Applied Analysis and Optimization
Volume 3, Number 1, 2019, 83–101

Yokohama Publishers
ISSN 2189-1664 Online Journal

PDE-CONSTRAINED VECTOR VARIATIONAL PROBLEMS GOVERNED BY CURVILINEAR INTEGRAL FUNCTIONALS

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Abstract. This paper is concerned with necessary and sufficient efficiency conditions for a new class of multiobjective optimization problems. More precisely, we formulate and prove efficiency conditions for a multidimensional multiobjective variational problem of minimizing a vector of path-independent curvilinear integral functionals subject to nonlinear equality and inequality constraints involving higher-order partial derivatives. Under generalized (ρ, b)-quasiinvexity assumptions, sufficient efficiency conditions for a feasible solution are established.

1. Introduction

According to Chinchuluun and Pardalos [5], most optimization problems arising in practice have several objectives that need to be optimized simultaneously. This kind of problem is of considerable interest and includes various branches of mathematical science, engineering design, portofolio selection and game theory. For example, some multiobjective optimization problems occur when the torsion of prismatic bars is described in the elastic case (Sauer [18]), as well as in the elastic-plastic case (Ting [21]). Multiobjective variational problems subject to nonlinear equality and inequality constraints have been formulated and studied by many researchers. Singh and Hanson [20] have derived duality results using invex functions in vector ratio problems. Jeyakumar and Mond [9] have generalized these results for V-invex functions. Later, a unified formulation of generalized convexity, in order to derive duality results and optimality conditions, has been provided by Liang et al. [11]. As well, recently, Antczak and Arana-Jiménez [1] have generalized the concept of B-(p, r)-invexity (for differentiable scalar constrained optimization problems) to the continuous vector case. But, since so many phenomena are subject to laws involving partial differential equations (PDE) and/or partial differential inequations (PDI), there is a need for a consistent analysis of scalar and multiobjective optimization problems with PDE and/or PDI constraints and curvilinear/multiple integral functionals.

The notion of *multi-time* has been introduced in physics by Dirac, Fock and Podolsky [6] and, later, this term has also been used in mathematics. For more contributions and various approaches about different aspects of this concept, the reader is directed to Saunders [19], Udrişte and Ţevy [29], Cardin and Viterbo [3], Prepeliţă [17], Pitea et al. [15], Petrat and Tumulka [14], Treanţă [22, 23, 24, 25], Lienert and Nickel [12], Jayswal et al. [8]. Quite recently, Treanţă and Arana-Jiménez [27], using multiple integral cost functionals and an appropriate invexity,

2010 *Mathematics Subject Classification.* 65K10, 26B25, 90C30, 49K20.
Key words and phrases. Multiple objective optimization, higher-order jet bundle, curvilinear integral functional, (ρ, b)-quasiinvexity.

have extended to the multidimensional case the study in Arana-Jiménez et al. [2]. Also, Mititelu and Treanţă [13] have introduced and performed a study on efficiency conditions in vector control problems governed by multiple integrals. As well, several duality models (involving (ρ, b)-quasiinvexity for multidimensional multiobjective fractional control problems) have been formulated in Treanţă and Mititelu [28].

The main purpose of this paper is to state and prove efficiency conditions for a new class of multidimensional multiobjective variational problems with nonlinear equality and inequality constraints involving higher-order partial derivatives. More exactly, taking into account the results recently derived in Treanţă [26] (see *Sect.* 4), we introduce a multiobjective optimization problem of minimizing a vector of path-independent curvilinear integral functionals on higher-order jet bundles. The nonlinear equality and inequality type constraints, involving higher-order partial derivatives, automatically generate an increase in complexity of the methods used to determine solutions. Also, this new class of multiobjective variational problems, in short (MVP), requests specific techniques and mathematical tools, such as: an extended invexity and an appropriate mathematical framework involving geometric objects. Under generalized (ρ, b)-quasiinvexity assumptions, we establish sufficient efficiency conditions for a feasible solution in (MVP). As it is well known, the functionals of mechanical work type, due to their physical meaning, are very important in applications. Thus, the importance of this paper is supported both from theoretical and practical reasonings. For other different points of view but related to this topic, the reader is directed to Preda [16], Ferrara [7] and Kimand and Kim [10].

The outline of the paper is as follows. *Sect.* 2 introduces the necessary tools that will be used to prove the main results. *Sect.* 3 starts with two auxiliary lemmas and contains one of the main results of this paper (necessary efficiency conditions for (MVP)); the sufficient efficiency conditions for (MVP) are established in *Sect.* 4. Finally, *Sect.* 5 provides the conclusions of this study.

Let us start with two Riemannian manifolds, $(T, \mathbf{h})$ and $(M, \mathbf{g})$, of dimensions m, respectively n. In addition, assume that M is a complete manifold. Denote by $t = (t^\beta)$, $\beta = \overline{1,m}$, and $x = (x^i)$, $i = \overline{1,n}$, the local coordinates on $(T, \mathbf{h})$ and $(M, \mathbf{g})$, respectively. By using the *product order relation* on R^m_+, the hyper-parallelepiped $\Omega_{t_0,t_1} \subset R^m_+$, with the diagonally opposite points $t_0 = \left(t_0^1, \dots, t_0^m\right)$ and $t_1 = \left(t_1^1, \dots, t_1^m\right)$, can be written as being the interval $[t_0; t_1]$. Let $J^{s-1}(T, M)$ be the $(s-1)$-th order jet bundle associated to T and M, where $s \geq 2$ is a fixed natural number, and let Γ_{t_0,t_1} be a piecewise C^{s-1}-class curve joining the points t_0 and t_1. Consider the following path-independent curvilinear integral functionals

$$F^l\left(x(\cdot)\right) := \int_{\Gamma_{t_0,t_1}} X^l_\beta\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) dt^\beta, \quad l = \overline{1,r},\ \beta = \overline{1,m},$$

determined by the (higher-order) closed Lagrange 1-form densities of C^∞-class

$$X_\beta = \left(X^l_\beta\right) : J^{s-1}(T, M) \to R^r, \quad l = \overline{1,r},\ \beta = \overline{1,m},$$

where we have used the notations:

$$\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t) := \left(t, x(t), x_{\alpha_1}(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t)\right), \quad t \in \Omega_{t_0,t_1},$$

with $x_{\alpha_1}(t) := \frac{\partial x}{\partial t^{\alpha_1}}(t), \dots, x_{\alpha_1\alpha_2\dots\alpha_{s-1}}(t) := \frac{\partial^{s-1}x}{\partial t^{\alpha_1}\partial t^{\alpha_2}\dots\partial t^{\alpha_{s-1}}}(t)$, $\alpha_j \in \{1,2,\dots,m\}$, $j = \overline{1,s-1}$, $x = (x^1,\dots,x^n) = \left(x^i\right)$, $i = \overline{1,n}$. The closeness conditions (complete integrability conditions) associated to $\left(X^l_\beta\right)$ are

$$D_\eta X^l_\beta = D_\beta X^l_\eta, \quad \beta,\eta = \overline{1,m},\ \beta \neq \eta,\ l = \overline{1,r},$$

where $D_\eta := \frac{\partial}{\partial t^\eta}$ denotes the total derivative operator.

Also, let us consider the following PDI, respectively PDE of evolution

$$g\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) \leq 0, \quad h\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) = 0, \quad t \in \Omega_{t_0,t_1}, \tag{1.1}$$

generated by the C^∞-class Lagrange matrix densities

$$g = \left(g^b_a\right) : J^{s-1}(T,M) \to R^{pq}, \quad a = \overline{1,q},\ b = \overline{1,p},\ p < n,$$

$$h = \left(h^b_a\right) : J^{s-1}(T,M) \to R^{de}, \quad a = \overline{1,e},\ b = \overline{1,d},\ d < n.$$

We point out that in (1.1) we have used the following assumptions:

$$u = v \Leftrightarrow u_i = v_i, \quad u \leq v \Leftrightarrow u_i \leq v_i,$$

$$u < v \Leftrightarrow u_i < v_i, \quad u \preceq v \Leftrightarrow u \leq v,\ u \neq v, \quad i = \overline{1,k},$$

for any two vectors $u = (u_1,\dots,u_k), v = (v_1,\dots,v_k)$ in R^k.

Using the space

$$C^\infty\left(\Omega_{t_0,t_1}, M\right) = \left\{x : \Omega_{t_0,t_1} \to M;\ x\ of\ C^\infty - class\right\},$$

endowed with the distance $d\left(x,x^0\right) = d\left(x(\cdot),x^0(\cdot)\right) = \sup_{t\in\Omega} d_{\mathbf{g}}\left(x(t),x^0(t)\right)$, where $d_{\mathbf{g}}\left(x(t),x^0(t)\right)$ is geodesic distance in $(M,\mathbf{g})$, we introduce the *set* $\mathtt{F}(\Omega_{t_0,t_1})$ *of all feasible solutions (domain)*

$$x \in C^\infty\left(\Omega_{t_0,t_1},M\right),\ g\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) \leq 0,\ h\left(\chi_{x_{\alpha_1\dots\alpha_{s-1}}}(t)\right) = 0,\ t \in \Omega_{t_0,t_1}$$

$$x(t_\xi) = x_\xi,\ x_{\alpha_1\dots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\dots\alpha_j\xi},\ \alpha_\zeta \in \{1,\dots,m\},\ \zeta,j = \overline{1,s-2},\ \xi \in \{0,1\},$$

for the following multidimensional multiobjective variational problem

$$(MVP) \qquad \min_{x(\cdot)} \left(F^1\left(x(\cdot)\right), F^2\left(x(\cdot)\right), \dots, F^r\left(x(\cdot)\right)\right)$$

$$\text{subject to } x(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1}),$$

a higher-order PDE&PDI-constrained vector optimization problem.

Note. We can consider the following constraints (boundary conditions)

$$x(t)|_{\partial\Omega_{t_0,t_1}} = \text{given},\ x_{\alpha_1}(t)|_{\partial\Omega_{t_0,t_1}} = \text{given}, \dots, x_{\alpha_1\dots\alpha_{s-2}}(t)|_{\partial\Omega_{t_0,t_1}} = \text{given},$$

instead of

$$x(t_\xi) = x_\xi,\ x_{\alpha_1\dots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\dots\alpha_j\xi},\ \alpha_\zeta \in \{1,\dots,m\},\ \zeta,j = \overline{1,s-2},\ \xi \in \{0,1\}.$$

2. Preliminaries

We start this section by considering the following scalar optimization problem

$$(SP) \qquad \min_{x(\cdot)} \int_{\Gamma_{t_0,t_1}} X_\beta \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) dt^\beta$$

$$\text{subject to } x(\cdot) \in F(\Omega_{t_0,t_1}).$$

The necessary conditions for the optimality of a feasible solution $x^0 \in F(\Omega_{t_0,t_1})$ in the problem (SP) are

$$\frac{\partial L_\beta}{\partial x^i} - D_{\alpha_1} \frac{\partial L_\beta}{\partial x^i_{\alpha_1}} + \frac{1}{n(\alpha_1, \alpha_2)} D^2_{\alpha_1 \alpha_2} \frac{\partial L_\beta}{\partial x^i_{\alpha_1 \alpha_2}} - \frac{1}{n(\alpha_1, \alpha_2, \alpha_3)} D^3_{\alpha_1 \alpha_2 \alpha_3} \frac{\partial L_\beta}{\partial x^i_{\alpha_1 \alpha_2 \alpha_3}}$$

$$+ \dots + (-1)^{s-1} \frac{1}{n(\alpha_1, \alpha_2, \dots, \alpha_{s-1})} D^{s-1}_{\alpha_1 \alpha_2 \dots \alpha_{s-1}} \frac{\partial L_\beta}{\partial x^i_{\alpha_1 \alpha_2 \dots \alpha_{s-1}}} = 0,$$

$$i \in \{1, 2, \dots, n\}, \ \beta \in \{1, 2, \dots, m\} \qquad \text{(higher-order Euler-Lagrange PDE)}$$

$$\mu_\beta(t) g \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) = 0, \quad \mu_\beta(t) \geq 0, \quad t \in \Omega_{t_0,t_1},$$

where:

$$\bullet \ D^{s-1}_{\alpha_1 \dots \alpha_{s-1}} := \frac{\partial^{s-1}}{\partial t^{\alpha_1} \dots \partial t^{\alpha_{s-1}}}$$

is the total derivative operator of order $s-1$;

$$\bullet \ L_\beta \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t), \mu(t), \nu(t), \lambda \right) := \lambda X_\beta \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right)$$

$$+ \mu_\beta(t) g \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) + \nu_\beta(t) h \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right), \quad \beta = \overline{1, m},$$

is an auxiliary Lagrange 1-form density, with λ a real number, and

$$\mu(t) := (\mu_\beta(t)) = \left(\mu^a_{\beta b}(t) \right), \quad a = \overline{1, q}, \ b = \overline{1, p},$$

$$\nu(t) := (\nu_\beta(t)) = \left(\nu^a_{\beta b}(t) \right), \quad a = \overline{1, e}, \ b = \overline{1, d},$$

the Lagrange multipliers subject to the condition that the 1-form L_β is closed. Also, we accept the following notations:

$$\bullet \ \chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) := \left(t, x^0(t), x^0_{\alpha_1}(t), \dots, x^0_{\alpha_1 \alpha_2 \dots \alpha_{s-1}}(t) \right), \quad t \in \Omega_{t_0,t_1};$$

$$\bullet \ n(\alpha_1, \alpha_2, \dots, \alpha_k) := \frac{| \, 1_{\alpha_1} + 1_{\alpha_2} + \dots + 1_{\alpha_k} \, |!}{(1_{\alpha_1} + 1_{\alpha_2} + \dots + 1_{\alpha_k})!}$$

represents the Saunders number (see Saunders [19]). For a better understanding of this number we consider some

Particular cases.

- $k = 1$ involves: $n(\alpha_1) = 1$
- $k = 2$ involves:

$n(\alpha_1, \alpha_2) = 1$, for $\alpha_1 = \alpha_2$
$n(\alpha_1, \alpha_2) = 2$, for $\alpha_1 \neq \alpha_2$

- $k = 3$ involves:

$n(\alpha_1, \alpha_2, \alpha_3) = 1$, for $\alpha_1 = \alpha_2 = \alpha_3$
$n(\alpha_1, \alpha_2, \alpha_3) = 3$, for $\alpha_1 = \alpha_2 \neq \alpha_3$
$n(\alpha_1, \alpha_2, \alpha_3) = 6$, for $\alpha_1 \neq \alpha_2 \neq \alpha_3$.

Above and below, the summation over the repeated indices is assumed. Now, according to Valentine [30], we reformulate the foregoing necessary optimality conditions in (SP) as follows:

Theorem 2.1. *Suppose that $x^0 \in \mathtt{F}(\Omega_{t_0,t_1})$, a feasible solution of the problem (SP), is an optimal solution and $X = (X_\beta)$ (a closed 1-form), g, h are functions of C^∞-class. Then, there exist the multipliers λ, $\mu(t)$ and $\nu(t)$, satisfying the following conditions*

$$\begin{aligned}
&\lambda \frac{\partial X_\beta}{\partial x^i}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \mu_\beta(t)\frac{\partial g}{\partial x^i}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \nu_\beta(t)\frac{\partial h}{\partial x^i}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\\
&\quad - D_{\alpha_1}\left\{\lambda \frac{\partial X_\beta}{\partial x^i_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \mu_\beta(t)\frac{\partial g}{\partial x^i_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}\\
&\quad - D_{\alpha_1}\left\{\nu_\beta(t)\frac{\partial h}{\partial x^i_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}\\
&\quad + \frac{1}{n(\alpha_1,\alpha_2)}D^2_{\alpha_1\alpha_2}\left\{\lambda\frac{\partial X_\beta}{\partial x^i_{\alpha_1\alpha_2}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}\\
&\quad + \frac{1}{n(\alpha_1,\alpha_2)}D^2_{\alpha_1\alpha_2}\left\{\mu_\beta(t)\frac{\partial g}{\partial x^i_{\alpha_1\alpha_2}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \nu_\beta(t)\frac{\partial h}{\partial x^i_{\alpha_1\alpha_2}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}\\
&\quad - \cdots + (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\lambda\frac{\partial X_\beta}{\partial x^i_{\alpha_1\alpha_2\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}\\
&\quad + (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\mu_\beta(t)\frac{\partial g}{\partial x^i_{\alpha_1\alpha_2\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}\\
&\quad + (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\nu_\beta(t)\frac{\partial h}{\partial x^i_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\} = 0
\end{aligned}$$

$$\text{(higher order Euler-Lagrange PDE)} \quad \beta = \overline{1,m}, \quad i = \overline{1,n}$$

$$\mu_\beta(t) g\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) = 0, \quad \mu_\beta(t) \geq 0, \quad t \in \Omega_{t_0,t_1}.$$

Definition 2.2. The optimal solution $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ of the problem (SP) is called normal optimal solution if $\lambda \neq 0$.

Without loss of generality, we can suppose that $\lambda = 1$.

Definition 2.3. A feasible solution $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ of the problem (MVP) is called efficient solution if there exists no other feasible solution $x(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ such that $F(x(\cdot)) \preceq F\left(x^0(\cdot)\right)$, where

$$F(x(\cdot)) := \left(\int_{\Gamma_{t_0,t_1}} X^1_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)dt^\beta, \ldots, \int_{\Gamma_{t_0,t_1}} X^r_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)dt^\beta\right).$$

3. The first main result: Necessary efficiency conditions in (MVP)

Now, we are in a position to prove a first part of the main results. In order to develop an optimization theory on higher-order jet bundles, in accordance with Chankong and Haimes [4], let us establish the following results.

Lemma 3.1. *Let x^0 be a feasible solution of (MVP). It is an efficient solution of the problem (MVP) if and only if $x^0 \in \mathtt{F}(\Omega_{t_0,t_1})$ is an optimal solution of each problem $P_l(x^0)$, $l = \overline{1,r}$,*

$$\min_{x(\cdot)} \int_{\Gamma_{t_0,t_1}} X^l_\beta \left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta$$

subject to

$$x(t_\xi) = x_\xi,\ x_{\alpha_1\ldots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\ldots\alpha_j\xi},\ \alpha_\zeta \in \{1,\ldots,m\},\ \zeta, j = \overline{1, s-2},\ \xi \in \{0,1\}$$

$$g\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) \leq 0, \quad h\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) = 0, \quad t \in \Omega_{t_0,t_1}$$

$$\int_{\Gamma_{t_0,t_1}} X^c_\beta \left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta \leq \int_{\Gamma_{t_0,t_1}} X^c_\beta \left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta,\ c = \overline{1,r},\ c \neq l.$$

Proof. " $\Longrightarrow$ " Let x^0 be an efficient solution of the problem (MVP). Assume there exists $k \in \{1,\ldots,r\}$ such that $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ is not an optimal solution of the problem $P_k(x^0)$. Thus, there exists a function $y(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ such that

$$\int_{\Gamma_{t_0,t_1}} X^c_\beta \left(\chi_{y_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta \leq \int_{\Gamma_{t_0,t_1}} X^c_\beta \left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta,\ c = \overline{1,r},\ c \neq k$$

and

$$\int_{\Gamma_{t_0,t_1}} X^k_\beta \left(\chi_{y_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta < \int_{\Gamma_{t_0,t_1}} X^k_\beta \left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta.$$

This contradicts the efficiency of the function $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ in (MVP) and, consequently, we have proved the direct implication.

" $\Longleftarrow$ " Consider $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ an optimal solution of each problem $P_l(x^0)$, $l = \overline{1,r}$, and suppose that $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ is not an efficient solution of the problem (MVP). Consequently, there exists a function $y(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ such that

$$\int_{\Gamma_{t_0,t_1}} X^c_\beta \left(\chi_{y_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta \leq \int_{\Gamma_{t_0,t_1}} X^c_\beta \left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta, \quad c = \overline{1,r}$$

and there exists $k \in \{1,\ldots,r\}$ such that

$$\int_{\Gamma_{t_0,t_1}} X^k_\beta \left(\chi_{y_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta < \int_{\Gamma_{t_0,t_1}} X^k_\beta \left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta.$$

Also, the function $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ minimizes the functional

$$\int_{\Gamma_{t_0,t_1}} X^k_\beta \left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta$$

on the set of all feasible solutions of the problem $P_k(x^0)$ and the proof is complete. □

Lemma 3.2. *Consider $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ an optimal solution of the problem $P_l(x^0)$, $l \in \{1,\ldots,r\}$ fixed. Then, there exist the multipliers $\lambda_{cl} \geq 0$, $c = \overline{1,r}$, $\mu_l(t)$ and $\nu_l(t)$ such that the following conditions are fulfilled:*

$$\sum_{c=1}^r \lambda_{cl} \frac{\partial X^c_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \mu_{l\beta}(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \nu_{l\beta}(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)$$

$$- \; D_{\alpha_1}\left\{\sum_{c=1}^{r}\lambda_{cl}\frac{\partial X^c_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)+\mu_{l\beta}(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}$$

$$- \; D_{\alpha_1}\left\{\nu_{l\beta}(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}$$

$$+\cdots+(-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\sum_{c=1}^{r}\lambda_{cl}\frac{\partial X^c_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}$$

$$+(-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\mu_{l\beta}(t)\frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}$$

$$+(-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\nu_{l\beta}(t)\frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}=0$$

(higher order Euler-Lagrange PDE), $\beta=\overline{1,m}$

$$\mu_{l\beta}(t)g\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)=0,\quad \mu_{l\beta}(t)\geq 0,\quad t\in\Omega_{t_0,t_1}.$$

Proof. Consider the C^∞-class functions

$$\phi^c=\left(\phi^c_\beta\right),\quad \phi^c\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\geq 0,\quad c=\overline{1,r},\ c\neq l,\ \beta=\overline{1,m},$$

defined as follows

$$G_c(x(t)):=\int_{\Gamma_{t_0,t_1}}\left[X^c_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)-R^c_0+\phi^c_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta=0,$$

with

$$R^l_0:=\int_{\Gamma_{t_0,t_1}}X^l_\beta\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)dt^\beta=\min_{x(\cdot)}\int_{\Gamma_{t_0,t_1}}X^l_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)dt^\beta,$$

$l=\overline{1,r},\ x^0(\cdot)\in F(\Omega_{t_0,t_1})$. Thus, the problem $P_l(x^0)$, $l\in\{1,\ldots,r\}$ fixed, can be changed into

$$\max_{x(\cdot)}\int_{\Gamma_{t_0,t_1}}X^l_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)dt^\beta$$

subject to

$$x\in F(\Omega_{t_0,t_1}),\quad G_c(x(t))=0$$

$$\phi^c\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\geq 0,\quad c=\overline{1,r},\ c\neq l,\quad \beta=\overline{1,m},$$

or

$$\max_{x(\cdot)}\int_{\Gamma_{t_0,t_1}}\left\{X^l_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}\right)\right.$$

$$\left.+\sum_{c=1;\,c\neq l}^{r}\lambda_{cl}\left[X^c_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}\right)-R^c_0+\phi^c_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}\right)\right]\right\}dt^\beta$$

subject to

$$x \in \mathtt{F}(\Omega_{t_0,t_1}), \quad \phi^c\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right) \geq 0, \quad c = \overline{1,r},\ c \neq l, \quad \beta = \overline{1,m},$$

equivalent to

$$\max_{x(\cdot)} \int_{\Gamma_{t_0,t_1}} \left\{ X^l_\beta\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right) + \sum_{c=1;\ c\neq l}^{r} \lambda_{cl}\left[X^c_\beta\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right) - R^c_0 + \phi^c_\beta\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right)\right]\right\} dt^\beta \tag{3.1}$$

subject to

$$x \in C^\infty(\Omega_{t_0,t_1}, M),\ g\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) \leq 0,\ h\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) = 0,\ t \in \Omega_{t_0,t_1}$$
$$x(t_\xi) = x_\xi,\ x_{\alpha_1 \ldots \alpha_j}(t_\xi) = \tilde{x}_{\alpha_1 \ldots \alpha_j \xi},\ \alpha_\zeta \in \{1, \ldots, m\},\ \zeta, j = \overline{1, s-2},\ \xi \in \{0,1\}$$
$$-\phi^c\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right) \leq 0, \quad c = \overline{1,r},\ c \neq l, \quad \beta = \overline{1,m}.$$

Define the following Lagrange function (or, *Lagrangian*)

$$V_l\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}, \mu_{l\beta}, \nu_{l\beta}, \gamma_l, a_c\right) := \gamma_l X^l_\beta\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right)$$
$$+ \sum_{c=1;\ c\neq l}^{r} \gamma_l \lambda_{cl}\left[X^c_\beta\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right) - R^c_0 + \phi^c_\beta\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right)\right]$$
$$+\mu_{l\beta}(t) g\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right) + \nu_{l\beta}(t) h\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right) - \sum_{c=1;\ c\neq l}^{r} a_c(t)\phi^c_\beta\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}\right).$$

We have used the following notations:

$$\gamma_l \in R,\ \gamma_l \geq 0, \quad a_c \in C^\infty(\Omega_{t_0,t_1}, R),\ a_c \geq 0,\ c = \overline{1,r},\ c \neq l$$
$$\mu_l(t) := (\mu_{l\beta}(t)) = \left(\mu^a_{l\beta b}(t)\right), \quad a = \overline{1,q},\ b = \overline{1,p},$$
$$\nu_l(t) := (\nu_{l\beta}(t)) = \left(\nu^a_{l\beta b}(t)\right), \quad a = \overline{1,e},\ b = \overline{1,d},\ l = \overline{1,r}\ \text{(fixed)}.$$

Since $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ is an optimal solution for the previous optimization problem (3.1), the following necessary conditions are satisfied

$$\frac{\partial V_l}{\partial x}\left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}, \mu_{l\beta}, \nu_{l\beta}, \gamma_l, a_c\right) - D_{\alpha_1}\frac{\partial V_l}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}, \mu_{l\beta}, \nu_{l\beta}, \gamma_l, a_c\right)$$
$$+ \frac{1}{n(\alpha_1, \alpha_2)} D^2_{\alpha_1\alpha_2}\frac{\partial V_l}{\partial x_{\alpha_1\alpha_2}}\left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}, \mu_{l\beta}, \nu_{l\beta}, \gamma_l, a_c\right)$$
$$- \frac{1}{n(\alpha_1, \alpha_2, \alpha_3)} D^3_{\alpha_1\alpha_2,\alpha_3}\frac{\partial V_l}{\partial x_{\alpha_1\alpha_2\alpha_3}}\left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}, \mu_{l\beta}, \nu_{l\beta}, \gamma_l, a_c\right) + \cdots + (-1)^{s-1}$$
$$\frac{1}{n(\alpha_1, \ldots, \alpha_{s-1})} D^{s-1}_{\alpha_1 \ldots \alpha_{s-1}}\frac{\partial V_l}{\partial x_{\alpha_1 \ldots \alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}, \mu_{l\beta}, \nu_{l\beta}, \gamma_l, a_c\right) = 0,$$

(higher-order Euler-Lagrange PDE)

$$\mu_{l\beta}(t) g\left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) = 0, \quad \mu_{l\beta}(t) \geq 0, \quad \beta = \overline{1,m}, \quad t \in \Omega_{t_0,t_1},$$
$$a_c(t)\phi^c_\beta\left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) = 0, \quad a_c(t) \geq 0, \quad c = \overline{1,r}, \quad c \neq l$$
$$\gamma_l \geq 0, \quad \lambda_{cl} \geq 0, \quad c = \overline{1,r}, \quad c \neq l.$$

Concretely, we have

$$\gamma_l \frac{\partial X_\beta^l}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) + \sum_{c=1;\ c\neq l}^{r} \gamma_l \lambda_{cl}\left[\frac{\partial X_\beta^c}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) + \frac{\partial \phi_\beta^c}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right]$$
$$+ \mu_{l\beta}(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) + \nu_{l\beta}(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) - \sum_{c=1;\ c\neq l}^{r} a_c(t)\frac{\partial \phi_\beta^c}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)$$
$$- D_{\alpha_1}\left\{\gamma_l \frac{\partial X_\beta^l}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) + \sum_{c=1;\ c\neq l}^{r} \gamma_l \lambda_{cl}\frac{\partial X_\beta^c}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}$$
$$- D_{\alpha_1}\left\{\sum_{c=1;\ c\neq l}^{r} \gamma_l \lambda_{cl}\frac{\partial \phi_\beta^c}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) + \mu_{l\beta}(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}$$
$$- D_{\alpha_1}\left\{\nu_{l\beta}(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) - \sum_{c=1;\ c\neq l}^{r} a_c(t)\frac{\partial \phi_\beta^c}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}$$
$$+ \cdots + (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\gamma_l \frac{\partial X_\beta^l}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}$$
$$+ (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\sum_{c=1;\ c\neq l}^{r} \gamma_l \lambda_{cl}\frac{\partial X_\beta^c}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}$$
$$+ (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\sum_{c=1;\ c\neq l}^{r} \gamma_l \lambda_{cl}\frac{\partial \phi_\beta^c}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}$$
$$+ (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\mu_{l\beta}(t)\frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}$$
$$+ (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\nu_{l\beta}(t)\frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}$$
$$+ (-1)^{s}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\sum_{c=1;\ c\neq l}^{r} a_c(t)\frac{\partial \phi_\beta^c}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\} = 0,$$

equivalent to

$$\gamma_l \frac{\partial X_\beta^l}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) + \sum_{c=1;\ c\neq l}^{r} \left[\gamma_l \lambda_{cl} - a_c(t)\right]\frac{\partial \phi_\beta^c}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)$$
$$+ \mu_{l\beta}(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) + \nu_{l\beta}(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)$$
$$+ \sum_{c=1;\ c\neq l}^{r} \gamma_l \lambda_{cl}\frac{\partial X_\beta^c}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)$$

$$
\begin{aligned}
&- D_{\alpha_1}\left\{\gamma_l \frac{\partial X^l_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) + \sum_{c=1;\ c\neq l}^{r} \gamma_l\lambda_{cl}\frac{\partial X^c_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}\\
&- D_{\alpha_1}\left\{\sum_{c=1;\ c\neq l}^{r}\left[\gamma_l\lambda_{cl} - a_c(t)\right]\frac{\partial \phi^c_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right) + \mu_{l\beta}(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}\\
&- D_{\alpha_1}\left\{\nu_{l\beta}(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}\\
&+ \cdots + (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\gamma_l\frac{\partial X^l_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}\\
&+ (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\sum_{c=1;\ c\neq l}^{r}\gamma_l\lambda_{cl}\frac{\partial X^c_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}\\
&+ (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\\
&\qquad\qquad\left\{\sum_{c=1;\ c\neq l}^{r}\left[\gamma_l\lambda_{cl} - a_c(t)\right]\frac{\partial \phi^c_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}\\
&+ (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\mu_{l\beta}(t)\frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\}\\
&+ (-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\nu_{l\beta}(t)\frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}\right)\right\} = 0.
\end{aligned}
$$

Imposing the conditions: $\gamma_l\lambda_{cl} - a_c(t) = 0,\ c = \overline{1,r},\ c \neq l$, for any $t \in \Omega_{t_0,t_1}$, $\gamma_l = \lambda_{ll} \geq 0,\ \lambda_{cl} = \gamma_l\lambda_{cl} \geq 0,\ c = \overline{1,r},\ c \neq l$, we obtain the conclusion of Lemma 3.2 and the proof is complete. □

Definition 3.3. The feasible solution $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ is called normal efficient solution of the problem (MVP) if it is a normal optimal solution for at least one of the scalar problems $P_l(x^0),\ l = \overline{1,r}$.

Now, we have all the necessary mathematical tools for establishing the main result of this section, namely, the normal necessary efficiency conditions of (MVP).

Theorem 3.4 (Normal necessary efficiency conditions for (MVP))**.** *Let $x^0(\cdot) \in \mathtt{F}(\Omega_{t_0,t_1})$ be a [normal] efficient solution of the problem (MVP). Then, there exist $\lambda \in R^r$, μ and ν such that the following conditions are satisfied:*

$$
\begin{aligned}
&\sum_{c=1}^{r}\lambda_c\frac{\partial X^c_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \mu_\beta(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \nu_\beta(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\\
&\qquad - D_{\alpha_1}\left\{\sum_{c=1}^{r}\lambda_c\frac{\partial X^c_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}\\
&\qquad\qquad - D_{\alpha_1}\left\{\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right\}
\end{aligned}
$$

$$+\cdots+(-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\sum_{c=1}^{r}\lambda_c\frac{\partial X^c_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right\}$$

$$+(-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right\}$$

$$+(-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right\}=0$$

$$\textit{(higher order Euler-Lagrange PDE)},\quad \beta=\overline{1,m}$$

$$\mu_\beta(t)g\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)=0,\quad \mu_\beta(t)\geq 0,\quad t\in\Omega_{t_0,t_1},\quad \beta=\overline{1,m}$$

$$\lambda\geq 0,\quad e^t\lambda=1,\quad e^t=(1,1,\ldots,1)\in R^r.$$

Proof. We have that $x^0(\cdot)\in F(\Omega_{t_0,t_1})$ is an optimal solution of each problem $P_l(x^0),\ l=\overline{1,r}$ (see Lemma 3.1). Thus, if $x^0(\cdot)\in F(\Omega_{t_0,t_1})$ is [normal] optimal solution in $P_l(x^0),\ l\in\{1,\ldots,r\}$ fixed, then the conditions which appear in Lemma 3.2 are fulfilled $[\lambda_{ll}=1]$. Making the sum from $l=1$ to $l=r$ of all relations in Lemma 3.2 and setting

$$\sum_{l=1}^{r}\lambda_{cl}=\tilde{\lambda}_c,\quad \sum_{l=1}^{r}\mu_{l\beta}(t)=\tilde{\mu}_\beta(t),\quad \sum_{l=1}^{r}\nu_{l\beta}(t)=\tilde{\nu}_\beta(t),$$

we obtain

$$\sum_{c=1}^{r}\tilde{\lambda}_c\frac{\partial X^c_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)+\tilde{\mu}_\beta(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)+\tilde{\nu}_\beta(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)$$

$$-D_{\alpha_1}\left\{\sum_{c=1}^{r}\tilde{\lambda}_c\frac{\partial X^c_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)+\tilde{\mu}_\beta(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right\}$$

$$-D_{\alpha_1}\left\{\tilde{\nu}_\beta(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right\}$$

$$+\cdots+(-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\sum_{c=1}^{r}\tilde{\lambda}_c\frac{\partial X^c_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right\}$$

$$+(-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\tilde{\mu}_\beta(t)\frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right\}$$

$$+(-1)^{s-1}\frac{1}{n(\alpha_1,\ldots,\alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\left\{\tilde{\nu}_\beta(t)\frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right\}=0$$

$$\text{(higher order Euler-Lagrange PDE)},\quad \beta=\overline{1,m}$$

$$\tilde{\mu}_\beta(t)g\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)=0,\quad \tilde{\mu}_\beta(t)\geq 0,\quad t\in\Omega_{t_0,t_1}$$

$$\tilde{\lambda}_c\geq 0,\quad \left[\tilde{\lambda}_c\geq 1\right],\quad c=\overline{1,r}.$$

Consider $S:=\sum_{c=1}^{r}\tilde{\lambda}_c\geq 1$ and denoting $\lambda_c=\tilde{\lambda}_c/S,\ \mu_\beta(t)=\tilde{\mu}_\beta(t)/S,\ \nu_\beta(t)=\tilde{\nu}_\beta(t)/S$, we get the conclusions of Theorem 3.4 are valid. □

4. The second main result: Sufficient efficiency conditions in (MVP)

In the following, in order to formulate and prove sufficient conditions of efficiency associated with the vector optimization problem (MVP), we will introduce a *generalized quasiinvexity*. Consider ρ a real number and $b\colon\ [C^\infty(\Omega_{t_0,t_1}, M)]^{2s} \to [0,\infty)$ a functional. Further, let us consider the notations

$$b\left(x(\cdot), x_{\alpha_1}(\cdot), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(\cdot), x^0(\cdot), x^0_{\alpha_1}(\cdot), \ldots, x^0_{\alpha_1\ldots\alpha_{s-1}}(\cdot)\right) := b_{xx^0}$$

$$\eta\left(t, x(t), x_{\alpha_1}(t), \ldots, x_{\alpha_1\ldots\alpha_{s-1}}(t), x^0(t), x^0_{\alpha_1}(t), \ldots, x^0_{\alpha_1\ldots\alpha_{s-1}}(t)\right) := \eta_{txx^0}, \quad t \in \Omega_{t_0,t_1}.$$

Also, let $a = (a_\beta) : J^{s-1}(T, M) \to \mathbf{R}^m$ be a closed Lagrange 1-form that determines the following path-independent curvilinear integral functional

$$A(x(\cdot)) = \int_{\Gamma_{t_0,t_1}} a_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta.$$

Definition 4.1. The functional A is [strictly] (ρ, b)-quasiinvex at $x^0(\cdot)$ if there exist the vector functions $\eta = (\eta_1, \ldots, \eta_n)$, with the property

$$\eta_{tx^0x^0} = 0,\ D_{\alpha_1}\eta_{tx^0x^0} = 0,\ \cdots,\ D^{s-2}_{\alpha_1\ldots\alpha_{s-2}}\eta_{tx^0x^0} = 0$$

$$\alpha_\zeta \in \{1, \ldots, m\}, \quad \zeta = \overline{1, s-2}, \quad t \in \Omega_{t_0,t_1},$$

and $\theta\colon\ [C^\infty(\Omega_{t_0,t_1}, M)]^{2s} \to \mathbf{R}^n$ such that, for any $x(\cdot)$ $[x(\cdot) \neq x^0(\cdot)]$, we have the following implication:

$$\left[A(x(\cdot)) \leq A(x^0(\cdot))\right]$$

$$\Longrightarrow [b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\eta_{txx^0}\frac{\partial a_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + (D_{\alpha_1}\eta_{txx^0})\frac{\partial a_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$\cdots + b_{xx^0}\int_{\Gamma_{t_0,t_1}}\left[\left(\frac{1}{n(\alpha_1, \ldots, \alpha_{s-1})}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right)\frac{\partial a_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$[<] \leq -\rho b_{xx^0} \parallel \theta_{xx^0} \parallel^2].$$

Example. Consider

$$x : [0,1] \to \mathfrak{M} \subseteq R^2, \quad x(t) = \left(x^1(t), x^2(t)\right),$$

a C^2-class function defined on the real interval $[0,1]$. Let $h : [0,1] \times \mathfrak{M} \to \mathbb{R}$ be a continuously differentiable function. The following functional of curvilinear integral type

$$H(x(\cdot)) = \int_0^1 h(t, \ddot{x}(t))\, dt$$

is, as it can be verified, $(\rho, 1)$-quasiinvex, for $\rho \leq 0$ and any θ, at $x^0(\cdot)$ with respect to

$$\begin{aligned}
&\eta\left(t, x(t), \dot{x}(t), \ddot{x}(t), x^0(t), \dot{x}^0(t), \ddot{x}^0(t)\right)\\
&\quad = \left(\eta_1(t, x(t), \dot{x}(t), \ddot{x}(t), x^0(t), \dot{x}^0(t), \ddot{x}^0(t)), \eta_2(t, x(t), \dot{x}(t), \ddot{x}(t), x^0(t), \dot{x}^0(t), \ddot{x}^0(t))\right)\\
&\quad = \left(H(x(\cdot)) - H\left(x^0(\cdot)\right)\right)\left(D^2\frac{\partial h}{\partial \ddot{x}^1}\left(t, \ddot{x}^0(t)\right), D^2\frac{\partial h}{\partial \ddot{x}^2}\left(t, \ddot{x}^0(t)\right)\right).
\end{aligned}$$

The previous example can be easily extended to n-dimensional vector valued functions and, by using normal coordinates, to the multidimensional case.

Now, we are able to prove the second part of the main results.

Theorem 4.2 (Sufficient efficiency conditions for (MVP))**.** *Let $x^0(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$, $\lambda \in R^r$, μ, ν be satisfying the conditions in Theorem 3.4. Further, assume that the following hypotheses are fulfilled:*

a) *the functionals*

$$\int_{\Gamma_{t_0,t_1}} X^l_\beta \left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) dt^\beta, \quad l=\overline{1,r},\ \beta=\overline{1,m}$$

are (ρ^1_l, b)-quasiinvex at $x^0(\cdot)$ with respect to η and θ;

b) $\int_{\Gamma_{t_0,t_1}} \mu_\beta(t) g\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) dt^\beta$ *is (ρ^2, b)-quasiinvex at $x^0(\cdot)$ with respect to η and θ;*

c) $\int_{\Gamma_{t_0,t_1}} \nu_\beta(t) h\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) dt^\beta$ *is (ρ^3, b)-quasiinvex at $x^0(\cdot)$ with respect to η and θ;*

d) *at least one of the integrals*

$$\int_{\Gamma_{t_0,t_1}} X^l_\beta \left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) dt^\beta, \quad l=\overline{1,r},\ \beta=\overline{1,m},$$

$$\int_{\Gamma_{t_0,t_1}} \mu_\beta(t) g\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) dt^\beta, \quad \int_{\Gamma_{t_0,t_1}} \nu_\beta(t) h\left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) dt^\beta$$

is strictly (ρ, b)-quasiinvex at $x^0(\cdot)$ with respect to η and θ (see $\rho = \rho^1_l, \rho^2$ or ρ^3);

e) $\sum_{l=1}^r \lambda_l \rho^1_l + \rho^2 + \rho^3 \geq 0 \quad (\rho^1_l,\ \rho^2,\ \rho^3 \in \mathbf{R})$.

Then the point $x^0(\cdot)$ is an efficient solution of the problem (MVP).

Proof. By reductio ad absurdum, suppose that $x^0(\cdot)$ is not an efficient solution for (MVP). Then, for $l=\overline{1,r}$, there exists $x(\cdot) \in \mathcal{F}(\Omega_{t_0,t_1})$ such that

$$\int_{\Gamma_{t_0,t_1}} X^l_\beta \left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) dt^\beta \leq \int_{\Gamma_{t_0,t_1}} X^l_\beta \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) dt^\beta$$

and there exists at least $k \in \{1, 2, \ldots, r\}$ with

$$\int_{\Gamma_{t_0,t_1}} X^k_\beta \left(\chi_{x_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) dt^\beta < \int_{\Gamma_{t_0,t_1}} X^k_\beta \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) dt^\beta.$$

Using a), we have

$$b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\eta_{txx^0} \frac{\partial X^l_\beta}{\partial x} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right) + \left(D_{\alpha_1} \eta_{txx^0}\right) \frac{\partial X^l_\beta}{\partial x_{\alpha_1}} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right)\right] dt^\beta \ldots$$

$$+ b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\left(\frac{1}{n\left(\alpha_1, \ldots, \alpha_{s-1}\right)} D^{s-1}_{\alpha_1 \ldots \alpha_{s-1}} \eta_{txx^0}\right) \frac{\partial X^l_\beta}{\partial x_{\alpha_1 \ldots \alpha_{s-1}}} \left(\chi_{x^0_{\alpha_1 \ldots \alpha_{s-1}}}(t)\right)\right] dt^\beta$$

$$\leq -\rho^1_l b_{xx^0} \parallel \theta_{xx^0} \parallel^2 .$$

Multiplying by $\lambda_l \geq 0$ and making summation over $l = \overline{1, r}$, we obtain

$$\begin{aligned} & b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\eta_{txx^0} \lambda \frac{\partial X_\beta}{\partial x} \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right. \\ & \qquad \left. + \left(D_{\alpha_1} \eta_{txx^0} \right) \lambda \frac{\partial X_\beta}{\partial x_{\alpha_1}} \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta \dots \\ & + b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\left(\frac{1}{n(\alpha_1, \dots, \alpha_{s-1})} D^{s-1}_{\alpha_1 \dots \alpha_{s-1}} \eta_{txx^0} \right) \right. \\ & \qquad \left. \lambda \frac{\partial X_\beta}{\partial x_{\alpha_1 \dots \alpha_{s-1}}} \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta \\ & \leq - \left(\sum_{l=1}^{r} \lambda_l \rho_l^1 \right) b_{xx^0} \parallel \theta_{xx^0} \parallel^2 . \end{aligned} \tag{4.1}$$

The following inequality

$$\int_{\Gamma_{t_0,t_1}} \mu_\beta(t) g \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) dt^\beta \leq \int_{\Gamma_{t_0,t_1}} \mu_\beta(t) g \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) dt^\beta,$$

according to b), leads us to

$$\begin{aligned} & b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\eta_{txx^0} \mu_\beta(t) \frac{\partial g}{\partial x} \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right. \\ & \qquad \left. + \left(D_{\alpha_1} \eta_{txx^0} \right) \mu_\beta(t) \frac{\partial g}{\partial x_{\alpha_1}} \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta \dots \\ & + b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\left(\frac{1}{n(\alpha_1, \dots, \alpha_{s-1})} D^{s-1}_{\alpha_1 \dots \alpha_{s-1}} \eta_{txx^0} \right) \right. \\ & \qquad \left. \mu_\beta(t) \frac{\partial g}{\partial x_{\alpha_1 \dots \alpha_{s-1}}} \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta \\ & \leq -\rho^2 b_{xx^0} \parallel \theta_{xx^0} \parallel^2 . \end{aligned} \tag{4.2}$$

Also, the equality (see c))

$$\int_{\Gamma_{t_0,t_1}} \nu_\beta(t) h \left(\chi_{x_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) dt^\beta = \int_{\Gamma_{t_0,t_1}} \nu_\beta(t) h \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) dt^\beta$$

gives

$$\begin{aligned} & b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\eta_{txx^0} \nu_\beta(t) \frac{\partial h}{\partial x} \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right. \\ & \qquad \left. + \left(D_{\alpha_1} \eta_{txx^0} \right) \nu_\beta(t) \frac{\partial h}{\partial x_{\alpha_1}} \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta \dots \\ & + b_{xx^0} \int_{\Gamma_{t_0,t_1}} \left[\left(\frac{1}{n(\alpha_1, \dots, \alpha_{s-1})} D^{s-1}_{\alpha_1 \dots \alpha_{s-1}} \eta_{txx^0} \right) \right. \\ & \qquad \left. \nu_\beta(t) \frac{\partial h}{\partial x_{\alpha_1 \dots \alpha_{s-1}}} \left(\chi_{x^0_{\alpha_1 \dots \alpha_{s-1}}}(t) \right) \right] dt^\beta \\ & \leq -\rho^3 b_{xx^0} \parallel \theta_{xx^0} \parallel^2 . \end{aligned} \tag{4.3}$$

Making the sum (4.1) + (4.2) + (4.3), side by side, and taking into account d), we have

$$
\begin{aligned}
& b_{xx^0}\int_{\Gamma_{t_0,t_1}} \eta_{txx^0}\left[\lambda\frac{\partial X_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)+\mu_\beta(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \\
& \quad + b_{xx^0}\int_{\Gamma_{t_0,t_1}} \eta_{txx^0}\left[\nu_\beta(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \\
& \quad + b_{xx^0}\int_{\Gamma_{t_0,t_1}} \left(D_{\alpha_1}\eta_{txx^0}\right)\left[\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right. \\
& \qquad\qquad\qquad \left. +\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \\
& \quad + b_{xx^0}\int_{\Gamma_{t_0,t_1}} \left(D_{\alpha_1}\eta_{txx^0}\right)\left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \\
& \quad \cdots + b_{xx^0}\int_{\Gamma_{t_0,t_1}} \left(\frac{1}{n\left(\alpha_1,\ldots,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right) \\
& \qquad\qquad\qquad \left[\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \\
& \quad + b_{xx^0}\int_{\Gamma_{t_0,t_1}} \left(\frac{1}{n\left(\alpha_1,\ldots,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right) \\
& \qquad\qquad\qquad \left[\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \\
& \quad + b_{xx^0}\int_{\Gamma_{t_0,t_1}} \left(\frac{1}{n\left(\alpha_1,\ldots,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right) \\
& \qquad\qquad\qquad \left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \\
& \quad < -\left(\sum_{l=1}^{r}\lambda_l\rho_l^1+\rho^2+\rho^3\right) b_{xx^0}\parallel\theta_{xx^0}\parallel^2 .
\end{aligned}
$$

This implies that $b_{xx^0} > 0$ and the foregoing inequality can be rewritten as

$$
\begin{aligned}
& \int_{\Gamma_{t_0,t_1}} \eta_{txx^0}\left[\lambda\frac{\partial X_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)+\mu_\beta(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \\
& \quad + \int_{\Gamma_{t_0,t_1}} \eta_{txx^0}\left[\nu_\beta(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \\
& \quad + \int_{\Gamma_{t_0,t_1}} \left(D_{\alpha_1}\eta_{txx^0}\right)\left[\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right. \\
& \qquad\qquad\qquad \left. +\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta \\
& \quad + \int_{\Gamma_{t_0,t_1}} \left(D_{\alpha_1}\eta_{txx^0}\right)\left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\cdots\alpha_{s-1}}}(t)\right)\right]dt^\beta
\end{aligned}
$$

$$+\cdots+\int_{\Gamma_{t_0,t_1}}\left(\frac{1}{n\left(\alpha_1,\ldots,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}\eta_{txx^0}\right)$$
$$\left[\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$+\int_{\Gamma_{t_0,t_1}}\left(\frac{1}{n\left(\alpha_1,\ldots,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\cdots\alpha_{s-1}}\eta_{txx^0}\right)$$
$$\left[\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$+\int_{\Gamma_{t_0,t_1}}\left(\frac{1}{n\left(\alpha_1,\ldots,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\cdots\alpha_{s-1}}\eta_{txx^0}\right)$$
$$\left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$<-\left(\sum_{l=1}^{r}\lambda_l\rho_l^1+\rho^2+\rho^3\right)\parallel\theta_{xx^0}\parallel^2,$$

or, after integrating by parts, we get

$$\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\left[\lambda\frac{\partial X_\beta}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)+\mu_\beta(t)\frac{\partial g}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$+\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\left[\nu_\beta(t)\frac{\partial h}{\partial x}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$-\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}D_{\alpha_1}\left[\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)+\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$-\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}D_{\alpha_1}\left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$\cdots+(-1)^{s-1}\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\frac{1}{n\left(\alpha_1,\ldots,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}$$
$$\left[\lambda\frac{\partial X_\beta}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$(-1)^{s-1}\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\frac{1}{n\left(\alpha_1,\ldots,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}$$
$$\left[\mu_\beta(t)\frac{\partial g}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$(-1)^{s-1}\int_{\Gamma_{t_0,t_1}}\eta_{txx^0}\frac{1}{n\left(\alpha_1,\ldots,\alpha_{s-1}\right)}D^{s-1}_{\alpha_1\ldots\alpha_{s-1}}$$
$$\left[\nu_\beta(t)\frac{\partial h}{\partial x_{\alpha_1\ldots\alpha_{s-1}}}\left(\chi_{x^0_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right]dt^\beta$$
$$<-\left(\sum_{l=1}^{r}\lambda_l\rho_l^1+\rho^2+\rho^3\right)\parallel\theta_{xx^0}\parallel^2.$$

The above given computation is obtained by using the boundary conditions $x(t_\xi) = x_\xi$, $x_{\alpha_1\ldots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\ldots\alpha_j\xi}$, $\alpha_\zeta \in \{1,\ldots,m\}$, $\zeta, j = \overline{1,s-2}$, $\xi \in \{0,1\}$, (see $x(t_\xi) = x_\xi = x^0(t_\xi)$, $x_{\alpha_1\ldots\alpha_j}(t_\xi) = \tilde{x}_{\alpha_1\ldots\alpha_j\xi} = x^0_{\alpha_1\ldots\alpha_j}(t_\xi)$), and the following conditions (see Definition 4.1),

$$\eta_{tx^0x^0} = 0,\ D_{\alpha_1}\eta_{tx^0x^0} = 0,\ \cdots,D^{s-2}_{\alpha_1\ldots\alpha_{s-2}}\eta_{tx^0x^0} = 0$$

$$\alpha_\zeta \in \{1,\ldots,m\},\quad \zeta = \overline{1,s-2},\quad t \in \Omega_{t_0,t_1}.$$

Theorem 3.4 leads us to

$$0 < -\left(\sum_{l=1}^{r} \lambda_l \rho_l^1 + \rho^2 + \rho^3\right) \parallel \theta_{xx^0} \parallel^2 .$$

Applying the hypothesis e) and $\parallel \theta_{xx^0} \parallel^2 \geq 0$, we get a contradiction. Thus, the point x^0 is an efficient solution for (MVP) and the proof is complete. ☐

Corollary 4.3 (Sufficient efficiency conditions for (MVP))**.** *Let $x^0(\cdot)$ be a feasible solution of the problem (MVP) and $\lambda \in R^r$, μ, ν satisfying the conditions in Theorem 3.4. Also, consider that the following statements hold:*

a) *the functionals*

$$\int_{\Gamma_{t_0,t_1}} X^l_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta,\quad l = \overline{1,r},\ \beta = \overline{1,m}$$

are (ρ^1_l, b)-quasiinvex at the point $x^0(\cdot)$ with respect to η and θ;

b') *the functional*

$$\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \nu_\beta(t) h\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

is (ρ^2, b)-quasi-invex at the point $x^0(\cdot)$ with respect to η and θ;

d') *at least one of the integrals*

$$\int_{\Gamma_{t_0,t_1}} X^l_\beta\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) dt^\beta,\quad l = \overline{1,r},\ \beta = \overline{1,m},$$

$$\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \nu_\beta(t) h\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

is strictly (ρ^1_l, b) or (ρ^2, b)-quasiinvex at the point $x^0(\cdot)$ with respect to η and θ;

e') $\sum_{l=1}^{r} \lambda_l \rho^1_l + \rho^2 \geq 0 \quad (\rho^1_l,\ \rho^2 \in \mathbf{R})$.

Then the point $x^0(\cdot)$ is an efficient solution of the problem (MVP).

Proof. By replacing the integrals from hypotheses b), c) of Theorem 4.2 by the integral

$$\int_{\Gamma_{t_0,t_1}} \left[\mu_\beta(t) g\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right) + \nu_\beta(t) h\left(\chi_{x_{\alpha_1\ldots\alpha_{s-1}}}(t)\right)\right] dt^\beta$$

the conclusion is obtained. ☐

5. Conclusions, future works and perspectives

In this paper, we have introduced and studied a new class of multidimensional multiobjective variational problems (see (MVP)) involving higher-order partial derivatives and path-independent curvilinear integral functionals. Within this framework, we have stated and proved necessary and sufficient efficiency conditions for (MVP), developing an optimization theory on the associated higher-order jet bundles.

Additional research and perspectives related to this paper: the study on various types of dualities associated with (MVP); the extension of this class of variational problems to control problems and, for this new class of problems, the development of a duality theory.

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Manuscript received October 2 2018
revised February 1 2019

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