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NONOSCILLATION AND EXPONENTIAL STABILITY OF THE SECOND ORDER DELAY DIFFERENTIAL EQUATION WITH DAMPING

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Abstract. For a delay differential equation

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)\dot{x}(g_k(t)) + \sum_{k=1}^{l} b_k(t)x(h_k(t)) = 0,\ g_k(t) \leq t, h_k(t) \leq t,$$

new sufficient nonoscillation and exponential stability conditions are obtained. Our main tools are a nonstandard transformation of differential equation of the second order to a system of two equations of the first order, connection between nonoscillation and exponential stability and Bohl-Perron theorem.

1. Introduction

Many models in applications are described by delay differential equations (DDE) of the second order, see for example monographs [21, 23, 25, 28, 29, 30] and references therein. Existence of positive solutions, oscillation of all solutions and stability are among the most important questions appearing in the study of applied models.

Nonoscillation properties were investigated in the monograph [1] and in the papers [5, 8, 16, 17]. Oscillation results for second order equations with a detailed bibliography one can find in [13, 19, 20, 24, 26]. For asymptotic and exponential stability various methods were used: investigations of a characteristic equation and some other applications of complex analysis [15, 22], method of Lyapunov functionals [27], fixed point method [14] and methods based on Bohl-Perron theorem [4, 9, 10, 12].

In [4, 12, 18] a connection between nonoscillation and exponential stability was studied for second order DDE. A new nonoscillation criterion was obtained in [12] by construction of a generalized Riccati inequality. Then on the basis of the Bohl-Perron theorem it was shown that a nonoscillatory equation (under some natural conditions) and some perturbations of this equation are exponentially stable. In the paper [10], a second order DDE was transformed by a some nonstandard substitution to a system of first order DDE. By an application of known stability tests for systems of DDE new exponential stability tests were obtained for second order DDE. Sturm separation theorems and growth of the Wronskian for unboundedness solutions and instability were used in [17].

In this paper we combine methods applied in papers [10, 12]. To obtain new nonoscillation results we transform second order DDE to a system of DDE as in [10] and then use a connection between nonoscillation and stability similar to [12].

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By this way we obtain both new nonoscillation and stability results for more general equations then in [10, 12].

The paper is organized as follows. Section 2 contains relevant definitions and notations. In section 3 we obtain nonoscillation results. Section 4 deals with exponential stability.

2. Preliminaries

We consider a scalar delay differential equation of the second order

$$\ddot{x}(t)+\sum_{k=1}^{m}a_k(t)\dot{x}(g_k(t))+\sum_{k=1}^{l}b_k(t)x(h_k(t))=0, \tag{2.1}$$

for $l \leq m$ under the following assumptions:

(a1) $a_k,\ b_k$ are Lebesgue measurable and locally essentially bounded functions,

$$0<a_k^0\leq a_k(t)\leq A_k^0, k=1,\dots,l, a_k(t)\geq 0, k=l+1,\dots,m,$$

$$0\leq b_k^0\leq b_k(t)\leq B_k^0, k=1,\dots,l;$$

(a2) h_k, g_k are Lebesgue measurable functions,

$0\leq \tau_k^0\leq t-h_k(t)\leq \tau_k, 0\leq \sigma_k^0\leq t-g_k(t)\leq \sigma_k; 0\leq g_k(t)-h_k(t)\leq \delta_k, k=1,\dots,l.$

Let us consider together with (2.1) the initial value problem with a right hand side

$$(Lx)(t):=\ddot{x}(t)+\sum_{k=1}^{m}a_k(t)\dot{x}(g_k(t))+\sum_{k=1}^{l}b_k(t)x(h_k(t))=f(t),\ t\geq t_0, \tag{2.2}$$

$$x(t)=\varphi(t), \dot{x}(t)=\xi(t)\ t<t_0;\ x(t_0)=x_0,\ \dot{x}(t_0)=x_0^{'}. \tag{2.3}$$

for each $t_0 \geq 0$.

We also assume that the following hypothesis holds

(a3) $f:[t_0,\infty)\to R$ is a Lebesgue measurable locally essentially bounded function, $\varphi,\xi:(-\infty,t_0)\to R$ are Borel measurable bounded functions.

Definition 2.1. Suppose a function $x:[t_0,\infty)\to R$ is differentiable and $\dot{x}$ is a locally absolutely continuous function. Extend the functions x and $\dot{x}$ for $t\leq t_0$ by equalities (2.3). We say that so extended function x is a *a solution* of problem (2.2), (2.3) if it satisfies equation (2.2) for almost every $t\in[t_0,\infty)$.

Definition 2.2. For each $s\geq 0$ the solution $X(t,s)$ of the problem

$$\begin{aligned}&\ddot{x}(t)+\sum_{k=1}^{m}a_k(t)\dot{x}(g_k(t))+\sum_{k=1}^{l}b_k(t)x(h_k(t))=0,\ t\geq s,\\ &x(t)=0,\ \dot{x}(t)=0,\ t<s;\ x(s)=0,\ \dot{x}(s)=1,\end{aligned} \tag{2.4}$$

is called *the fundamental function* of equation (2.1).

Remark 2.3. In the literature [3] the fundamental function is also called *the Cauchy function.*

We assume $X(t,s)=0,\ 0\le t<s$.

Let functions x_1 and x_2 be the solutions of the following problems

$$\ddot{x}(t)+\sum_{k=1}^{m} a_k(t)\dot{x}(g_k(t))+\sum_{k=1}^{l} b_k(t)x(h_k(t))=0,\ t\ge t_0;$$

$$x(t)=0,\ \dot{x}(t)=0,\ t<t_0,$$

with the initial conditions $x(t_0)=1,\ \dot{x}(t_0)=0$ for x_1 and $x(t_0)=0,\ \dot{x}(t_0)=1$ for x_2, respectively. Note that by Definition 2.2 $x_2(t)=X(t,t_0)$.

Lemma 2.4. [3] *Let (a1)-(a3) hold. Then there exists one and only one solution of problem (2.2), (2.3) and it can be presented in the form*

$$x(t)=x_1(t)x_0+x_2(t)x_0^{'}+\int_{t_0}^{t}X(t,s)f(s)ds$$

$$-\int_{t_0}^{t}X(t,s)\left[\sum_{k=1}^{m}a_k(s)\xi(g_k(s))+\sum_{k=1}^{l}b_k(s)\varphi(h_k(s))\right]ds.$$

Definition 2.5. Eq. (2.1) is *(uniformly) exponentially stable*, if there exist $M>0$, $\mu>0$, such that the solution of problem (2.2),(2.3) with $f=0$ and its derivative have the estimate

$$\max\{|x(t)|,|\dot{x}(t)|\}\le M\ e^{-\mu(t-t_0)}\left[|x(t_0)|+|\dot{x}(t_0)|+\max\{\sup_{t<t_0}|\xi(t)|,\sup_{t<t_0}|\varphi(t)|\}\right],\quad t\ge t_0,$$

where M and μ do not depend on t_0,ξ,φ.

Definition 2.6. The fundamental function $X(t,s)$ of (2.1) and its derivative on t *have an exponential estimate* if there exist positive numbers $K>0,\lambda>0$, such that

$$\max\{|X(t,s)|,|X_t'(t,s)|\}\le K\ e^{-\lambda(t-s)},\ t\ge s\ge 0.$$

For the linear equation (2.1) with bounded delays ((a2) holds) the last two definitions are equivalent.

Under (a2) the exponential stability does not depend on values of equation parameters on any finite interval.

Let us introduce some function spaces on the semi-axis. Denote by $\mathbf{L}_\infty[t_0,\infty)$ the space of all essentially bounded on $[t_0,\infty)$ scalar functions and by $\mathbf{C}[t_0,\infty)$ the space of all continuous bounded on $[t_0,\infty)$ scalar functions with the supremum norm. By $\|x\|$ denote the norm of x in any of these spaces.

Lemma 2.7. [3] *Suppose there exists* $t_0\ge 0$ *such that for every* $f\in\mathbf{L}_\infty[t_0,\infty)$ *both the solution* x *of the problem*

$$(Lx)(t)=f(t),t>0;\ x(\xi)=\dot{x}(\xi)=0,\xi\le t_0 \tag{2.5}$$

belongs to $\mathbf{C}[t_0,\infty)$ *and its derivative* $\dot{x}$ *belongs to* $\mathbf{L}_\infty[t_0,\infty)$. *Then equation (2.1) is exponentially stable.*

Remark 2.8. Lemma 2.7 is true if we take instead of $f \in \mathbf{L}_\infty[t_0, \infty)$ all functions from this space equal 0 on some fixed interval $[t_0, t_0 + \epsilon]$.

Lemma 2.9. [12] *Suppose for equation (2.1) $b_k(t) \geq 0, t - h_k(t) \leq \tau$, the fundamental function of (2.1) is positive $X(t,s) > 0, t > s > t_0$. Then*

$$\int_{t_0+\tau}^{t} X(t,s) \sum_{k=1}^{l} b_k(s) ds \leq 1.$$

Consider now a system of DDE of the first order

$$\dot{x}_i(t) + \sum_{k=1}^{m} \sum_{j=1}^{n} a_{ij}^k(t) x(h_{ij}^k(t)) = 0, \; i = 1, \ldots, n \tag{2.6}$$

under the following conditions:

(A1) coefficients a_{ij}^k are Lebesgue measurable locally essentially bounded functions;

(A2) delays $h_{ij}^k : [0, \infty) \to \mathbf{R}$ are Lebesgue measurable functions, $h_{ij}^k(t) \leq t$, $\lim_{t \to \infty} h_{ij}^k(t) = \infty$, $k = 1, \cdots, m$, $i, j = 1, \cdots, n$.

Denote $X(t) = [x_1(t), \cdots, x_n(t)]^T$, let A_{ij}^k be an $n \times n$ matrix with the only nonzero entry a_{ij}^k. Then (2.6) can be rewritten in the vector form

$$\dot{X}(t) + \sum_{k=1}^{m} \sum_{i,j=1}^{n} A_{ij}^k(t) X(h_{ij}^k(t)) = 0. \tag{2.7}$$

The fundamental matrix $C(t,s)$ of equation (2.7) will be called the fundamental matrix of system (2.6).

Lemma 2.10. [1, Theorem 9.2] *Suppose*

a) $a_{ii}^k(t) \geq 0, a_{ij}^k(t) \leq 0$, $i \neq j$, $k = 1, \cdots, m$, $t \geq t_0$, and

b) the fundamental functions $Y_i(t,s)$ of the scalar equations

$$\dot{y}(t) + \sum_{k=1}^{m} a_{ii}^k(t) y(h_{ii}^k(t)) = 0, \; i = 1, \cdots, n \tag{2.8}$$

are positive for $t \geq s \geq t_0$.

Then for the fundamental matrix of the system (2.6) we have $C(t,s) \geq 0$, $t \geq s \geq t_0$.

Corollary 2.11. *Suppose $a_{ii}^k(t) \geq 0, a_{ij}^k(t) \leq 0$, $i \neq j$, $t \geq t_0$ and*

$$\int_{\max\{t_0, \min_k h_{ii}^k(t)\}}^{t} \sum_{k=1}^{m} a_{ii}^k(s) ds \leq \frac{1}{e}, \; i = 1, \cdots, n. \tag{2.9}$$

Then the fundamental matrix of the system (2.6) satisfies the inequality $C(t,s) \geq 0$, $t \geq s \geq t_0$.

3. Nonoscillation Criteria

Denote $a_0 = \sum_{k=1}^{l} a_k^0, B_0 = \sum_{k=1}^{l} B_k^0$.

Theorem 3.1. *Assume that $a_0^2 \geq 4B_0$ and the following condition holds for $t \geq t_0$:*

$$\int_{g_0(t)}^{t} \left[\sum_{j=1}^{m} e^{-\frac{a_0\sigma_j^0}{2}} a_j(s) + \sum_{i=1}^{l} \left(e^{-\frac{a_0\sigma_i^0}{2}} \frac{a_0}{2}\delta_i a_i(s) + e^{-\frac{a_0\tau_i^0}{2}} \frac{a_0^2}{4l} \right) \right] ds \leq \frac{1}{e} \tag{3.1}$$

where $g_0(t) = \min_k g_k(t), 1 \leq k \leq l$.

Then the fundamental function $X(t,s)$ of equation (2.1) is nonnegative: $X(t,s) \geq 0, t \geq s \geq t_0$.

Proof. We will show that $x(t) = X(t,t_0) \geq 0$. The general case $X(t,s) \geq 0, t \geq s \geq t_0$ is considered similarly. From the definition of the fundamental function we have $x(t) = \dot{x}(t) = 0, t < t_0, x(t_0) = 0, \dot{x}(t_0) = 1$. We will transform equation (2.1) to a system of two DDE equations of the first order. After the substitution

$$\dot{x} = -\frac{a_0}{2}x + y, \ddot{x} = \frac{a_0^2}{4}x - \frac{a_0}{2}y + \dot{y}$$

equation (2.1) has a form

$$\dot{y}(t) = -\frac{a_0^2}{4}x(t) + \sum_{k=1}^{m} \frac{a_0}{2} a_k(t)x(g_k(t)) - \sum_{k=1}^{l} b_k(t)x(h_k(t))$$
$$+\frac{a_0}{2}y(t) - \sum_{k=1}^{m} a_k(t)y(g_k(t)).$$

Rewrite the previous equation

$$\dot{y}(t) = -\frac{a_0^2}{4}x(t) - \sum_{k=1}^{l} \frac{a_0}{2} a_k(t)(x(h_k(t)) - x(g_k(t))) + \sum_{k=1}^{l} \left(\frac{a_0}{2}a_k(t) - b_k(t)\right) x(h_k(t))$$
$$+ \sum_{k=l+1}^{m} \frac{a_0}{2} a_k(t)x(g_k(t)) + \frac{a_0}{2}y(t) - \sum_{k=1}^{m} a_k(t)y(g_k(t)).$$

We continue transformations:

$$\dot{y}(t) = \sum_{k=1}^{l} \left(\frac{a_0}{2}a_k(t) - \frac{a_0^2}{4l} - b_k(t)\right) x(h_k(t) - \sum_{k=1}^{l} \frac{a_0^2}{4l}(x(t) - x(h_k(t)))$$
$$- \sum_{k=1}^{l} \frac{a_0}{2} a_k(t)(x(h_k(t)) - x(g_k(t))) + \sum_{k=l+1}^{m} \frac{a_0}{2} a_k(t)x(g_k(t))$$
$$+\frac{a_0}{2}y(t) - \sum_{k=1}^{m} a_k(t)y(g_k(t)).$$

Hence

$$\dot{y}(t) = -\sum_{k=1}^{l} \frac{a_0^2}{4l} \int_{h_k(t)}^{t} \dot{x}(s)ds - \sum_{k=1}^{l} \frac{a_0}{2} a_k(t) \int_{g_k(t)}^{h_k(t)} \dot{x}(s)ds$$

$$+\sum_{k=1}^{l}\left(\frac{a_0}{2}a_k(t)-\frac{a_0^2}{4l}-b_k(t)\right)x(h_k(t))+\sum_{k=l+1}^{m}\frac{a_0}{2}a_k(t)x(g_k(t))$$

$$+\frac{a_0}{2}y(t)-\sum_{k=1}^{m}a_k(t)y(g_k(t)).$$

Since $\dot{x}=-\frac{a_0}{2}x+y$, then

$$\dot{y}(t)=\sum_{k=1}^{l}\frac{a_0^3}{8l}\int_{h_k(t)}^{t}x(s)ds+\sum_{k=1}^{l}\frac{a_0^2}{4}a_k(t)\int_{g_k(t)}^{h_k(t)}x(s)ds \tag{3.2}$$

$$+\sum_{k=1}^{l}\left(\frac{a_0}{2}a_k(t)-\frac{a_0^2}{4l}-b_k(t)\right)x(h_k(t))+\sum_{k=l+1}^{m}\frac{a_0}{2}a_k(t)x(g_k(t))$$

$$+\frac{a_0}{2}y(t)-\sum_{k=1}^{m}a_k(t)y(g_k(t))-\sum_{k=1}^{l}\frac{a_0^2}{4l}\int_{h_k(t)}^{t}y(s)ds-\sum_{k=1}^{l}\frac{a_0}{2}a_k(t)\int_{g_k(t)}^{h_k(t)}y(s)ds.$$

Denote $X(t)=\{x(t),y(t)\}^T, x(t)=y(t)=0, t<t_0, x(t_0)=y(t_0)=1$ the solution of the system: equation (3.2) and the equation

$$\dot{x}(t)=-\frac{a_0}{2}x(t)+y(t). \tag{3.3}$$

By Corollary 9 [6] there exist measurable functions $\tilde{h}_k^1,\tilde{h}_k^2,\tilde{g}_k^1,\tilde{g}_k^2$ such that $h_k(t)\le \tilde{h}_k^i(t)\le t, g_k(t)\le \tilde{g}_k^i(t)\le t, i=1,2$ and

$$\int_{h_k(t)}^{t}x(s)ds=(t-h_k(t))x(\tilde{h}_k^1(t)), \int_{h_k(t)}^{t}y(s)ds=(t-h_k(t))y(\tilde{h}_k^2(t)),$$

$$\int_{g_k(t)}^{h_k(t)}x(s)ds=(h_k(t)-g_k(t))x(\tilde{g}_k^1(t)), \int_{g_k(t)}^{h_k(t)}y(s)ds=(h_k(t)-g_k(t))y(\tilde{g}_k^2(t)).$$

Hence $X(t)=\{x(t),y(t)\}^T$ is a solution of (3.3) and the following equation

$$\dot{y}(t)=\sum_{k=1}^{l}\frac{a_0^3}{8l}(t-h_k(t))x(\tilde{h}_k^1(t))+\sum_{k=1}^{l}\frac{a_0^2}{4}a_k(t)(g_k(t)-h_k(t))x(\tilde{g}_k^1(t)) \tag{3.4}$$

$$+\sum_{k=1}^{l}\left(\frac{a_0}{2}a_k(t)-\frac{a_0^2}{4l}-b_k(t)\right)x(h_k(t))+\sum_{k=l+1}^{m}\frac{a_0}{2}a_k(t)x(g_k(t))$$

$$+\frac{a_0}{2}y(t)-\sum_{k=1}^{m}a_k(t)y(g_k(t))-\sum_{k=1}^{l}\frac{a_0^2}{4l}(t-h_k(t))y(\tilde{h}_k^2(t))-\sum_{k=1}^{l}\frac{a_0}{2}a_k(t)(g_k(t)-h_k(t))y(\tilde{g}_k^2(t)).$$

For system (3.3), (3.4) we will check conditions of Lemma 2.10. We have

$$\frac{a_0}{2}a_k(t)-\frac{a_0^2}{4l}-b_k(t)\ge\frac{a_0^2}{2l}-\frac{a_0^2}{4l}-\frac{B_0}{l}=\frac{a_0^2}{4l}-\frac{B_0}{l}\ge 0.$$

Hence, condition a) of Lemma 2.10 holds.

To check condition b) of the lemma, consider the equation

$$\dot{y}(t) = \frac{a_0}{2}y(t) - \sum_{k=1}^{m} a_k(t)y(g_k(t)) - \sum_{k=1}^{l} \frac{a_0^2}{4l}(t - h_k(t))y(\tilde{h}_k^2(t)) \tag{3.5}$$

$$-\sum_{k=1}^{l} \frac{a_0}{2} a_k(t)(g_k(t) - h_k(t))y(\tilde{g}_k^2(t)).$$

After a substitution $y(t) = e^{\frac{a_0 t}{2}} z(t)$ equation (3.5) has a form

$$\dot{z}(t) = -\sum_{k=1}^{m} e^{-\frac{a_0(t-g_k(t)}{2}} a_k(t)z(g_k(t)) - \sum_{k=1}^{l} e^{-\frac{a_0(t-\tilde{h}_k^2(t)}{2}} \frac{a_0^2}{4l}(t - h_k(t))z(\tilde{h}_k^2(t)) \tag{3.6}$$

$$-\sum_{k=1}^{l} e^{-\frac{a_0(t-\tilde{g}_k^2(t)}{2}} \frac{a_0}{2} a_k(t)(h_k(t) - g_k(t))z(\tilde{g}_k^2(t)).$$

Equation (3.5) is nonoscillatory if equation (3.6) is nonoscillatory. We have $t - \tilde{g}_k^2(t) \geq t - g_k(t) \geq \sigma_k^0, t - \tilde{h}_k^2(t) \geq t - h_k(t) \geq \tau_k^0$,

$$\sum_{k=1}^{m} e^{-\frac{a_0(t-g_k(t)}{2}} a_k(t) + \sum_{k=1}^{l} e^{-\frac{a_0(t-\tilde{h}_k^2(t)}{2}} \frac{a_0^2}{4l}(t - h_k(t))$$

$$+\sum_{k=1}^{l} e^{-\frac{a_0(t-\tilde{g}_k^2(t)}{2}} \frac{a_0}{2} a_k(t)(h_k(t) - g_k(t))$$

$$\leq \sum_{j=1}^{m} e^{-\frac{a_0\sigma_j^0}{2}} a_j(s) + \sum_{i=1}^{l} \left(e^{-\frac{a_0\sigma_i^0}{2}} \frac{a_0}{2}\delta_i a_i(s) + e^{-\frac{a_0\tau_i^0}{2}} \frac{a_0^2}{4l} \right).$$

Corollary 2.11 and condition (3.1) imply that equation (3.6) is nonoscillatory.

Hence equation (3.5) is nonoscillatory and then condition b) of Lemma 2.10 holds.

Therefore the fundamental matrix $C(t,s)$ of the system (3.3), (3.4) is nonnegative.

Suppose now that $x(t), x(t_0) = 0, \dot{x}(t_0) = 1, x(t) = 0, t < t_0$ is the solution of (2.1). Hence $X(t) = \{x(t), y(t)\}^T, \{x(t_0), y(t_0)\}^T = \{0,1\}^T, X(t) = 0, t < t_0$ is a solution of system (3.3), (3.4). We have $X(t) = C(t,t_0)\{0,1\}^T \geq 0$. Hence $x(t) \geq 0$. But $x(t) = X(t,t_0)$ where $X(t,s)$ is the fundamental function of (2.1). Then $x(t) = X(t,t_0) \geq 0$. By the same way we can show that $X(t,s) \geq 0$ for any $s, t \geq s \geq t_0$.

The theorem is proven. □

Consider the equation with two delays

$$\ddot{x}(t) + a(t)\dot{x}(g(t)) + b(t)x(h(t)) = 0 \tag{3.7}$$

where for a, b, g, h conditions (a1)-(a2) hold.

Corollary 3.2. *Assume that for some $t_0 \geq 0$ and $t \geq t_0$*

$$0 < a_0 \leq a(t) \leq A_0, 0 \leq b_0 \leq b(t) \leq B_0, a_0^2 \geq 4B_0,$$

$$\tau^0 \leq t - h(t) \leq \tau, \sigma_0 \leq t - g(t) \leq \sigma, 0 \leq g(t) - h(t) \leq \delta.$$

If

$$\int_{g(t)}^{t}\left[e^{-\frac{a_0\sigma_0}{2}}a(s)\left(1+\frac{a_0\delta}{2}\right)+e^{-\frac{a_0\tau_0}{2}}\frac{a_0^2\tau}{4}\right]ds\le\frac{1}{e}, \tag{3.8}$$

then equation (3.7) has a nonnegative fundamental function.

Corollary 3.3. *Assume that for some $t_0\ge 0$ and $t\ge t_0$*

$$0<a_0\le a(t)\le A_0, 0\le b_0\le b(t)\le B_0, a_0^2\ge 4B_0, t-g(t)\le\sigma.$$

If

$$\int_{g(t)}^{t}e^{-\frac{a_0\sigma_0}{2}}a(s)\left(1+\frac{a_0\sigma}{2}\right)ds\le\frac{1}{e},$$

then equation

$$\ddot{x}(t)+a(t)\dot{x}(g(t))+b(t)x(t)=0 \tag{3.9}$$

has a nonnegative fundamental function.

Corollary 3.4. *Assume that for some $t_0\ge 0$ and $t\ge t_0$*

$$0<a_0\le a(t)\le A_0, 0\le b(t)\le B_0, a_0^2\ge 4B_0, c(t)\ge 0,$$

$$\int_{g(t)}^{t}\left(a(s)+e^{-\frac{a_0\sigma}{2}}c(s)\right)ds\le\frac{1}{e}.$$

Then the fundamental function of the equation

$$\ddot{x}(t)+a(t)\dot{x}(t)+c(t)\dot{x}(g(t))+b(t)x(t)=0 \tag{3.10}$$

is nonnegative.

4. Exponential Stability

Theorem 4.1. *Assume the fundamental function $X(t,s)$ of equation (2.1) is nonnegative, $\sum_{k=1}^m b_k(t)\ge b_0>0$ and the first order equation*

$$\dot{y}(t)+\sum_{k=1}^{m}a_k(t)y(g_k(t))=0 \tag{4.1}$$

is exponentially stable. Then equation (2.1) is exponentially stable.

Proof. We apply Lemma 2.7. Let f be an essentially bounded on $[t_0,\infty)$ function where $f(t)=0, t<\tau=\max\{\tau_k\}, k=1,\dots,m$ and x is the solution of problem (2.5). Then by Lemmas 2.4 and 2.9

$$x(t)=\int_{t_0+\tau}^{t}X(t,s)f(s)ds=\int_{t_0+\tau}^{t}X(t,s)\sum_{k=1}^{l}b_k(s)\frac{f(s)}{\sum_{k=1}^{l}b_k(s)}ds.$$

Hence $|x(t)|\le\|\frac{f}{\sum_{k=1}^{l}b_k}\|_{[t_0,\infty)}$. Then x is a bounded function on the interval $[t_0,\infty)$.

We will prove that $\dot{x}$ is also a bounded function. Denote $y=\dot{x}$. Then

$$\dot{y}(t)+\sum_{k=1}^{l}a_k(t)y(g_k(t))=f_1(t) \tag{4.2}$$

where $f_1(t) = f(t) - \sum_{k=1}^{m} b_k(t)x(h_k(t))$. Since x and f are bounded functions on the semi-axes, then f_1 is also a bounded function. Equation (4.2) is exponentially stable. Then the solution of (4.2) is a bounded on the semi-axes function. By Lemma 2.7 equation (2.1) is exponentially stable. □

Corollary 4.2. *Assume conditions of Corollary 3.2 hold, $b(t) \geq b_0 > 0$ and*

$$\limsup_{t\to\infty} \int_{g(t)}^{t} a(s)ds < 1 + \frac{1}{e} \tag{4.3}$$

then equation (3.7) is exponentially stable.

Proof. By Corollary 3.2 the fundamental function of equation (3.7) is nonnegative. Condition (4.3) implies [7] that the DDE of the first order

$$\dot{y}(t) + a(t)y(g(t)) = 0$$

is exponentially stable. By Theorem 4.1 equation (3.7) is exponentially stable. □

Corollary 4.3. *Assume conditions of Corollary 3.3, condition (4.3) hold, $b(t) \geq b_0 > 0$ then equation (3.9) is exponentially stable.*

Corollary 4.4. *Assume conditions of Corollary 3.4 hold, $b(t) \geq b_0 > 0$ and*

$$\int_{g(t)}^{t} c(s)ds \leq \frac{1}{e}, \ \frac{1}{\beta}e^{-g_0} > \ln\frac{\beta^2+\beta}{\beta^2+1}, \tag{4.4}$$

where $g_0 = \limsup_{t\to\infty} \int_{g(t)}^{t} a(s)ds, \beta = \limsup_{t\to\infty} \frac{c(t)}{a(t)}$. Then equation (3.10) is exponentially stable.

Proof. Corollary 3.4 implies that the fundamental function of (3.10) is nonnegative. By [7, Theorem 2] equation

$$\dot{y}(t) + a(t)y(t) + c(t)y(g(t)) = 0$$

is exponentially stable. Theorem 4.1 implies this corollary. □

We will now obtain exponential stability conditions for an equation (without assumption that this equation is nonoscillatory.

Theorem 4.5. *Assume that all conditions of Theorem 4.1 hold for the equation (2.1) and*

$$\left\| \frac{\sum_{k=1}^{l} c_k}{\sum_{k=1}^{l} b_k} - 1 \right\| < 1. \tag{4.5}$$

Then the following equation

$$\ddot{x}(t) + \sum_{k=1}^{m} a_k(t)\dot{x}(g_k(t)) + \sum_{k=1}^{l} c_k(t)x(h_k(t)) = 0 \tag{4.6}$$

is exponentially stable.

Proof. Consider for $t \geq t_0$ the problem

$$(Lx)(t) = f(t), t > 0;\ x(t) = \dot{x}(t) = 0, t \leq t_0 \tag{4.7}$$

with $f \in \mathbf{L}_\infty$. Equation (4.7) one can rewrite in the form

$$\begin{aligned} &\ddot{x}(t) + \sum_{k=1}^{m} c_k(t)\dot{x}(g_k(t)) + \sum_{k=1}^{l} b_k(t)x(h_k(t)) \\ &= -\sum_{k=1}^{l}(c_k(t) - b_k(t))x(h_k(t)) + f(t). \end{aligned} \tag{4.8}$$

Denote $X(t,s) \geq 0$ the fundamental function of equation (2.1). From (4.8) we have

$$x(t) = -\int_{t_0}^{t} X(t,s)\sum_{k=1}^{l}(c_k(s) - b_k(s))x(h_k(s))ds + f_1(t)$$

where $f_1(t) = \int_{t_0}^t X(t,s)f(s)ds$. Since equation (2.1) is exponentially stable then $f_1 \in L_\infty$. In the space L_∞ denote the operator

$$(Hx)(t) = -\int_{t_0}^{t} X(t,s)\sum_{k=1}^{l}(c_k(s) - b_k(s))x(h_k(s))ds.$$

We have

$$|(Hx)(t)| \leq \int_{t_0}^{t} X(t,s)\sum_{k=1}^{l} b_k(s)\left|\frac{\sum_{k=1}^{l} c_k}{\sum_{k=1}^{l} b_k} - 1\right| |x(h_k(s))|ds.$$

Lemma 2.9 implies that

$$\|H\| \leq \left\|\frac{\sum_{k=1}^{l} c_k}{\sum_{k=1}^{l} b_k} - 1\right\| < 1.$$

Hence the solution x of problem (4.7) is a bounded on the semi-axes $[t_0, \infty)$ function. Similarly to the proof of Theorem 4.1 we can show that $\dot{x} \in L_\infty$. By Lemma 2.7 equation (4.6) is exponentially stable. □

Corollary 4.6. *Assume that conditions of Corollary 3.3 hold,* $b(t) \geq b_0 > 0$ *and instead of condition* $a_0^2 \geq 4B_0$ *the ineaquality* $a_0^2 \geq 2B_0$ *holds. then equation (3.9) is exponentially stable.*

.

Proof. By corollary 3.3 the equation

$$\ddot{x}(t) + a(t)\dot{x}(g(t)) + \frac{a_0^2}{4}x(h(t)) = 0$$

is exponentially stable. Condition $a_0^2 \geq 2B_0$ implies that $\left\|\frac{b}{\frac{a_0^2}{4}} - 1\right\| < 1$. By Theorem 4.5 equation (3.9) is exponentially stable. □

Example 4.7. Consider the following equation

$$\ddot{x}(t) + \dot{x}(t - \alpha|\sin t|) + bx(t) = 0. \tag{4.9}$$

First find b and α for which equation (4.9) has a positive fundamental function. If we take take $\sigma = \alpha, \sigma_0 = 0$ then conditions of Corollary 3.3 hold for $0 \le b \le 0.25, \alpha < 0.3$.

Corollary 4.6 implies exponential stability of (4.9) for $0 < b \le 0.5, \alpha < 0.3$.

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