) = F_1, \quad P_2(t_f) = F_2, \quad P_3(t_f) = F_3. \tag{4.14}$$

It is clear that the terminal-value problem (4.11)-(4.13),(4.14) is equivalent to the terminal-value problem (4.6).

4.1.2. *Solution of the problem (4.6).* Consider the following $(n+m) \times (n+m)$-matrices:

$$\mathcal{A}_1(t) = \begin{pmatrix} A_1(t) & A_{21}(t) \\ 0 & A_3(t) \end{pmatrix}, \tag{4.15}$$

$$\mathcal{A}_2(t) = \begin{pmatrix} 0 & A_{22}(t) \\ 0 & 0 \end{pmatrix}. \tag{4.16}$$

Due to (4.4),(4.15)-(4.16),

$$A(t, t_{\rm im}) = \mathcal{A}_1(t) + \mathcal{A}_2(t)\delta(t - t_{\rm im}), \quad t \in [0, t_f]. \tag{4.17}$$

Using the matrix (4.15), we consider the following terminal-value problem with respect to a symmetric matrix-valued function $P_{\rm r}(t)$ in the interval $[t_{\rm im}, t_f]$:

$$\begin{aligned} \frac{dP_{\rm r}(t)}{dt} &= -P_{\rm r}(t)\mathcal{A}_1(t) - \mathcal{A}_1^T(t)P_{\rm r}(t) \\ &\quad + P_{\rm r}(t)\big(S_{\rm u}(t) - S_{\rm v}(t)\big)P_{\rm r}(t) - D(t), \quad P_{\rm r}(t_f) = F. \end{aligned} \tag{4.18}$$

Assume that this problem has the solution $P_{\rm r}(t)$, $t \in [t_{\rm im}, t_f]$.

Now, using the matrix (4.16), we consider the following initial-value problem with respect to a symmetric matrix-valued function $\Delta_P(\theta)$ in the interval $[0,1]$:

$$\begin{aligned} \frac{d\Delta_P(\theta)}{d\theta} &= -\Delta_P(\theta)\mathcal{A}_2(t_{\rm im}) - \mathcal{A}_2^T(t_{\rm im})\Delta_P(\theta), \\ \Delta_P(0) &= P_{\rm r}(t_{\rm im} + 0). \end{aligned} \tag{4.19}$$

This problem has the unique solution $\Delta_P(\theta)$, $\theta \in [0,1]$.

Let us obtain this solution. For this purpose, we partition the solutions $P_{\rm r}(t)$ and $\Delta_P(\theta)$ of the problems (4.18) and (4.19), respectively, into blocks as:

$$P_{\rm r}(t) = \begin{pmatrix} P_{{\rm r},1}(t) & P_{{\rm r},2}(t) \\ P_{{\rm r},2}^T(t) & P_{{\rm r},3}(t) \end{pmatrix}, \tag{4.20}$$

$$\Delta_P(\theta) = \begin{pmatrix} \Delta_{P,1}(\theta) & \Delta_{P,2}(\theta) \\ \Delta_{P,2}^T(\theta) & \Delta_{P,3}(\theta) \end{pmatrix}, \tag{4.21}$$

where the blocks $P_{{\rm r},1}(t)$, $\Delta_{P,1}(\theta)$ are of the dimension $n \times n$, the blocks $P_{{\rm r},2}(t)$, $\Delta_{P,2}(\theta)$ are of the dimension $n \times m$, and the blocks $P_{{\rm r},3}(t)$, $\Delta_{P,3}(\theta)$ are of the dimension $m \times m$.

Using the block representations (4.20) and (4.21), we convert the problem (4.19) to the following initial-value problem with respect to the blocks $\Delta_{P,1}(\theta)$, $\Delta_{P,2}(\theta)$

and $\Delta_{P,2}(\theta)$ of the matrix $\Delta_P(\theta)$:

$$\begin{aligned}
\frac{d\Delta_{P,1}(\theta)}{d\theta} &= 0, \quad \theta \in [0,1],\\
\frac{d\Delta_{P,2}(\theta)}{d\theta} &= -\Delta_{P,1}(\theta)A_{22}(t_{\rm im}), \quad \theta \in [0,1],\\
\frac{d\Delta_{P,3}(\theta)}{d\theta} &= -\Delta_{P,2}^T(\theta)A_{22}(t_{\rm im}) - A_{22}^T(t_{\rm im})\Delta_{P,2}(\theta), \quad \theta \in [0,1],\\
\Delta_{P,1}(0) &= P_{r,1}(t_{\rm im}+0),\\
\Delta_{P,2}(0) &= P_{r,2}(t_{\rm im}+0),\\
\Delta_{P,3}(0) &= P_{r,2}(t_{\rm im}+0).
\end{aligned} \tag{4.22}$$

Solving this problem, we directly obtain the blocks of the matrix $\Delta_P(\theta)$

$$\begin{aligned}
\Delta_{P,1}(\theta) &= P_{\rm r,1}(t_{\rm im}+0), \quad \theta \in [0,1],\\
\Delta_{P,2}(\theta) &= P_{\rm r,2}(t_{\rm im}+0) - P_{\rm r,1}(t_{\rm im}+0)A_{22}(t_{\rm im})\theta, \quad \theta \in [0,1],\\
\Delta_{P,3}(\theta) &= P_{\rm r,3}(t_{\rm im}+0)\\
&\quad -P_{\rm r,2}^T(t_{\rm im}+0)A_{22}(t_{\rm im})\theta - A_{22}^T(t_{\rm im})P_{\rm r,2}(t_{\rm im}+0)\theta\\
&\quad +A_{22}^T(t_{\rm im})P_{\rm r,1}(t_{\rm im}+0)A_{22}(t_{\rm im})\theta^2, \quad \theta \in [0,1].
\end{aligned} \tag{4.23}$$

In addition to the problems (4.18) and (4.19), we consider the following terminal-value problem with respect to a symmetric matrix-valued function $P_{\rm l}(t)$ in the interval $[0, t_{\rm im}]$:

$$\begin{aligned}
\frac{dP_{\rm l}(t)}{dt} &= -P_{\rm l}(t)\mathcal{A}_1(t) - \mathcal{A}_1^T(t)P_{\rm l}(t)\\
&\quad +P_{\rm l}(t)\big(S_{\rm u}(t) - S_{\rm v}(t)\big)P_{\rm l}(t) - D(t), \quad P_{\rm l}(t_{\rm im}) = \Delta_P(1).
\end{aligned} \tag{4.24}$$

Assume that this problem has the solution $P_{\rm l}(t)$, $t \in [0, t_{\rm im}]$.

Based on the results of [6] and using the solutions of the problems (4.18), (4.19) and (4.24), we define the solution of the problem (4.6).

Definition 4.1. For any given $t_{\rm im} \in (0, t_f)$, the following function $P(t; t_{\rm im})$ is called the solution of the terminal-value problem (4.6) in the interval $[0, t_f]$:

$$P(t; t_{\rm im}) = \begin{cases} P_{\rm r}(t), & t \in (t_{\rm im}, t_f],\\ P_{\rm l}(t), & t \in [0, t_{\rm im}]. \end{cases} \tag{4.25}$$

4.1.3. *Saddle-point solvability of the game (2.1)-(2.2),(2.3) with a given* $t_{\rm im}$.

Theorem 4.2. *Let for a given* $t_{\rm im} \in (0, t_f)$, *the terminal-value problem (4.6) has the solution (4.25). Then, for this* $t_{\rm im}$, *the equality (4.1) is fulfilled in the game (2.1)-(2.2),(2.3). The value of this game is*

$$J^0(t_{\rm im}) = z_0^T P(0; t_{\rm im}) z_0. \tag{4.26}$$

Moreover, the state-feedback saddle point of this game also exists and has the form

$$\begin{aligned} u^0(z,t) &= u^0(z,t;t_{\rm im}) \stackrel{\triangle}{=} -R_{\rm u}^{-1}(t)B_{\rm u}^T(t)P(t;t_{\rm im})z, \\ v^0(z,t) &= v^0(z,t;t_{\rm im}) \stackrel{\triangle}{=} R_{\rm v}^{-1}(t)B_{\rm v}^T(t)P(t;t_{\rm im})z, \\ & \quad z \in E^{n+m}, \quad t \in [0,t_f]. \end{aligned} \tag{4.27}$$

Proof. First of all, let us note the following. Since the vector-valued functions $u^0(z,t)$ and $v^0(z,t)$ are linear with respect to z with piecewise continuous matrix-valued coefficients, then $u^0(z,t) \in \mathcal{U}$, $v^0(z,t) \in \mathcal{V}$. Moreover, due to Definition 3.1, the initial-value problem (2.1)-(2.2) with $u(t) = u^0(z,t)$, $v(t) = v^0(z,t)$ and any z_0 has the unique solution, and this solution is a piecewise continuous vector-valued function of $t \in [0,t_f]$. Therefore, due to Definition 3.2, $\big(u^0(z,t), v^0(z,t)\big) \in UV$, and the sets $\mathcal{F}_u\big(v^0(z,t)\big)$, $\mathcal{F}_v\big(u^0(z,t)\big)$, $\mathcal{H}_u$, $\mathcal{H}_v$ are nonempty.

Let $\big(u(z,t), v(z,t)\big)$ be any given element of the set UV.

Consider the Lyapunov-like function

$$V\big(z(t;t_{\rm im}),t\big) \stackrel{\triangle}{=} z^T(t;t_{\rm im})P(t;t_{\rm im})z(t;t_{\rm im}), \tag{4.28}$$

where $z(t;t_{\rm im}) = {\rm col}\big(x(t;t_{\rm im}), y(t;t_{\rm im})\big)$; the pair $\big(x(t;t_{\rm im}), y(t;t_{\rm im})\big)$ is the solution (3.5) of the problem (2.1)-(2.2) with $u(t) = u(z,t)$, $v(t) = v(z,t)$.

The function $V\big(z(t;t_{\rm im}),t\big)$ can be rewritten in the form

$$V\big(z(t;t_{\rm im}),t\big) = \begin{cases} V_1\big(z_1(t;t_{\rm im}),t\big), & t \in [0,t_{\rm im}), \\ V_2\big(z_2(t;t_{\rm im}),t\big), & t \in (t_{\rm im},t_f], \end{cases} \tag{4.29}$$

where

$$\begin{aligned} V_1\big(z_1(t;t_{\rm im}),t\big) &= z_1^T(t;t_{\rm im})P_{\rm l}(t)z_1(t;t_{\rm im}), \\ z_1(t;t_{\rm im}) &= {\rm col}\big(x_1(t;t_{\rm im}), y_1(t;t_{\rm im})\big), \end{aligned} \tag{4.30}$$

$$\begin{aligned} V_2\big(z_2(t;t_{\rm im}),t\big) &= z_2^T(t;t_{\rm im})P_{\rm r}(t)z_2(t;t_{\rm im}), \\ z_2(t;t_{\rm im}) &= {\rm col}\big(x_2(t;t_{\rm im}), y_2(t;t_{\rm im})\big). \end{aligned} \tag{4.31}$$

Remember that $z_1(t;t_{\rm im})$ and $z_2(t;t_{\rm im})$ are the solutions of the initial-value problems (3.1)-(3.2) and (3.3)-(3.4), respectively.

Using the initial conditions in (3.3)-(3.4) and the terminal condition in (4.24), one can show by a routine algebra that

$$V_1(z_1(t;t_{\rm im}),t)\big|_{t=t_{\rm im}-0} - V_2(z_2(t;t_{\rm im}),t)\big|_{t=t_{\rm im}+0} = 0. \tag{4.32}$$

Let us calculate $dV_1\big(z_1(t;t_{\rm im}),t\big)/dt$, $t\in[0,t_{\rm im})$. We have

$$\begin{aligned}\frac{dV_1\big(z_1(t;t_{\rm im}),t\big)}{dt} &= \left(\frac{dz_1(t;t_{\rm im})}{dt}\right)^T P_1(t)z_1(t;t_{\rm im})\\ &+z_1^T(t;t_{\rm im})\frac{dP_1(t)}{dt}z_1(t;t_{\rm im})+z_1^T(t;t_{\rm im})P_1(t)\left(\frac{dz_1(t;t_{\rm im})}{dt}\right).\end{aligned} \tag{4.33}$$

Since $z_1(t;t_{\rm im})$ is the solution of the initial-value problem (3.1)-(3.2), then using the equations (4.4) and (4.5), we obtain

$$\begin{aligned}\frac{dz_1(t;t_{\rm im})}{dt} &= A(t,t_{\rm im})z_1(t;t_{\rm im})+B_{\rm u}(t)u_1(t;t_{\rm im})\\ &+B_{\rm v}(t)v_1(t;t_{\rm im}),\quad t\in[0,t_{\rm im}),\end{aligned} \tag{4.34}$$

where

$$u_1(t;t_{\rm im})=u\big(z_1(t;t_{\rm im}),t\big),\quad v_1(t;t_{\rm im})=v\big(z_1(t;t_{\rm im}),t\big). \tag{4.35}$$

Substitution of the expressions for $dP_1(t)/dt$ and $dz_1(t;t_{\rm im})/dt$ (see the equations (4.24) and (4.34)) into (4.33) yields after a routine rearrangement

$$\begin{aligned}\frac{dV_1\big(z_1(t;t_{\rm im}),t\big)}{dt} &= u_1^T(t;t_{\rm im})B_u^T(t)P_1(t)z_1(t;t_{\rm im})\\ &+v_1^T(t;t_{\rm im})B_v^T(t)P_1(t)z_1(t;t_{\rm im})\\ &+z_1^T(t;t_{\rm im})P_1(t)S_u(t)P_1(t)z_1(t;t_{\rm im})\\ &-z_1^T(t;t_{\rm im})P_1(t)S_v(t)P_1(t)z_1(t;t_{\rm im})\\ &-z_1^T(t;t_{\rm im})D(t)z_1(t;t_{\rm im})\\ &+z_1^T(t;t_{\rm im})P_1(t)B_u(t)u_1(t;t_{\rm im})\\ &+z_1^T(t;t_{\rm im})P_1(t)B_v(t)v_1(t;t_{\rm im}),\\ &\qquad t\in[0,t_{\rm im}).\end{aligned} \tag{4.36}$$

Using (4.7)-(4.8) and (4.27), we can rewrite the equation (4.36) in the form

$$\begin{aligned}\frac{dV_1\big(z_1(t;t_{\rm im}),t\big)}{dt} &= \big(u_1(t;t_{\rm im})-u_1^0(t;t_{\rm im})\big)^T R_u(t)\big(u_1(t;t_{\rm im})-u_1^0(t;t_{\rm im})\big)\\ &-\big(v_1(t;t_{\rm im})-v_1^0(t;t_{\rm im})\big)^T R_v(t)\big(v_1(t;t_{\rm im})-v_1^0(t;t_{\rm im})\big)\\ &-u_1^T(t;t_{\rm im})R_u(t)u_1(t;t_{\rm im})+v_1^T(t;t_{\rm im})R_v(t)v_1(t;t_{\rm im})\\ &-z_1^T(t;t_{\rm im})D(t)z_1(t;t_{\rm im}),\quad t\in[0,t_{\rm im}),\end{aligned} \tag{4.37}$$

where

$$\begin{aligned}u_1^0(t;t_{\rm im}) &= u^0\big(z_1(t;t_{\rm im}),t;t_{\rm im}\big),\\ v_1^0(t;t_{\rm im}) &= v^0\big(z_1(t;t_{\rm im}),t;t_{\rm im}\big).\end{aligned} \tag{4.38}$$

Quite similarly to (4.37)-(4.38), we obtain

$$\begin{aligned}\frac{dV_2\big(z_2(t;t_{\rm im}),t\big)}{dt} &= \big(u_2(t;t_{\rm im})-u_2^0(t;t_{\rm im})\big)^T R_u(t)\big(u_2(t;t_{\rm im})-u_2^0(t;t_{\rm im})\big)\\ &\quad -\big(v_2(t;t_{\rm im})-v_2^0(t;t_{\rm im})\big)^T R_v(t)\big(v_2(t;t_{\rm im})-v_2^0(t;t_{\rm im})\big)\\ &\quad -u_2^T(t;t_{\rm im})R_u(t)u_2(t;t_{\rm im})+v_2^T(t;t_{\rm im})R_v(t)v_2(t;t_{\rm im})\\ &\quad -z_2^T(t;t_{\rm im})D(t)z_2(t;t_{\rm im}), \qquad t\in(t_{\rm im},t_f],\end{aligned} \tag{4.39}$$

where

$$\begin{aligned}u_2(t;t_{\rm im}) &= u\big(z_2(t;t_{\rm im}),t\big),\\ v_2(t;t_{\rm im}) &= v\big(z_2(t;t_{\rm im}),t\big),\end{aligned} \tag{4.40}$$

$$\begin{aligned}u_2^0(t;t_{\rm im}) &= u^0\big(z_2(t;t_{\rm im}),t;t_{\rm im}\big),\\ v_2^0(t;t_{\rm im}) &= v^0\big(z_2(t;t_{\rm im}),t;t_{\rm im}\big).\end{aligned} \tag{4.41}$$

Integrating the equation (4.37) from $t=0$ to $t=t_{\rm im}$, we obtain

$$\begin{aligned}&V_1\big(z_1(t;t_{\rm im}),t\big)|_{t=t_{\rm im}-0}-V_1\big(z_1(0;t_{\rm im}),0\big)\\ &\quad=\int_0^{t_{\rm im}}\Big[\big(u_1(t;t_{\rm im})-u_1^0(t;t_{\rm im})\big)^T R_u(t)\big(u_1(t;t_{\rm im})-u_1^0(t;t_{\rm im})\big)\\ &\quad -\big(v_1(t;t_{\rm im})-v_1^0(t;t_{\rm im})\big)^T R_v(t)\big(v_1(t;t_{\rm im})-v_1^0(t;t_{\rm im})\big)\Big]dt\\ &\quad -\int_0^{t_{\rm im}}\Big[z_1^T(t;t_{\rm im})D(t)z_1(t;t_{\rm im})+u_1^T(t;t_{\rm im})R_u(t)u_1(t;t_{\rm im})\\ &\quad -v_1^T(t;t_{\rm im})R_v(t)v_1(t;t_{\rm im})\Big]dt.\end{aligned} \tag{4.42}$$

Similarly, integrating the equation (4.39) from $t=t_{\rm im}$ to $t=t_f$, we have

$$\begin{aligned}&V_2\big(z_2(t_f;t_{\rm im}),t_f\big)-V_2\big(z_2(t;t_{\rm im}),t\big)|_{t=t_{\rm im}+0}\\ &\quad=\int_{t_{\rm im}}^{t_f}\Big[\big(u_2(t;t_{\rm im})-u_2^0(t;t_{\rm im})\big)^T R_u(t)\big(u_2(t;t_{\rm im})-u_2^0(t;t_{\rm im})\big)\\ &\quad -\big(v_2(t;t_{\rm im})-v_2^0(t;t_{\rm im})\big)^T R_v(t)\big(v_2(t;t_{\rm im})-v_2^0(t;t_{\rm im})\big)\Big]dt\\ &\quad -\int_{t_{\rm im}}^{t_f}\Big[z_2^T(t;t_{\rm im})D(t)z_2(t;t_{\rm im})+u_2^T(t;t_{\rm im})R_u(t)u_2(t;t_{\rm im})\\ &\quad -v_2^T(t;t_{\rm im})R_v(t)v_2(t;t_{\rm im})\Big]dt.\end{aligned} \tag{4.43}$$

Adding the equations (4.42) and (4.43), and using the terminal condition in (4.18) and the equations (2.3),(3.5),(4.25),(4.26)-(4.27),(4.28)-(4.31), (4.32),(4.35),(4.38), (4.40),(4.41), we obtain

$$J\big(u(t;t_{\rm im}),v(t;t_{\rm im})\big) = z_0^T P(0;t_{\rm im})z_0 + \int_0^{t_f} \Big[\big(u(t;t_{\rm im})-u^0(t;t_{\rm im})\big)^T R_u(t)\big(u(t;t_{\rm im})-u^0(t;t_{\rm im})\big) - \big(v(t;t_{\rm im})-v^0(t;t_{\rm im})\big)^T R_v(t)\big(v(t;t_{\rm im})-v^0(t;t_{\rm im})\big)\Big]dt, \tag{4.44}$$

where

$$u(t;t_{\rm im}) = u\big(z(t;t_{\rm im}),t\big), \quad v(t;t_{\rm im}) = v\big(z(t;t_{\rm im}),t\big), \quad t\in[0,t_f], \tag{4.45}$$

$$u^0(t;t_{\rm im}) = u^0\big(z(t;t_{\rm im}),t\big), \quad v^0(t;t_{\rm im}) = v^0\big(z(t;t_{\rm im}),t\big), \quad t\in[0,t_f]. \tag{4.46}$$

Now, the equations (4.44)-(4.46), along with the positive definiteness of the matrices $R_u(t)$, $R_v(t)$, $t\in[0,t_f]$, directly yield the fulfilment of the condition (4.1) for $u(z,t)=u^0(z,t)$, $v(z,t)=v^0(z,t)$. The latter, along with the equations (4.2)-(4.3) and (4.26)-(4.27), proves the theorem. □

4.2. **Obtaining the minimum guaranteed outcome and the minimum guaranteeing minimizer's strategy in the ODG.**

Theorem 4.3. *Let for any* $t_{\rm im}\in(0,t_f)$*, the problem (4.6) has the solution (4.25). Then,* $J_u^* = \inf_{t_{\rm im}\in(0,t_f)} J^0(t_{\rm im})$*. Moreover, if there exists* $t_{\rm im}^*\in(0,t_f)$ *such that* $J^0(t_{\rm im}^*) = \inf_{t_{\rm im}\in(0,t_f)} J^0(t_{\rm im})$*, then the pair* $\big(t_{\rm im}^*, u^0(z,t;t_{\rm im}^*)\big)$ *constitutes the minimum guaranteeing minimizer's strategy in the ODG.*

Proof. The statements of the theorem directly follow from the equations (3.6) and (3.7), and Theorem 4.2. □

5. Linear-quadratic pursuit-evasion game with hybrid pursuer's dynamics

5.1. **Formulation of initial game.** In this section, the above obtained theoretical results are applied to solution of some pursuit-evasion game. Namely, we consider an engagement between two flying vehicles, a pursuer (an interceptor) and an evader (a target). It is well known that in a small vicinity of the collision course a nonlinear three-dimensional motion of the vehicles can be linearized and decoupled into two planar motions in perpendicular planes (see e.g. [8, 13]). Due to this observation, in what follows, we consider the linear planar model of the engagement. In Fig. 1, the planar engagement geometry is shown. The points (x_p,y_p) and (x_e,y_e) are current coordinates of the pursuer and the evader, respectively; V_p and V_e are constant magnitudes of the velocity vectors of the pursuer and the evader, respectively; a_p, a_e are their lateral accelerations; φ_p,φ_e are the respective angles between the velocity vectors and the X-axis (the initial line of sight).

In the linear model, the duration of the engagement is $t_f = R_0/V_c$, where R_0 is a known initial distance between the vehicles and V_c is the closing speed of the engagement. Thus, in the case of the first-order dynamics of the pursuer and the

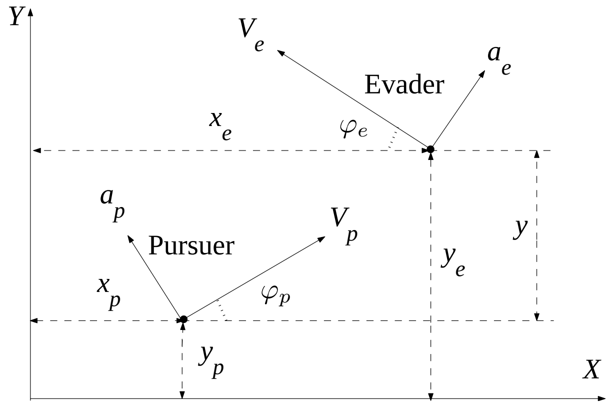

FIGURE 1. Engagement geometry

evader, the set of linear differential equations, modeling the engagement, has the following form [13]:

$$\frac{dx(t)}{dt} = Ax(t) + bu_p(t) + cu_e(t), \quad x(0) = x_0, \quad t \in [0, t_f], \tag{5.1}$$

where $x(t) = \operatorname{col}\big(x_1(t), x_2(t), x_3(t), x_4(t)\big)$ is the state vector; $u_p(t)$ and $u_e(t)$ are scalar controls (the acceleration commands in y-direction) of the pursuer and the evader, respectively; $x_0 = \operatorname{col}\big(x_{10}, x_{20}, x_{30}, x_{40}\big)$ is a given vector;

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1/\tau_e & 0 \\ 0 & 0 & 0 & -1/\tau_p \end{bmatrix}, \tag{5.2}$$

$$b = \operatorname{col}(0, 0, 0, 1/\tau_p), \quad c = \operatorname{col}(0, 0, 1/\tau_e, 0), \tag{5.3}$$

$x_1(t) = y_e(t) - y_p(t)$ is the relative separation of the vehicles normal to the initial line of sight (the X-axis); $x_2(t)$ is the relative velocity of the vehicles normal to the initial line of sight; $x_3(t)$ and $x_4(t)$ are the lateral accelerations of the evader and the pursuer, respectively, both normal to the initial line of sight; τ_e, τ_p are the respective time constants.

Consider the differential game with the dynamics (5.1) and the cost functional

$$\mathcal{J}(u_p, u_e) = x_1^2(t_f) + \int_0^{t_f} \big[\alpha u_p^2(t) - \beta u_e^2(t)\big] dt, \tag{5.4}$$

where $\alpha > 0$ and $\beta > 0$ are given constants.

The cost functional (5.4) is minimized by the pursuer (by $u_p(t)$) and maximized by the evader (by $u_e(t)$) along trajectories of (5.1).

In what follows, we assume that the evader's time constant τ_e is fixed. The pursuer has two values $\tau_{p,1}$ and $\tau_{p,2}$ of the time constant on its disposal, and the pursuer can change its time constant from $\tau_{p,1}$ to $\tau_{p,2}$ (or from $\tau_{p,2}$ to $\tau_{p,1}$) once during the game at a non-prescribed in advance time momentum $t_{\text{sw}} \in (0, t_f)$.

5.2. **Game with impulsive dynamics.** Let us choose some order of the pursuer's time constant values

$$\tau_p = \begin{cases} \tau_{p,i_1}, & t \in [0, t_{\rm sw}), \\ \tau_{p,i_2}, & t \in [t_{\rm sw}, t_f], \end{cases} \tag{5.5}$$

where $i_1, i_2 \in \{1, 2\}$.

Let us introduce the variable

$$\begin{aligned} Z(t) &= x_1(t) + (t_f - t)x_2(t) \\ &+\tau_e^2 \Psi\big((t_f - t)/\tau_e\big)x_3(t) - \tau_p^2 \Psi\big((t_f - t)/\tau_p\big)x_4(t), \end{aligned} \tag{5.6}$$

where

$$\Psi(\xi) \stackrel{\triangle}{=} \exp(-\xi) + \xi - 1 > 0, \quad \xi > 0. \tag{5.7}$$

From (5.6)-(5.7), one can observe two important features of $Z(t)$. First,

$$Z(t_f) = x_1(t_f). \tag{5.8}$$

Second, due to the switch (5.5) in the pursuer's time constant, the variable $Z(t)$ has a finite break at $t = t_{\rm sw}$:

$$\Delta Z \stackrel{\triangle}{=} Z(t_{\rm sw}) - Z(t_{\rm sw} - 0) = \gamma(t_{\rm sw})x_4(t_{\rm sw}), \tag{5.9}$$

where

$$\gamma(t) = \tau_{p,i_1}^2 \Psi\big((t_f - t)/\tau_{p,i_1}\big) - \tau_{p,i_2}^2 \Psi\big((t_f - t)/\tau_{p,i_2}\big). \tag{5.10}$$

Using $Z(t)$ as a new state variable and taking into account its features (5.8) and (5.9), we can transform the game (5.1),(5.4) to a new pursuit-evasion game of a lower dimension but with impulsive dynamics

$$\begin{aligned} \frac{dZ(t)}{dt} &= -h(t, \tau_p)u_p(t) + h(t, \tau_e)u_e(t) \\ &+\gamma(t)\delta(t - t_{\rm sw})x_4(t), \quad Z(0) = Z_0, \quad t \in [0, t_f], \end{aligned} \tag{5.11}$$

$$\frac{dx_4(t)}{dt} = \frac{u_p(t) - x_4(t)}{\tau_p}, \quad x_4(0) = x_{40}, \quad t \in [0, t_f], \tag{5.12}$$

where τ_p is given by (5.5) with an unspecified $t_{\rm sw} \in (0, t_f)$;

$$h(t, \tau) = \tau \Psi\big((t_f - t)/\tau\big); \tag{5.13}$$

$$Z_0 = x_{10} + t_f x_{20} + \tau_e^2 \Psi\big(t_f/\tau_e\big)x_{30} - \tau_p^2 \Psi\big(t_f/\tau_p\big)x_{40}. \tag{5.14}$$

The cost functional in the new game has the form

$$J(u_p, u_e, t_{\rm sw}) = Z^2(t_f) + \int_0^{t_f} \big[\alpha u_p^2(t) - \beta u_e^2(t)\big]dt. \tag{5.15}$$

The cost functional (5.15) is minimized by a proper choice of the pair $\big(t_{\rm sw}, u_p\big)$ and maximized by a proper choice of u_e.

Like in the theoretical results of the previous sections, we solve the game (5.11)-(5.12),(5.15) from the minimizer's (pursuer's) viewpoint. Namely, we look for the minimum guaranteed outcome of this game

$$J_p^* \stackrel{\triangle}{=} \inf_{t_{\rm sw}\in(0,t_f)} \inf_{u_p\in\mathcal{H}_{\rm u_p}} \sup_{u_e\in\mathcal{F}_{\rm u_e}(u_p)} J\big(u_p,u_e,t_{\rm sw}\big), \tag{5.16}$$

and the minimum guaranteeing pursuer's strategy $\big(t_{\rm sw}^*,u_p^*(z,t)\big)$, $\big(t_{\rm sw}^*\in(0,t_f),\ u_p^*(z,t)\in\mathcal{H}_{\rm u_p}\big)$:

$$\sup_{u_e\in\mathcal{F}_{\rm u_e}(u_p^*)} J\big(u_p^*(z,t),u_e,t_{\rm sw}^*\big)=J_p^*. \tag{5.17}$$

In (5.16)-(5.17), $z=\mathrm{col}(Z,x_4)$, and $\mathcal{H}_{\rm u_p}$ and $\mathcal{F}_{\rm u_e}(u_p)$ are the sets of the pursuer's and evader's state-feedback controls similar to the sets $\mathcal{H}_{\rm u}$ and $\mathcal{F}_{\rm v}(u)$ of the minimizer's and maximizer's state-feedback controls defined in Section 3.

5.3. **Solution of the impulsive dynamics game (5.11)-(5.17).** First, let us obtain the solution of the problem (4.18), corresponding to the game (5.11)-(5.17). This solution can be represented in the form

$$P_{\rm r}(t)=\begin{pmatrix} P_{\rm r,1}(t) & P_{\rm r,2}(t)\\ P_{\rm r,2}(t) & P_{\rm r,3}(t)\end{pmatrix}, \tag{5.18}$$

where $P_{\rm r,j}(t)$, $(j=1,2,3)$ are scalar functions.

Due to this representation, the problem (4.18), corresponding to the game (5.11)-(5.17), can be rewritten in the following form:

$$\begin{aligned}\frac{dP_{r,1}(t)}{dt} &= H_{1,i_2}(t)P_{r,1}^2(t)-2H_{2,i_2}(t)P_{r,1}(t)P_{r,2}(t)\\ &\quad +H_{3,i_2}P_{r,2}^2(t),\quad t\in[t_{\rm sw},t_f],\end{aligned} \tag{5.19}$$

$$\begin{aligned}\frac{dP_{r,2}(t)}{dt} &= -\frac{1}{\tau_{p,i_2}}P_{r,2}(t)+H_{1,i_2}(t)P_{r,1}(t)P_{r,2}(t)-H_{2,i_2}(t)P_{r,2}^2(t)\\ &\quad -H_{2,i_2}(t)P_{r,1}(t)P_{r,3}(t)+H_{3,i_2}P_{r,2}(t)P_{r,3}(t),\quad t\in[t_{\rm sw},t_f],\end{aligned} \tag{5.20}$$

$$\begin{aligned}\frac{dP_{r,3}(t)}{dt} &= -\frac{2}{\tau_{p,i_2}}P_{r,3}(t)+H_{1,i_2}(t)P_{r,2}^2(t)\\ &\quad -2H_{2,i_2}(t)P_{r,2}(t)P_{r,3}(t)+H_{3,i_2}P_{r,3}^2(t),\quad t\in[t_{\rm sw},t_f],\end{aligned} \tag{5.21}$$

$$P_{r,1}(t_f)=1,\quad P_{r,2}(t_f)=0,\quad P_{r,3}(t_f)=0, \tag{5.22}$$

where

$$\begin{aligned} H_{1,i_2}(t)&=\frac{h^2(t,\tau_{p,i_2})}{\alpha}-\frac{h^2(t,\tau_e)}{\beta},\\ H_{2,i_2}(t)&=\frac{h(t,\tau_{p,i_2})}{\alpha\tau_{p,i_2}},\qquad H_{3,i_2}=\frac{1}{\alpha\tau_{p,i_2}^2}.\end{aligned} \tag{5.23}$$

It is verified directly by a routine algebra, that the solution of the problem (5.19)-(5.21),(5.22) is:

$$P_{\mathrm{r},1}(t) = \left(1 + \int_t^{t_f} H_{1,i_2}(\sigma) d\sigma\right)^{-1},$$
$$P_{\mathrm{r},k}(t) = 0, \quad k = 2,3, \quad t \in [t_{\mathrm{sw}}, t_f]. \tag{5.24}$$

To provide the existence of this solution for any $t_{\mathrm{sw}} \in (0, t_f)$, in what follows we assume:
(A2) For given $\tau_{p,1}$, $\tau_{p,2}$, t_f, α, β, order $\{i_1, i_2\}$, the following inequality is valid

$$\int_t^{t_f} H_{1,i_2}(\sigma) d\sigma \neq -1, \quad t \in (0, t_f).$$

The solution of the problem (4.19), corresponding to the game (5.11)-(5.17), has the form (4.21) where, by virtue of (4.23) and (5.24), the scalar blocks are

$$\Delta_{P,1}(\theta) = P_{\mathrm{r},1}(t_{\mathrm{sw}}), \quad \Delta_{P,2}(\theta) = -P_{\mathrm{r},1}(t_{\mathrm{sw}})\gamma(t_{\mathrm{sw}})\theta,$$
$$\Delta_{P,3}(\theta) = P_{\mathrm{r},1}(t_{\mathrm{sw}})\gamma^2(t_{\mathrm{sw}})\theta^2, \quad \theta \in [0,1], \tag{5.25}$$

$\gamma(t)$ is given in (5.10), $P_{\mathrm{r},1}(t)$ is given in (5.24).

Proceed to solution of the problem (4.24) corresponding to the game (5.11)-(5.17). To solve this problem, we introduce into the consideration several matrices.

First of all, let us note that the matrix $\mathcal{A}_1(t)$, $t \in [0, t_{sw})$ in this game has the form

$$\mathcal{A}_1(t) = \mathcal{A}_{1,i_1} \stackrel{\triangle}{=} \begin{pmatrix} 0 & 0 \\ 0 & -1/\tau_{p,i_1} \end{pmatrix}. \tag{5.26}$$

Thus,

$$\exp\big(-\mathcal{A}_{1,i_1}(t - t_{sw})\big) = \begin{pmatrix} 1 & 0 \\ 0 & \exp\big((t - t_{sw})/\tau_{p,i_1}\big) \end{pmatrix}. \tag{5.27}$$

Also, consider the matrices

$$\Delta_P(1) = P_{\mathrm{r},1}(t_{\mathrm{sw}}) \begin{pmatrix} 1 & -\gamma(t_{\mathrm{sw}}) \\ -\gamma(t_{\mathrm{sw}}) & \gamma^2(t_{\mathrm{sw}}) \end{pmatrix}, \tag{5.28}$$

$$S_{p,e}(t) = \begin{pmatrix} H_{1,i_1}(t) & -H_{2,i_1}(t) \\ -H_{2,i_1}(t) & H_{3,i_1} \end{pmatrix}, \tag{5.29}$$

$$\Gamma(t) = I_2 - \int_{t_{\mathrm{sw}}}^t \exp\big(-\mathcal{A}_{1,i_1}(\sigma - t_{sw})\big) S_{p,e}(\sigma) \exp\big(-\mathcal{A}_{1,i_1}(\sigma - t_{sw})\big) d\sigma \Delta_P(1), \tag{5.30}$$

where $t \in [0, t_{\rm sw}]$, I_2 is the 2-dimensional identity matrix;

$$H_{1,i_1}(t) = \frac{h^2(t, \tau_{p,i_1})}{\alpha} - \frac{h^2(t, \tau_e)}{\beta},$$

$$H_{2,i_1}(t) = \frac{h(t, \tau_{p,i_1})}{\alpha \tau_{p,i_1}}, \qquad H_{3,i_1} = \frac{1}{\alpha \tau_{p,i_1}^2}. \tag{5.31}$$

In what follows, we assume:

(A3) For given $\tau_{p,1}$, $\tau_{p,2}$, t_f, α, β, order $\{i_1, i_2\}$ and any $t_{\rm sw} \in (0, t_f)$, the matrix $\Gamma(t)$ is invertible for all $t \in [0, t_{\rm sw}]$.

Due to this assumption, the solution of the problem (4.24), corresponding to the game (5.11)-(5.17), is

$$P_{\rm l}(t) = \exp\big(-\mathcal{A}_{1,i_1}(t - t_{sw})\big)\Delta_P(1)\big(\Gamma(t)\big)^{-1}\exp\big(-\mathcal{A}_{1,i_1}(t - t_{sw})\big), \quad t \in [0, t_{\rm sw}]. \tag{5.32}$$

Remark 5.1. Note, that the positive semi-definiteness of the matrix $S_{p,e}(t)$ for both, $\tau_{p,i_1} = \tau_{p,1}$, $\tau_{p,i_1} = \tau_{p,2}$, and any $t \in [0, t_f]$ is a sufficient condition for the validity of the assumption A3. In such a case, the assumption A2 also is fulfilled.

Consider the value

$$J^0_{p,e}(t_{\rm sw}) \stackrel{\triangle}{=} \big(Z_0, x_{40}\big)P_{\rm l}(0){\rm col}\big(Z_0, x_{40}\big), \tag{5.33}$$

and the pursuer's state-feedback control

$$u^0_p(Z, x_4, t; t_{\rm sw}) \stackrel{\triangle}{=} \begin{cases} \Big(\frac{h(t,\tau_{p,i_1})}{\alpha}, -\frac{1}{\alpha\tau_{p,i_1}}\Big) P_{\rm l}(t){\rm col}\big(Z, x_4\big), & t \in [0, t_{\rm sw}), \\ \Big(\frac{h(t,\tau_{p,i_2})}{\alpha}, -\frac{1}{\alpha\tau_{p,i_2}}\Big) P_{\rm r}(t){\rm col}\big(Z, x_4\big), & t \in [t_{\rm sw}, t_f], \end{cases} \tag{5.34}$$

where the matrix $P_{\rm r}(t)$ is given by the equations (5.18),(5.24), while the matrix $P_{\rm l}(t)$ is given by the equations (5.27)-(5.32).

Based on Theorems 4.2 and 4.3, we obtain the following assertion.

Theorem 5.2. *Let the assumptions A2 and A3 be valid. Then, $J^*_p = \inf_{t_{\rm sw}\in(0,t_f)} J^0_{p,e}(t_{\rm sw})$. Moreover, if there exists $t^*_{\rm sw} \in (0, t_f)$ such that $J^0_{p,e}(t^*_{\rm sw}) = \inf_{t_{\rm sw}\in(0,t_f)} J^0_{p,e}(t_{\rm sw})$, then the pair $\big(t^*_{\rm sw}, u^0_p(Z, x_4, t; t^*_{\rm sw})\big)$ constitutes the minimum guaranteeing pursuer's strategy in the impulsive dynamics pursuit-evasion game (5.11)-(5.17).*

Remark 5.3. If for both orders of the pursuer's time constant values there exist the minimum guaranteeing pursuer's strategies in the game (5.11)-(5.17), then the pursuer can choose that order and the corresponding strategy, for which the minimum guaranteed outcome of the game is smaller.

6. Conclusions

In the paper, a finite horizon zero-sum linear-quadratic differential game with impulsive dynamics was considered. The value of the impulse depends on its momentum and on values of some game's state coordinates at this momentum. The momentum of the impulse is not prescribed, and it is at the disposal of the minimizing player (the minimizer). Thus, this momentum is an additional minimizer's control. This game was treated from the minimizer's viewpoint. Namely, the minimum guaranteed outcome of the game was derived, and the minimum guaranteeing minimizer's strategy (impulse momentum, state-feedback control) was designed. These results are based on the solution of a terminal-value problem for a matrix differential Riccati equation with a state-dependent impulse, obtained in this paper. Then, a given duration planar pursuit-evasion differential game with hybrid pursuer's dynamics and quadratic cost functional was considered. The pursuer can switch from one dynamics to the other once during the game. Using a new state variable (the zero-effort miss distance), this game was reduced to a lower dimension linear-quadratic pursuit-evasion game with impulsive dynamics. The value of the impulse depends on the switch momentum of the pursuer's dynamics and on the value of the pursuer's lateral acceleration at this momentum. This new pursuit-evasion game was solved by application of the theoretical results obtained earlier in the paper for the general game.

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