Applied Analysis and Optimization
Volume 3, Number 1, 2019, 57–82

Yokohama Publishers
ISSN 2189-1664 Online Journal

ON THREE-STEP ITERATION SCHEMES

ARIANA PITEA

Abstract. In this paper, a study of some three-step iteration methods for the determination of a fixed point is done. The setting chosen here is that of Hadamard manifolds. Convergence results for nine numerical methods are stated and proved. Furthermore, their stability is studied. An analysis with respect to the data (in)dependency is also presented.

1. Introduction and preliminaries

The study of numerical approximation of fixed points of some adequate mappings has been developed extensively. After Picard [12] proposed his famous iteration process, in 1890, it took quite a while since Mann [11] imposed his iterative process. Since then, this direction has been thoroughly searched. Ishikawa [6] moved forward by introducing a scheme in two steps. Since then, this field has developed significantly. In [4], Gdawiec and Kotarski present a table with the (inter)dependencies between such kind of methods. They also connect them to finding the maximum modulus of a suitable complex polynomial over a unit disc onto the complex plane and creating beautiful images.

The setting we use is that of Hadamard manifolds, which are complete, simply connected Riemannian manifolds, possessing a nonpositive sectional curvature.

Let $(M, \langle\cdot,\cdot\rangle)$ be a connected Riemannian manifold. For each $x \in M$, we denote by T_xM the tangent space of M at x. Let $\gamma\colon [a,b] \to M$ be a piecewise smooth curve joining the points $p = \gamma(a)$, and $q = \gamma(b)$. The length of the curve γ is $L(\gamma) = \int_a^b \|\gamma'(t)\|\,\mathrm{d}t$. The Riemannian distance d is then the minimal length of a curve joining the two considered points, over the set of all such curves.

The notion of convexity has to be stated also in the setting of the Riemannian manifolds, as in the next definition.

Definition 1.1 ([22]). Consider that $(M, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold and C is a nonempty subset of M. C is called geodesically convex if any geodesic which joins the arbitrary points p and q from C is contained in C, that is if $\gamma\colon [a,b] \to M$ is a geodesic with $p = \gamma(a) \in C$, and $q = \gamma(b) \in C$, then $\gamma((1-t)a + tb) \in C$, for any $t \in [0,1]$.

The exponential map at $p \in M$ is a function $\exp\colon T_pM \to M$, $\exp_p v = \gamma_{p,v}(1)$, where $\gamma_{p,v}$ is the geodesic starting from p, of velocity v. We emphasize that $\exp_p(tv) = \gamma_{p,v}(t)$.

A Riemannian manifold is complete if for any $p \in M$ any geodesic emerging from p is defined on the whole real line $\mathbb{R}$. In this case, any two points of M can be connected by a minimizing geodesic, and M is complete with respect to the distance

2010 *Mathematics Subject Classification.* 65L20,49Q99, 90C99.
Key words and phrases. Iteration scheme, fixed point, convergence, stability, data analysis.

d. A complete simply connected Riemannian manifold with nonpositive sectional curvature is called a Hadamard manifold. On such kind of manifolds, the following result holds.

Proposition 1.2 ([15]). *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, and $p \in M$. Then the exponential mapping $\exp_p\colon T_pM \to M$ is a diffeomorphism. More than that, for any two points p, $q \in M$, there exists a unique minimal geodesic which joins them.*

A function $f\colon M \to \mathbb{R}$ on a Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$ is called geodesically convex if $f \circ \gamma\colon \mathbb{R} \to \mathbb{R}$ is a convex function for any geodesic γ in M, that is

$$(f \circ \gamma)((1-t)a + tb) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b) \quad a, b \in \mathbb{R},\ t \in [0,1].$$

In the sequel, we shall need the following property of the Hadamard manifolds (which is, actually, a feature of the hyperbolic spaces, see [15]).

Proposition 1.3 ([15]). *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, and d the geodesic distance. Suppose γ_1, γ_2 are geodesics in M joining the points p_1, q_1, and p_2, q_2. Then*

$$d(\gamma_1(\lambda), \gamma_2(\lambda)) \leq (1-\lambda)d(\gamma_1(0), \gamma_2(0)) + \lambda d(\gamma_1(1), \gamma_2(1)),\ \lambda \in [0,1].$$

Hence, the distance function on a Riemannian manifold is geodesically convex; in particular, for any $p \in M$, the function $d(\cdot, p)\colon M \to \mathbb{R}$ is geodesically convex.

If $(M, \langle\cdot,\cdot\rangle)$ is a Riemannian manifold, it is said that $\Delta(p_1, p_2, p_3)$ is a geodesic triangle if it is formed by the vertices p_1, p_2, p_3 and minimizing geodesics which join these vertices.

One of the main tools we need is that of comparison triangles. The next lemma states the existence of such triangles.

Lemma 1.4 ([2]). *Let $(M, \langle,\cdot,\cdot\rangle)$ be a Hadamard manifold, and $\Delta(p, q, r)$ a geodesic triangle in it. Then there are $\tilde{p}$, $\tilde{q}$, $\tilde{r} \in \mathbb{R}^2$ so that*

$$d(p,q) = \|\tilde{p} - \tilde{q}\|, \qquad d(q,r) = \|\tilde{q} - \tilde{r}\|, \qquad d(r,p) = \|\tilde{r} - \tilde{p}\|.$$

We recall a property of the corresponding geodesic triangles which refers to parts of the lengths of their edges, which is of crucial importance in the development of our results.

Lemma 1.5 ([15]). *Let $(M, \langle,\cdot,\cdot\rangle)$ be a Hadamard manifold, $\Delta(p, q, r)$ a geodesic triangle in it, and $\Delta(\tilde{p}, \tilde{q}, \tilde{r})$ be the corresponding comparison triangle. Let z be a point on the geodesic segment $[p, q]$, and $\tilde{z}$ its comparison point on $[\tilde{p}, \tilde{q}]$. Then $d(z, r) \leq \|\tilde{z} - \tilde{r}\|$.*

In the proof of our results, we need a specific type of convergence, namely the Fejér convergence, see, for example, [10] and the references therein.

Definition 1.6. Let (X, d) be a metric space, and F a nonempty subset of X. A sequence $\{x_n\}$ is called Fejér convergent to F if for each element $p \in F$, $d(x_{n+1}, p) \leq d(x_n, p)$, $n \geq 0$.

Having in mind these helpful connections with the space $\mathbb{R}^2$, we need the following property (which, in fact, is a feature of uniformly convex Banach spaces).

Lemma 1.7 ([23, p. 484]). *Let $(\mathbb{R}^2, \|\cdot\|)$ be the usual Euclidean space, $t_n \in [a,b] \subset (0,1)$, for all $n \geq 0$. Presume that $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{R}^2$, such that*

- $\limsup_{n\to\infty} \|x_n\| \leq r$;
- $\limsup_{n\to\infty} \|y_n\| \leq r$;
- $\limsup_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$.

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

2. Convergence study

Consider that $(M, \langle\cdot,\cdot,\rangle)$ is a Hadamard manifold, C a nonempty, closed, and geodesically convex subset of M, and $T\colon C \to C$ is a nonexpansive mapping. $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences from $(0,1)$.

The first part of the study is motivated by a numerical method proposed in [3].

Consider $x_0 \in C$, and define

$$\begin{aligned} x_{n+1} &= \exp_{z_n} a_n \exp_{z_n}^{-1} Tz_n, \\ y_n &= \exp_{x_n} b_n \exp_{x_n}^{-1} Tx_n, \\ z_n &= \exp_{Tx_n} c_n \exp_{Tx_n}^{-1} Ty_n, \ n \geq 0. \end{aligned} \tag{2.1}$$

For $n \geq 0$, consider

- $\gamma_1\colon [0,1] \to M$ the geodesic which joins z_n with Tz_n.
- $\gamma_2\colon [0,1] \to M$ the geodesic which joins x_n with Tx_n,
- $\gamma_3\colon [0,1] \to M$ the geodesic joining the points Tx_n with Ty_n.

In the geodesics language, for a given value $x_0 \in C$, the iteration procedure is

$$\begin{aligned} x_{n+1} &= \gamma_1(a_n), \\ y_n &= \gamma_2(b_n), \\ z_n &= \gamma_3(c_n), \ n \geq 0. \end{aligned}$$

With respect to this iterative process, we have the following convergence result.

Theorem 2.1. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, d the Riemannian distance, and C a nonempty, closed, and geodesically convex subset of M. Suppose that F, the set of fixed points of T, is nonempty, and that $\{a_n\}$, $\{c_n\} \subset (0,1)$, and $b_n \in [a,b] \subset (0,1)$, $n \geq 0$. Then the sequence generated by the iterative procedure (2.1) converges to a fixed point of T.*

Proof. Let p be a fixed point of the mapping T.

Having in mind that the Riemannian distance d is a convex function, we obtain

$$\begin{aligned} d(y_n,p) = d(\gamma_2(b_n),p) &\leq (1-b_n)d(x_n,p) + b_n d(Tx_n,p) \\ &\leq (1-b_n)d(x_n,p) + b_n d(x_n,p) \\ &= d(x_n,p), \ n \geq 0. \end{aligned} \tag{2.2}$$

Also, by the use of the same property, we obtain

$$\begin{aligned} d(z_n,p) &= d(\gamma_3(c_n),p) \leq (1-c_n)d(Tx_n,p) + c_n d(Ty_n,p) \\ &\leq (1-c_n)d(x_n,p) + c_n d(y_n,p) \leq d(x_n,p), \ n \geq 0. \end{aligned} \tag{2.3}$$

Taking advantage of these two relations, the definition of the sequence $\{x_n\}$ and the nonexpansivness of T, it follows

$$(2.4)\qquad \begin{aligned} d(x_{n+1},p)=d(\gamma_1(a_n),p) &\leq (1-a_n)d(z_n,p)+a_nd(Tz_n,p)\\ &\leq (1-a_n)d(z_n,p)+a_nd(z_n,p)\\ &\leq d(x_n,p),\ n\geq 0. \end{aligned}$$

We have obtained $d(x_{n+1},p)\leq d(x_n,p)$, for $n\geq 0$. Let $r=\lim_{n\to\infty}d(x_n,p)$.

By applying lim sup in inequalities (2.2) and (2.3), we obtain

$$(2.5)\qquad \begin{aligned} &\limsup_{n\to\infty}d(y_n,p)\leq r,\\ &\limsup_{n\to\infty}d(z_n,p)\leq r. \end{aligned}$$

Taking advantage of (2.4), we obtain

$$d(x_{n+1},p)\leq d(z_n,p),\ n\geq 0.$$

Combining this inequality with relations (2.5), we get

$$r=\lim_{n\to\infty}d(x_{n+1},p)\leq\liminf_{n\to\infty}d(z_n,p)\leq\limsup_{n\to\infty}d(z_n,p)\leq r.$$

It follows that $\lim_{n\to\infty}d(z_n,p)$ exists and equals r. Focusing now on $d(y_n,p)$, from (2.3) we get

$$d(z_n,p)-d(x_n,p)\ \leq\ c_n(d(y_n,p)-d(x_n,p))\leq d(y_n,p)-d(x_n,p),\ n\geq 0,$$

so

$$d(z_n,p)\leq d(y_n,p),\ n\geq 0.$$

This leads to $r\leq\liminf_{n\to\infty}d(y_n,p)\leq r$; furthermore $r=\lim_{n\to\infty}d(y_n,p)$.

Let $n\geq 0$ be fixed, and the geodesic triangle $\Delta(x_n,Tx_n,p)$ with the vertices x_n, Tx_n, and p. By Lemma 1.4, there exists a corresponding comparison triangle $\Delta(\tilde{x}_n,\widetilde{Tx_n},\tilde{p})$. The corresponding comparison point of y_n, which is a point on the geodesic γ_2 joining the points x_n and Tx_n, is $\tilde{y}_n=(1-b_n)\tilde{x}_n+b_n\widetilde{Tx_n}$. Using Lemma 1.5, we obtain

$$\begin{aligned} d(y_n,p) &\leq \|\tilde{y}_n-\tilde{p}\|=\left\|(1-b_n)\tilde{x}_n+b_n\widetilde{Tx_n}-\tilde{p}\right\|\\ &\leq (1-b_n)\|\tilde{x}_n-\tilde{p}\|+b_n\left\|\widetilde{Tx_n}-\tilde{p}\right\|\\ &= (1-b_n)d(x_n,p)+b_nd(Tx_n,p)\\ &\leq d(x_n,p),\ n\geq 0. \end{aligned}$$

Taking $n\to\infty$, it follows

$$(2.6)\qquad r=\lim_{n\to\infty}\left\|(1-b_n)(\tilde{x}-\tilde{p})+b_n(\widetilde{Tx_n}-\tilde{p})\right\|.$$

The nonexpansiveness of the mapping T implies $d(Tx_n,p)\leq d(x_n,p)$, and it follows

$$\limsup_{n\to\infty}\left\|\widetilde{Tx_n}-\tilde{p}\right\|=\limsup_{n\to\infty}d(Tx_n,p)\leq r.$$

This relation, together with the equalities $\lim_{n\to\infty}\|\tilde{x}_n-\tilde{p}\|=r$ and (2.6), compel $\lim_{n\to\infty}\left\|\widetilde{Tx_n}-\tilde{x}_n\right\|=0$, after using Lemma 1.7. Eventually, we obtain that $\lim_{n\to\infty}d(Tx_n,x_n)=0$.

Relation (2.4) states that the sequence $\{x_n\}$ is Fejér convergent. Let x be a cluster point of $\{x_n\}$, and $\{x_{n_k}\}$ be a subsequence of it, which has the limit x. Keeping in mind that

$$\begin{aligned} d(x,Tx) &\leq d(x,x_{n_k}) + d(x_{n_k},Tx_{n_k}) + d(Tx_{n_k},Tx) \\ &\leq d(x,x_{n_k}) + d(x_{n_k},Tx_{n_k}) + d(x_{n_k},x),\ k \geq 0. \end{aligned}$$

and considering $k \to \infty$, we obtain that x is a fixed point of the mapping T. □

We continue our study with a process inspired by [9] and [21]. For $x_0 \in C$, define

$$\begin{aligned} x_{n+1} &= \exp_{Tx_n} a_n \exp^{-1}_{Tx_n} Tz_n, \\ y_n &= \exp_{x_n} b_n \exp^{-1}_{x_n} Tx_n, \\ z_n &= \exp_{Ax_n} c_n \exp^{-1}_{Ax_n} Ty_n,\ n \geq 0, \end{aligned} \tag{2.7}$$

where $Ax_n = x_n$, for all $n \geq 0$, or $Ax_n = Tx_n$, for all $n \geq 0$. Here $\{a_n\}$, $\{b_n\}$, and $\{c_n\} \subset (0,1)$.

Let $n \geq 0$, and denote by

- $\gamma_1 \colon [0,1] \to M$ the geodesic which joins Tx_n with Tz_n.
- $\gamma_2 \colon [0,1] \to M$ the geodesic which joins x_n with Tx_n,
- $\gamma_3 \colon [0,1] \to M$ the geodesic joining the points Ax_n with Ty_n.

Using these tools, for a given value $x_0 \in C$, the iteration procedure can be described as

$$\begin{aligned} x_{n+1} &= \gamma_1(a_n), \\ y_n &= \gamma_2(b_n), \\ z_n &= \gamma_3(c_n),\ n \geq 0. \end{aligned}$$

We are ready now to state our result on this iteration procedure.

Theorem 2.2. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, d the Riemannian distance, and C a nonempty, closed, and convex subset of M. Suppose that F, the set of fixed points of T, is nonempty, and that $\{a_n\}$, $\{c_n\} \subset (0,1)$, and $b_n \in [a,b] \subset (0,1)$. Then the sequence generated by the iteration process (2.7) is convergent to a fixed point of T.*

Proof. Let p be a fixed point of the mapping T.

As in Theorem 2.1, it can be shown that

$$d(y_n,p) \leq d(x_n,p),\ n \geq 0. \tag{2.8}$$

Furthermore,

$$\begin{aligned} d(z_n,p) &= d(\gamma_3(c_n),p) \leq (1-c_n)d(Ax_n,p) + c_n d(y_n,p) \\ &\leq (1-c_n)d(x_n,p) + c_n d(y_n,p) \leq d(x_n,p),\ n \geq 0. \end{aligned} \tag{2.9}$$

By combining these relations with the definition of the sequence $\{x_n\}$ and the nonexpansivness of T, we have

$$\begin{aligned} d(x_{n+1},p) = d(\gamma_1(a_n),p) &\leq (1-a_n)d(Tx_n,p) + a_n d(Tz_n,p) \\ &\leq (1-a_n)d(x_n,p) + a_n d(z_n,p) \end{aligned} \tag{2.10}$$

$$\leq \quad d(x_n,p),\ n\geq 0.$$

We have proved that $d(x_{n+1},p)\leq d(x_n,p)$, for $n\geq 0$. Let $r=\lim_{n\to\infty} d(x_n,p)$.

By applying lim sup in inequalities (2.8) and (2.9), we obtain

$$\begin{array}{l}\limsup_{n\to\infty} d(y_n,p)\leq r,\\ \limsup_{n\to\infty} d(z_n,p)\leq r.\end{array} \tag{2.11}$$

Taking advantage of (2.10), we obtain

$$d(x_{n+1},p)\leq (1-a_n)d(x_n,p)+a_n d(z_n,p),\ n\geq 0.$$

Furthermore,

$$d(x_{n+1},p)-d(x_n,p)\leq a_n(d(z_n,p)-d(x_n,p))\leq d(z_n,p)-d(x_n,p),\ n\geq 0.$$

Therefore, $d(x_{n+1},p)\leq d(z_n,p)$, $n\geq 0$. Having in mind also relations (2.11), we get

$$r=\lim_{n\to\infty} d(x_{n+1},p)\leq \liminf_{n\to\infty} d(z_n,p)\leq \limsup_{n\to\infty} d(z_n,p)\leq r.$$

Therefore, $\lim_{n\to\infty} d(z_n,p)$ exists and its value is r. Moving now towards $d(y_n,p)$, from (2.9) we get

$$d(z_n,p)-d(x_n,p) \quad\leq\quad c_n(d(y_n,p)-d(x_n,p))\leq d(y_n,p)-d(x_n,p),\ n\geq 0,$$

that is

$$d(z_n,p)\leq d(y_n,p),\ n\geq 0.$$

It follows that $r\leq \liminf_{n\to\infty} d(y_n,p)\leq r$, so $r=\lim_{n\to\infty} d(y_n,p)$.

For $n\geq 0$ fixed, consider the geodesic triangle $\Delta(x_n,Tx_n,p)$ with the vertices x_n, Tx_n, and p. According to Lemma 1.4, there is a corresponding comparison triangle $\Delta(\tilde{x}_n,\widetilde{Tx_n},\tilde{p})$. The corresponding comparison point of y_n, a point on the geodesic γ_2, which joins the points x_n and Tx_n, is $\tilde{y}_n=(1-b_n)\tilde{x}_n+b_n\widetilde{Tx_n}$. Applying Lemma 1.5, we are led to the following relations

$$\begin{array}{rcl} d(y_n,p) & \leq & \|\tilde{y}_n-\tilde{p}\|=\left\|(1-b_n)\tilde{x}_n+b_n\widetilde{Tx_n}-\tilde{p}\right\| \\ & \leq & (1-b_n)\left\|\tilde{x}_n-\tilde{p}\right\|+b_n\left\|\widetilde{Tx_n}-\tilde{p}\right\| \\ & = & (1-b_n)d(x_n,p)+b_n d(Tx_n,p) \\ & \leq & d(x_n,p),\ n\geq 0. \end{array}$$

If $n\to\infty$, we can draw the conclusion that

$$r=\lim_{n\to\infty}\left\|(1-b_n)(\tilde{x}-\tilde{p})+b_n(\widetilde{Tx_n}-\tilde{p})\right\|. \tag{2.12}$$

Having in mind the nonexpansiveness of the mapping T, we get $d(Tx_n,p)\leq d(x_n,p)$, hence $\limsup_{n\to\infty}\left\|\widetilde{Tx_n}-\tilde{p}\right\|=\limsup_{n\to\infty} d(Tx_n,p)\leq r$. Taking advantage of this relation and also combining the fact that $\lim_{n\to\infty}\|\tilde{x}_n-\tilde{p}\|=r$ with equality (2.12), by applying Lemma 1.7, we obtain $\lim_{n\to\infty}\left\|\widetilde{Tx_n}-\tilde{x}_n\right\|=0$, which compels $\lim_{n\to\infty} d(Tx_n,x_n)=0$.

Using relation (2.10) we obtain that the sequence $\{x_n\}$ is Fejér convergent. Let x be a cluster point of $\{x_n\}$, and $\{x_{n_k}\}$ be a subsequence of it, which has the limit x. By taking $k \to \infty$ in the following inequalities

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tx) \\ &\leq d(x, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, x),\ k \geq 0, \end{aligned}$$

we get $Tx = x$, and the theorem has been proved. □

The next proposed iteration procedure is motivated by [14], and [17], as in the next lines; $x_0 \in C$ fixed.

$$\begin{aligned} x_{n+1} &= \exp_{Ty_n} a_n \exp^{-1}_{Ty_n} Tz_n, \\ y_n &= \exp_{x_n} b_n \exp^{-1}_{x_n} Tx_n, \\ z_n &= \exp_{y_n} c_n \exp^{-1}_{y_n} A_n,\ n \geq 0, \end{aligned} \tag{2.13}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ are control sequences in $(0, 1)$, and $A_n = x_n$, for all $n \geq 0$, or $A_n = Ty_n$, for all $n \geq 0$.

Plainly speaking, let us fix $n \geq 0$. Denote by

- $\gamma_1 \colon [0, 1] \to M$ the geodesic which joins Ty_n with Tz_n.
- $\gamma_2 \colon [0, 1] \to M$ the geodesic which joins x_n with Tx_n,
- $\gamma_3 \colon [0, 1] \to M$ the geodesic joining the points y_n with A_n.

If $x_0 \in C$, in terms of geodesics, the above iteration scheme shows as

$$\begin{aligned} x_{n+1} &= \gamma_1(a_n), \\ y_n &= \gamma_2(b_n), \\ z_n &= \gamma_3(c_n),\ n \geq 0. \end{aligned}$$

We introduce now a convergence theorem with respect to this numerical algorithm.

Theorem 2.3. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold and C a nonempty, closed, and convex subset of M, and d the Riemannian distance. Suppose that F, the set of fixed points of T, is nonempty, and that $\{a_n\}$, $\{c_n\} \subset (0, 1)$, $b_n \in [a, b] \subset (0, 1)$. Then the sequence generated by the iteration process (2.13) is convergent to a fixed point of T.*

Proof. Let p be a fixed point of the mapping T.

Similarly to the proof in Theorem 2.1, it can be proved that

$$d(y_n, p) \leq d(x_n, p),\ n \geq 0. \tag{2.14}$$

Moving now on to the sequence $\{z_n\}$, keeping in mind the previous relation and the fact that T is nonexpansive, we get

$$\begin{aligned} d(z_n, p) &= d(\gamma_3(c_n), p) \leq (1 - c_n)d(y_n, p) + c_n d(A_n, p) \\ &\leq (1 - c_n)d(x_n, p) + c_n d(y_n, p) \\ &\leq d(x_n, p),\ n \geq 0. \end{aligned} \tag{2.15}$$

Using relations (2.14), and (2.15), we obtain

$$d(x_{n+1}, p) = d(\gamma_1(a_n), p) \leq (1 - a_n)d(Ty_n, p) + a_n d(Tz_n, p)$$

$$\begin{aligned}(2.16)\qquad &\le (1-a_n)d(y_n,p)+a_n d(z_n,p)\\ &\le d(x_n,p),\ n\ge 0.\end{aligned}$$

It follows that $d(x_{n+1},p) \le d(x_n,p)$, for $n \ge 0$. Denote by r the limit of the nonincreasing sequence $\{d(x_n,p)\}$.

Having in mind again inequalities (2.14) and (2.15), we get

$$\begin{aligned}&\limsup_{n\to\infty} d(y_n,p) \le r,\\ &\limsup_{n\to\infty} d(z_n,p) \le r.\end{aligned} \tag{2.17}$$

From relation (2.16), it follows that

$$d(x_{n+1},p) \le (1-a_n)d(y_n,p)+a_n d(z_n,p),\ n\ge 0,$$

therefore

$$d(x_{n+1},p)-d(y_n,p) \le a_n(d(z_n,p)-d(y_n,p)) \le d(z_n,p)-d(y_n,p),\ n\ge 0.$$

It follows that $d(x_{n+1},p) \le d(z_n,p)$, $n \ge 0$. Since from (2.15) we also have that $d(z_n,p) \le d(y_n,p)$, $n \ge 0$, we get that $d(x_{n+1},p) \le d(y_n,p)$, $n \ge 0$. By applying inequalities (2.17), it follows

$$r=\lim_{n\to\infty} d(x_{n+1},p) \le \liminf_{n\to\infty} d(y_n,p) \le \limsup_{n\to\infty} d(y_n,p) \le r.$$

Hence, $\lim_{n\to\infty} d(y_n,p)=r$.

Following the footsteps of the proof of Theorem 2.1 and Theorem 2.2, it can be seen that $\lim_{n\to\infty} d(Tx_n,\, x_n)=0$.

From (2.16) follows that $\{x_n\}$ is a Fejér convergent sequence. Let x be a cluster point of $\{x_n\}$, and $\{x_{n_k}\}$ be a subsequence with the limit x. Since

$$\begin{aligned}d(x,Tx) &\le d(x,x_{n_k})+d(x_{n_k},Tx_{n_k})+d(Tx_{n_k},Tx)\\ &\le d(x,x_{n_k})+d(x_{n_k},Tx_{n_k})+d(x_{n_k},x),\ k\ge 0,\end{aligned}$$

it follows that $d(Tx,x)=0$, so x is a fixed point of the mapping T. □

We continue our study being inspired by a numerical method proposed in [18]. Let $x_0 \in C$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{b_n\}$, $\{A_n\}$, and $\{B_n\}$ be sequences in $(0,1)$, such that $\alpha_n+\beta_n<1$, $n\ge 0$, and $A_n+B_n<1$, $n\ge 0$. Define

$$\begin{aligned}x_{n+1} &= \exp_{x_n}(\alpha_n+\beta_n)\exp_{x_n}^{-1}\bar{x}_n,\\ \bar{x}_n &= \exp_{Tz_n}\frac{\beta_n}{\alpha_n+\beta_n}\exp_{Tz_n}^{-1}Ty_n\\ y_n &= \exp_{x_n} b_n \exp_{x_n}^{-1}Tx_n,\\ z_n &= \exp_{x_n}(A_n+B_n)\exp_{x_n}^{-1}\bar{z}_n,\\ \bar{z}_n &= \exp_{Ty_n}\frac{B_n}{A_n+B_n}\exp_{Ty_n}^{-1}Tx_n,\ n\ge 0.\end{aligned} \tag{2.18}$$

For $n \ge 0$, consider

- $\gamma_1\colon [0,1]\to M$ the geodesic which joins x_n with $\bar{x}_n$,
- $\bar{\gamma}_1\colon [0,1]\to M$ the geodesic which joins Tz_n with Ty_n,
- $\gamma_2\colon [0,1]\to M$ the geodesic which joins x_n with Tx_n,
- $\gamma_3\colon [0,1]\to M$ the geodesic joining the points x_n with $\bar{z}_n$,
- $\bar{\gamma}_3\colon [0,1]\to M$ the geodesic joining the points Ty_n with Tx_n.

For a given value $x_0 \in C$, the iteration procedure can be expressed by the use of geodesics, as seen

$$\begin{aligned} x_{n+1} &= \gamma_1(\alpha_n + \beta_n), \\ \bar{x}_n &= \bar{\gamma}_1\left(\frac{\beta_n}{\alpha_n + \beta_n}\right), \\ y_n &= \gamma_2(b_n), \\ z_n &= \gamma_3(A_n + B_n), \\ \bar{z}_n &= \bar{\gamma}_3\left(\frac{B_n}{A_n + B_n}\right), \ n \geq 0. \end{aligned}$$

With respect to this iterative process, we have the following convergence result. Here we have used the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{A_n\}$, and $\{B_n\}$ from the interval $(0, 1)$, such that $\alpha_n + \beta_n < 1$, $n \geq 0$, and $A_n + B_n < 1$, $n \geq 0$.

Theorem 2.4. *Let $(M, \langle \cdot, \cdot \rangle)$ be a Hadamard manifold, d the Riemannian distance, and C a nonempty, closed, and convex subset of M. Suppose that F, the set of fixed points of T, is nonempty, and that $b_n \in [a, b] \subset (0, 1)$, $n \geq 0$. Then the sequence generated by the iterative procedure (2.18) converges to a fixed point of T.*

Proof. Consider p a fixed point of the mapping T.

Analogously to the proof of Theorem 2.1, it can be proved that

$$d(y_n, p) \leq d(x_n, p), \quad n \geq 0. \tag{2.19}$$

Also, by the use of the same property, and the nonexpansiveness of the mapping T, we get

$$\begin{aligned} d(\bar{z}_n, p) &\leq \frac{A_n}{A_n + B_n} d(Ty_n, p) + \frac{B_n}{A_n + B_n} d(Tx_n, p) \\ &\leq \frac{A_n}{A_n + B_n} d(y_n, p) + \frac{B_n}{A_n + B_n} d(x_n, p) \\ &\leq d(x_n, p), \quad n \geq 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} d(z_n, p) &= d(\gamma_3(A_n + B_n), p) \\ &\leq (1 - A_n - B_n) d(x_n, p) + (A_n + B_n) d(\bar{z}_n, p) \leq d(x_n, p), \ n \geq 0. \end{aligned} \tag{2.20}$$

Moreover, the nonexpansivness of T compels

$$\begin{aligned} d(\bar{x}_n, p) &= d\left(\bar{\gamma}_1\left(\frac{\beta_n}{\alpha_n + \beta_n}\right), p\right) \\ &\leq \frac{\alpha_n}{\alpha_n + \beta_n} d(Tz_n, p) + \frac{\beta_n}{\alpha_n + \beta_n} d(Ty_n, p) \\ &\leq \frac{\alpha_n}{\alpha_n + \beta_n} d(z_n, p) + \frac{\beta_n}{\alpha_n + \beta_n} d(y_n, p) \\ &\leq d(x_n, p), \quad n \geq 0. \end{aligned} \tag{2.21}$$

This leads to

$$d(x_{n+1}, p) = d(\gamma_1(\alpha_n + \beta_n), p) \ \leq \ (1 - \alpha_n - \beta_n) d(x_n, p) + (\alpha_n + \beta_n) d(\bar{x}_n, p)$$

$$\leq \quad d(x_n, p), \qquad n \geq 0. \tag{2.22}$$

We have obtained that $d(x_{n+1}, p) \leq d(x_n, p)$, $n \geq 0$. Let $r = \lim_{n\to\infty} d(x_n, p)$.

Taking the superior limit in inequalities (2.19), (2.20), and (2.21), we get that

$$\begin{array}{l} \limsup_{n\to\infty} d(y_n, p) \leq r, \\ \limsup_{n\to\infty} d(z_n, p) \leq r, \\ \limsup_{n\to\infty} d(\bar{x}_n, p) \leq r. \\ . \end{array} \tag{2.23}$$

Having in mind relation (2.22), we obtain

$$d(x_{n+1}, p) - d(x_n, p) \leq (\alpha_n + \beta_n)(d(\bar{x}_n, p) - d(x_n, p)) \leq d(\bar{x}_n, p) - d(x_n, p), n \geq 0,$$

hence $d(x_{n+1}, p) \leq d(\bar{x}_n, p)$, for all $n \geq 0$. Taking also into consideration relations (2.23), it follows

$$r = \lim_{n\to\infty} d(x_{n+1}, p) \leq \liminf_{n\to\infty} d(\bar{x}_n, p) \leq \limsup_{n\to\infty} d(\bar{x}_n, p) \leq r.$$

It follows that $\lim_{n\to\infty} d(\bar{x}_n, p)$ exists and equals r.

Taking into consideration relations (2.21), we have

$$d(\bar{x}_n, p) - d(z_n, p) \leq \frac{b_n}{a_n + b_n}(d(y_n, p) - d(z_n, p)) \leq d(y_n, p) - d(z_n, p), \ n \geq 0,$$

hence $d(\bar{x}_n, p) \leq d(y_n, p)$, for any $n \geq 0$. Proceeding as previously, we obtain $r = \lim_{n\to\infty} d(y_n, p)$, and also $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Relation (2.22) imposes the Fejér convergence of the sequence $\{x_n\}$. Let x be a cluster point of $\{x_n\}$, and $\{x_{n_k}\}$ be a subsequence of it, which has the limit x. By using the next relations

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tx) \\ &\leq d(x, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, x), \ k \geq 0, \end{aligned}$$

and taking $k \to \infty$, we get that x is a fixed point of the mapping T. □

We move on now and propose an iteration process on Hadamard manifolds, motivated by [8]. We take $x_0 \in C$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{b_n\}$, $\{A_n\}$, and $\{B_n\}$ sequences in $(0, 1)$ such that $\alpha_n + \beta_n < 1$, $n \geq 0$, $A_n + B_n < 1$, $n \geq 0$. Consider

$$\begin{aligned} x_{n+1} &= \exp_{z_n}(\alpha_n + \beta_n)\exp^{-1}_{z_n} \bar{x}_n, \\ \bar{x}_n &= \exp_{Tz_n} \frac{\beta_n}{\alpha_n + \beta_n} \exp^{-1}_{Tz_n} Ty_n \\ y_n &= \exp_{x_n} b_n \exp^{-1}_{x_n} Tx_n, \\ z_n &= \exp_{y_n}(A_n + B_n)\exp^{-1}_{y_n} \bar{z}_n, \\ \bar{z}_n &= \exp_{Ty_n} \frac{B_n}{A_n + B_n} \exp^{-1}_{Ty_n} Tx_n, \ n \geq 0. \end{aligned} \tag{2.24}$$

Let us fix $n \geq 0$, and define the next geodesics.

- $\gamma_1 \colon [0, 1] \to M$ the geodesic which joins z_n with $\bar{x}_n$,
- $\bar{\gamma}_1 \colon [0, 1] \to M$ the geodesic which joins Tz_n with Ty_n,
- $\gamma_2 \colon [0, 1] \to M$ the geodesic which joins x_n with Tx_n,
- $\gamma_3 \colon [0, 1] \to M$ the geodesic joining the points y_n with $\bar{z}_n$,

• $\bar{\gamma}_3 \colon [0,1] \to M$ the geodesic joining the points Ty_n with Tx_n.
For a given value $x_0 \in C$, the "geodesic" form of the above iteration is

$$\begin{aligned} x_{n+1} &= \gamma_1(\alpha_n + \beta_n), \\ \bar{x}_n &= \bar{\gamma}_1\left(\frac{\beta_n}{\alpha_n + \beta_n}\right), \\ y_n &= \gamma_2(b_n), \\ z_n &= \gamma_3(A_n + B_n), \\ \bar{z}_n &= \bar{\gamma}_3\left(\frac{B_n}{A_n + B_n}\right), \ n \geq 0. \end{aligned}$$

We state and prove a convergence result for the proposed scheme; the control sequences are $\{\alpha_n\}$, $\{\beta_n\}$, $\{a_n\}$, $\{b_n\}$, $\{A_n\}$, and $\{B_n\}$, all from the interval $(0,1)$, such that $\alpha_n + \beta_n < 1$, $n \geq 0$, and $A_n + B_n < 0$, $n \geq 0$.

Theorem 2.5. *Consider the Hadamard manifold $(M, \langle \cdot, \cdot \rangle)$, d the Riemannian distance, and C a nonempty, closed, and convex subset of M. Suppose that F, the set of fixed points of T, is nonempty, and that $b_n \in [a,b] \subset (0,1)$, $n \geq 0$. Then the sequence generated by the iterative procedure (2.24) converges to a fixed point of T.*

Proof. Let p be a fixed point of the mapping T.
As previously,

$$d(y_n, p) \leq d(x_n, p), \qquad n \geq 0. \tag{2.25}$$

Applying the convexity of the distance function d, and the nonexpansiveness of the mapping T, it yields

$$\begin{aligned} d(\bar{z}_n, p) &\leq \frac{A_n}{A_n + B_n} d(Ty_n, p) + \frac{B_n}{A_n + B_n} d(Tx_n, p) \\ &\leq \frac{A_n}{A_n + B_n} d(y_n, p) + \frac{B_n}{A_n + B_n} d(x_n, p) \\ &\leq d(x_n, p), \qquad n \geq 0. \end{aligned}$$

Moreover,

$$\begin{aligned} d(z_n, p) &= d(\gamma_3(A_n + B_n), p) \\ &\leq (1 - A_n - B_n) d(y_n, p) + (A_n + B_n) d(\bar{z}_n, p) \leq d(x_n, p), \ n \geq 0. \end{aligned}$$

The nonexpansivness of T enables the next relations

$$\begin{aligned} d(\bar{x}_n, p) &= d\left(\bar{\gamma}_1\left(\frac{\beta_n}{\alpha_n + \beta_n}\right), p\right) \\ &\leq \frac{\alpha_n}{\alpha_n + \beta_n} d(Tz_n, p) + \frac{\beta_n}{\alpha_n + \beta_n} d(Ty_n, p) \\ &\leq \frac{\alpha_n}{\alpha_n + \beta_n} d(z_n, p) + \frac{\beta_n}{\alpha_n + \beta_n} d(y_n, p) \\ &\leq d(x_n, p), \qquad n \geq 0. \end{aligned}$$

It follows that

$$d(x_{n+1}, p) = d(\gamma_1(\alpha_n + \beta_n), p) \leq (1 - \alpha_n - \beta_n) d(x_n, p) + (\alpha_n + \beta_n) d(\bar{x}_n, p)$$

$$\leq \quad d(x_n, p), \qquad n \geq 0. \tag{2.26}$$

Therefore, $d(x_{n+1}, p) \leq d(x_n, p)$, for $n \geq 0$. Let $r = \lim_{n\to\infty} d(x_n, p)$ be the limit of this nonincreasing sequence.

As in the previous theorems, it can be proved that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

From (2.26), the sequence $\{x_n\}$ is Fejér convergent. Let x be a cluster point of $\{x_n\}$, and $\{x_{n_k}\}$ be a subsequence of it with the limit x. As

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(Tx_{n_k}, Tx) \\ &\leq d(x, x_{n_k}) + d(x_{n_k}, Tx_{n_k}) + d(x_{n_k}, x), \; k \geq 0, \end{aligned}$$

taking $k \to \infty$, we obtain that $Tx = x$. □

Following the footsteps of [7], we introduce another iterative method, which might be considered a four-step one. Let $\{a_n\}$, and $\{b_n\}$ be sequences in $(0, 1)$. We take $x_0 \in C$, and

$$\begin{aligned} x_{n+1} &= T(\exp_{y_n} a_n \exp_{y_n}^{-1} Ty_n), \\ y_n &= T(\exp_{x_n} b_n \exp_{x_n}^{-1} Tx_n), \qquad n \geq 0. \end{aligned} \tag{2.27}$$

Consider $n \geq 0$, and denote by

• $\gamma_1 \colon [0, 1] \to M$ the geodesic which joins y_n with Ty_n,

• $\gamma_2 \colon [0, 1] \to M$ the geodesic which joins x_n with Tx_n.

For $x_0 \in C$, let us express the iteration formulae in terms of geodesics.

$$\begin{aligned} x_{n+1} &= T\gamma_1(a_n), \\ y_n &= T\gamma_2(b_n), \qquad n \geq 0. \end{aligned}$$

Theorem 2.6. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold and C a nonempty, closed, and convex subset of M, and d the Riemannian distance. Suppose that F, the set of fixed points of T, is nonempty, and that $\{a_n\} \subset (0, 1)$, $b_n \in [a, b] \subset (0, 1)$, $n \geq 0$. Then the sequence generated by the iteration process (2.27) is convergent to a fixed point of T.*

Proof. Consider p a point from F.

Having in mind the nonexpansiveness of T, and the fact that d is a convex function, we obtain

$$\begin{aligned} d(y_n, p) = d(T\gamma_2(b_n), p) &\leq d(\gamma_2(b_n), p) \\ &\leq (1 - b_n)d(x_n, p) + b_n d(Tx_n, p) \\ &\leq (1 - b_n)d(x_n, p) + b_n d(x_n, p) \\ &= d(x_n, p), \; n \geq 0. \end{aligned} \tag{2.28}$$

Taking advantage of the previous relation, we get

$$\begin{aligned} d(x_{n+1}, p) = d(T\gamma_1(a_n), p) &\leq d(\gamma_1(a_n), p) \\ &\leq (1 - a_n)d(y_n, p) + a_n d(Ty_n, p) \\ &\leq d(y_n, p) \leq d(x_n, p), \; n \geq 0. \end{aligned}$$

It follows that $d(x_{n+1}, p) \leq d(x_n, p)$, for $n \geq 0$. Denote $r = \lim_{n\to\infty} d(x_n, p)$.

From (2.28), it follows that

$$\limsup_{n\to\infty} d(y_n, p) \le r. \tag{2.29}$$

From relation (2.29), it follows that $d(x_{n+1}, p) \le d(y_n, p)$, $n \ge 0$, which, combined with the above relation leads to

$$r = \lim_{n\to\infty} d(x_{n+1}, p) \le \liminf_{n\to\infty} d(y_n, p) \le \limsup_{n\to\infty} d(y_n, p) \le r.$$

Therefore, there exists $\lim_{n\to\infty} d(y_n, p) = r$.

The last part of the proof is similar to that of the previous theorem, that is why we omit it. □

The last iteration method proposed here is motivated by [20]. For a start value $x_0 \in C$, and the control sequences $\{b_n\}$, $\{c_n\}$ in the interval $(0, 1)$, define

$$\begin{aligned} x_{n+1} &= T z_n, \\ y_n &= \exp_{x_n} b_n \exp^{-1}_{x_n} T x_n, \\ z_n &= T(\exp_{x_n} c_n \exp^{-1}_{x_n} y_n), \quad n \ge 0. \end{aligned} \tag{2.30}$$

Consider $n \ge 0$, and denote by

• $\gamma_2 \colon [0, 1] \to M$ the geodesic which joins x_n with $T x_n$,

• $\gamma_3 \colon [0, 1] \to M$ the geodesic which connects x_n with y_n.

Let $x_0 \in C$. Then the iterative scheme can be written as follows.

$$\begin{aligned} x_{n+1} &= T z_n, \\ y_n &= \gamma_2(b_n), \quad , \\ z_n &= T\gamma_3(c_n), \quad n \ge 0. \end{aligned} \tag{2.31}$$

The next result assures the convergence of the proposed algorithm.

Theorem 2.7. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold and C a nonempty, closed, and convex subset of M, and d the Riemannian distance. Suppose that F, the set of fixed points of T, is nonempty, and that $b_n \in [a, b] \subset (0, 1)$, $c_n \in (0, 1)$, $n \ge 0$. Then the sequence generated by the iteration process (2.30) converges to a fixed point of T.*

Proof. Let p be a point from F.

The definition of the sequence $\{y_n\}$ allows the following relations

$$\begin{aligned} d(y_n, p) = d(\gamma_2(b_n), p) &\le (1 - b_n) d(x_n, p) + b_n d(T x_n, p) \\ &\le (1 - b_n) d(x_n, p) + b_n d(x_n, p) \\ &= d(x_n, p), \ n \ge 0. \end{aligned} \tag{2.32}$$

This inequalities lead us to

$$\begin{aligned} d(z_n, p) &\le (1 - c_n) d(x_n, p) + c_n d(y_n, p) \\ &\le d(x_n, p), \quad n \ge 0. \end{aligned}$$

Furthermore,

$$d(x_{n+1}, p) = d(T z_n, p) \le d(z_n, p) \le d(x_n, p), \ n \ge 0. \tag{2.33}$$

It follows that
$$d(x_{n+1},p) \le d(z_n,p) \le d(x_n,p),$$
for $n \ge 0$. Denote by r the limit of the nondecreasing sequence $\{d(x_n,p)\}$. Also, we get that $\{d(z_n,p)\}$ is convergent to r.

Inequalities (2.32) compel
$$d(z_n,p) - d(x_n,p) \le c_n(d(y_n,p) - d(x_n,p)) \le d(y_n,p) - d(x_n,p),\ n \ge 0.$$
Therefore, $d(z_n,p) \le d(y_n,p)$, $n \ge 0$. Keeping in mind that $d(y_n,p) \le d(x_n,p)$, $n \ge 0$, we obtain that $\lim_{n\to\infty} d(y_n,p) = r$.

The last part of the proof being analogous to the proof of the theorem regarding the convergence of iteration (2.24), we skip it. □

3. Stability analysis

An important issue in studying the properties of an iteration scheme is its numerical stability, namely deciding if small variations of the initial values considered in the process lead to small changes in the numerical value of the approximated fixed point. The notion of T-stability was first introduced by Harder and Hicks [5], in the setting of normed spaces.

In the following, $(M, \langle\cdot,\cdot\rangle)$ is a Hadamard manifold, d is the geodesic distance, and C is a nonempty, closed, and geodesically convex subset of M.

Definition 3.1 ([5])**.** Let $\{t_n\}$ be an arbitrary sequence in C. Consider an iteration process $x_{n+1} = f(T, x_n)$ converging to a unique fixed point p, and $\varepsilon_n = d(t_{n+1}, f(T, t_n))$, $n \ge 0$, where $\{t_n\}$ is a sequence from C. This procedure is called T-stable (or stable with respect to T), if the following equivalence holds
$$\lim_{n\to\infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} t_n = p$$

This time we need a stronger type of mappings instead of the nonexpansive ones, namely the contraction mappings.

Our first result on stability is with regard to iteration (2.1).

Theorem 3.2. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, d the geodesic distance, C be a nonempty, closed, and geodesically convex subset of M. $T\colon C \to C$ is a contraction of constant k, with the fixed point p. Then iteration (2.1) is T-stable.*

Proof. Consider that $\{x_n\}$ is the sequence obtained by applying the iteration procedure (2.1).

Having in mind the definition of the considered iteration procedure, the convexity of d, and the form of the generalized contraction, we obtain
$$\begin{aligned} d(y_n,p) &\le (1-b_n)d(x_n,p) + b_n d(Tx_n,p) \\ &\le (1-b_n)d(x_n,p) + k b_n d(x_n,p) \\ &\le d(x_n,p),\ n \ge 0. \end{aligned}$$

Furthermore,
$$\begin{aligned} d(z_n,p) &\le (1-c_n)d(Tx_n,p) + c_n d(Ty_n,p) \\ &\le k(1-c_n)d(x_n,p) + k c_n d(y_n,p) \end{aligned}$$

$$\leq \quad kd(x_n, p), \; n \geq 0.$$

By using this inequality, and the convexity of d, it follows

$$\begin{aligned} d(x_{n+1}, p) &\leq (1 - a_n)d(z_n, p) + a_n d(Tz_n, p) \\ &\leq (1 - a_n)d(z_n, p) + ka_n d(z_n, p) \\ &\leq d(z_n, p) \leq kd(x_n, p), \; n \geq 0. \end{aligned} \tag{3.1}$$

Let $\{t_n\}$ be an arbitrary sequence from C. Denote by $f(T, t_n)$ the iteration scheme (2.1), and $\varepsilon_n = d(t_{n+1}, f(T, t_n))$, $n \geq 0$. We are ready now to move on to the stability analysis. First, we presume that $\lim_{n\to\infty} \varepsilon_n = 0$, and prove that $\lim_{n\to\infty} t_n = p$.

Keeping in mind inequality (3.1), the next inequalities hold true

$$\begin{aligned} d(t_{n+1}, p) &\leq d(t_{n+1}, f(T, t_n)) + d(f(T, t_n), p) \\ &\leq \varepsilon_n + kd(t_n, p), \; n \geq 0. \end{aligned}$$

Let $l = \limsup_{n\to\infty} d(t_n, p)$. Since $\varepsilon_n \to 0$, it follows that $l = kl$, therefore $l = 0$. We have proved that $\lim_{n\to\infty} t_n = p$.

Conversely, let us assume that $\lim_{n\to\infty} t_n = p$; we prove that $\lim_{n\to\infty} \varepsilon_n = 0$.

The following relations hold true

$$\begin{aligned} \varepsilon_n &= d(t_{n+1}, f(T, t_n)) \\ &\leq d(t_{n+1}, p) + d(f(T, t_n), p) \\ &\leq d(t_{n+1}, p) + kd(t_n, p), \; n \geq 0. \end{aligned}$$

Taking into consideration the hypothesis, we get $\varepsilon_n \to 0$, as $n \to \infty$.

It has been proved that the process (2.1) is T-stable. □

We move on now to iterations of type (2.7), by stating the next result.

Theorem 3.3. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, d the geodesic distance, C be a nonempty, closed, and geodesically convex subset of M. Consider that $T\colon C \to C$ is a contraction on C, of constant k. Then process (2.7) is T-stable.*

Proof. Let us denote the unique fixed point of the mapping T by p, and by $\{x_n\}$ the sequence obtained by applying the iteration procedure (2.7).

The iteration procedure, combined with the convexity of d, and the form of the generalized contraction, compel

$$\begin{aligned} d(y_n, p) &\leq (1 - b_n)d(x_n, p) + b_n d(Tx_n, p) \\ &\leq (1 - b_n)d(x_n, p) + kd(x_n, p) \\ &\leq d(x_n, p), \; n \geq 0. \end{aligned}$$

Taking advantage of the fact that T is also a contractive mapping, we obtain

$$\begin{aligned} d(z_n, p) &\leq (1 - c_n)d(Ax_n, p) + c_n d(Ty_n, p) \\ &\leq (1 - c_n)d(x_n, p) + kc_n d(y_n, p) \\ &\leq d(x_n, p), \; n \geq 0. \end{aligned}$$

By using this inequality, we get

$$d(x_{n+1}, p) \leq (1 - a_n)d(Tx_n, p) + a_n d(Ty_n, p)$$

$$\le \; k(1-a_n)d(x_n,p) + ka_n d(y_n,p)$$

$$\le \; kd(x_n,p),\; n \ge 0. \tag{3.2}$$

Let $\{t_n\}$ be an arbitrary sequence from C. Denote by $f(T,t_n)$ the iteration scheme (2.7), and $\varepsilon_n = d(t_{n+1}, f(T,t_n))$, $n \ge 0$. Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$.

Relation (3.2) imposes

$$\begin{aligned} d(t_{n+1},p) &\le d(t_{n+1}, f(T,t_n)) + d(f(T,t_n),p) \\ &\le \varepsilon_n + kd(t_n,p),\; n \ge 0. \end{aligned}$$

We have obtained that $\{d(t_n,p)\}$ is a sequence in the same conditions as Theorem 3.2. Analogously, it follows $d(t_n,p) \to 0$.

Conversely, suppose that $\lim_{n\to\infty} t_n = p$.

The following relations hold true

$$\begin{aligned} \varepsilon_n &= d(t_{n+1}, f(T,t_n)) \\ &\le d(t_{n+1},p) + d(f(T,t_n),p) \\ &\le d(t_{n+1},p) + kd(t_n,p),\; n \ge 0. \end{aligned}$$

Our assumption compels $\varepsilon_n \to 0$, as $n \to \infty$.

We have proved that iterations of type (2.7) are T-stable. □

Next, we refer to iterations of type (2.13).

Theorem 3.4. *Consider $(M, \langle\cdot,\cdot\rangle)$ a Hadamard manifold, d the geodesic distance. Let C be a nonempty, closed, and geodesically convex subset of M. Consider that $T\colon C \to C$ is a contraction on C, with constant k. Then the iteration process (2.13) is T-stable.*

Proof. Consider that p is the unique fixed point of the mapping T, and $\{x_n\}$ is the sequence obtained with the help of the iteration procedure (2.13).

The convexity of d, and the form of the generalized contraction, compel

$$\begin{aligned} d(y_n,p) &\le (1-b_n)d(x_n,p) + b_n d(Tx_n,p) \\ &\le (1-b_n)d(x_n,p) + b_n d(x_n,p) \\ &\le d(x_n,p),\; n \ge 0. \end{aligned}$$

Furthermore,

$$\begin{aligned} d(z_n,p) &\le (1-c_n)d(y_n,p) + c_n d(A_n,p) \\ &\le (1-c_n)d(x_n,p) + c_n d(x_n,p) \\ &= d(x_n,p),\; n \ge 0. \end{aligned}$$

Having in mind the last two inequalities, we get

$$\begin{aligned} d(x_{n+1},p) &\le (1-a_n)d(Ty_n,p) + a_n d(Tz_n,p) \\ &\le k(1-a_n)d(y_n,p) + ka_n d(z_n,p) \\ &\le kd(x_n,p),\; n \ge 0. \end{aligned}$$

The rest of the proof is similar to those of Theorem 3.2, and Theorem 3.3, therefore we omit it. □

With respect to process (2.18), we have obtained the next result.

Theorem 3.5. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, d the geodesic distance, C be a nonempty, closed, and geodesically convex subset of M. Consider that $T\colon C \to C$ is a contraction of constant k. If $\alpha_n + \beta_n \in (A, 1) \subset (0, 1)$, for all $n \geq 0$, then (2.18) is T-stable.*

Proof. Let us denote the unique fixed point of the mapping T by p, and by $\{x_n\}$ the sequence obtained by applying the iteration procedure (2.18).

The iteration procedure, combined with the convexity of d, and the form of the generalized contraction, compel

$$\begin{aligned} d(y_n, p) &\leq (1 - b_n)d(x_n, p) + b_n d(Tx_n, p) \\ &\leq (1 - b_n)d(x_n, p) + kb_n d(x_n, p) \\ &\leq d(x_n, p), \ n \geq 0. \end{aligned}$$

By the nonexpansiveness of T, we obtain

$$\begin{aligned} d(z_n, p) &\leq (1 - A_n - B_n)d(x_n, p) + A_n d(Ty_n, p) + B_n d(Tx_n, p) \\ &\leq (1 - A_n - B_n)d(x_n, p) + kA_n d(y_n, p) + kB_n d(x_n, p) \\ &\leq d(x_n, p), \ n \geq 0. \end{aligned}$$

By using these inequalities, it follows

$$\begin{aligned} d(x_{n+1}, p) &\leq (1 - \alpha_n - \beta_n)d(x_n, p) + \alpha_n d(Tz_n, p) + \beta_n d(Ty_n, p) \\ &\leq (1 - \alpha_n - \beta_n)d(x_n, p) + k\alpha_n d(z_n, p) + k\beta_n d(y_n, p) \\ &\leq (1 - \alpha_n - \beta_n)d(x_n, p) + k(\alpha_n + \beta_n)d(x_n, p) \\ &\leq \bar{k}d(x_n, p), \ n \geq 0, \end{aligned} \tag{3.3}$$

where $\bar{k} < 1$.

Let $\{t_n\}$ be an arbitrary sequence from C. Denote by $f(T, t_n)$ the iteration scheme (2.18), and $\varepsilon_n = d(t_{n+1}, f(T, t_n))$, $n \geq 0$. Suppose that $\lim_{n\to\infty} \varepsilon_n = 0$.

A relation similar to (3.3) imposes

$$\begin{aligned} d(t_{n+1}, p) &\leq d(t_{n+1}, f(T, t_n)) + d(f(T, t_n), p) \\ &\leq \varepsilon_n + \bar{k}d(t_n, p), \ n \geq 0. \end{aligned}$$

It follows that $t_n \to p$, as $n \to \infty$.

Assume now that $d(t_n, p) \to 0$. It follows

$$\begin{aligned} \varepsilon_n &= d(t_{n+1}, f(T, t_n)) \\ &\leq d(t_{n+1}, p) + d(f(T, t_n), p) \\ &\leq d(t_{n+1}, p) + \bar{k}d(t_n, p), \ n \geq 0. \end{aligned}$$

Our assumption compels $\varepsilon_n \to 0$, as $n \to \infty$. The proof is complete. □

With respect to process (2.24), we have obtained the next result.

Theorem 3.6. *Consider that $(M, \langle\cdot,\cdot\rangle)$ is a Hadamard manifold, d the geodesic distance, and C is a nonempty, closed, and geodesically convex subset of M. $T\colon C \to C$ is a contraction of constant k. If $\alpha_n + \beta_n \in (A, 1) \subset (0, 1)$, for all $n \geq 0$, then the iterative process (2.24) is T-stable.*

Proof. Let p be the unique fixed point of the mapping T, and $\{x_n\}$ the sequence obtained by applying the iteration procedure (2.24).

Keeping in mind the hypotheses of the theorem, it follows

$$\begin{aligned} d(y_n,p) &\leq (1-b_n)d(x_n,p)+b_n d(Tx_n,p) \\ &\leq (1-b_n)d(x_n,p)+kb_n d(x_n,p) \\ &\leq d(x_n,p),\ n\geq 0. \end{aligned}$$

Moreover,

$$\begin{aligned} d(z_n,p) &\leq (1-A_n-B_n)d(y_n,p)+A_n d(Ty_n,p)+B_n d(Tx_n,p) \\ &\leq (1-A_n-B_n)d(x_n,p)+kA_n d(y_n,p)+kB_n d(x_n,p) \\ &\leq d(x_n,p),\ n\geq 0. \end{aligned}$$

Taking advantage of the previous inequalities, we obtain

$$\begin{aligned} d(x_{n+1},p) &\leq (1-\alpha_n-\beta_n)d(z_n,p)+\alpha_n d(Tz_n,p)+\beta_n d(Ty_n,p) \\ &\leq (1-\alpha_n-\beta_n)d(z_n,p)+k\alpha_n d(z_n,p)+k\beta_n d(y_n,p) \\ &\leq (1-\alpha_n-\beta_n)d(x_n,p)+k(\alpha_n+\beta_n)d(x_n,p),\ n\geq 0. \qquad (3.4) \end{aligned}$$

The inequality (3.4) being similar to that in (3.3), the rest of the proof follows as in the case of that of Theorem 3.5. □

With respect to iteration (2.27), the following theorem holds true.

Theorem 3.7. *Let $(M,\langle\cdot,\cdot\rangle)$ be a Hadamard manifold with d its geodesic distance, C be a nonempty, closed, and geodesically convex subset of M. Consider that $T\colon C\to C$ is a contraction of constant k on C. Then procedure (2.27) is T-stable.*

Proof. Let p be the unique fixed point of T, and $\{x_n\}$ be the sequence obtained after applying the iteration procedure (2.27).

From the hypotheses, we get

$$\begin{aligned} d(y_n,p) &= d(\gamma_2(b_n),p) \\ &\leq (1-b_n)d(x_n,p)+b_n d(x_n,p) \\ &\leq d(x_n,p),\ n\geq 0. \end{aligned}$$

This inequality, combined with the monotone of φ compel

$$\begin{aligned} d(x_{n+1},p) &= d(T\gamma_1(a_n),p)\leq k(1-a_n)d(y_n,p)+ka_n d(Ty_n,p) \\ &\leq k(1-a_n)d(y_n,p)+ka_n d(y_n,p) \\ &\leq kd(x_n,p),\ n\geq 0. \end{aligned}$$

The rest of the proof is similar to those of the previous theorems, therefore we omit it. □

Furthermore, for process (2.30), we obtained the following result.

Theorem 3.8. *Let C be a nonempty, closed, and geodesically convex subset of M, where $(M,\langle\cdot,\cdot\rangle)$ is a Hadamard manifold with d its geodesic distance. Consider that $T\colon C\to C$ is a contraction on C of constant k. Then procedure (2.30) is T-stable.*

Proof. Let p be the unique fixed point of T, and $\{x_n\}$ be the sequence obtained by using the iteration method (2.30).

The following relations hold true

$$\begin{aligned} d(y_n, p) &\leq (1-b_n)d(x_n, p) + b_n d(Tx_n, p)) \\ &\leq 1-b_n)d(x_n, p) + b_n d(x_n, p) \\ &\leq d(x_n, p),\ n \geq 0. \end{aligned}$$

Having in mind our assumptions, we get

$$\begin{aligned} d(z_n, p) &\leq d(\gamma_3(c_n), p) \\ &\leq (1-c_n)d(x_n, p) + c_n d(y_n, p) \\ &\leq d(y_n, p) \leq d(x_n, p),\ n \geq 0. \end{aligned}$$

We obtain

$$d(x_{n+1}, p) = d(Tz_n, p) \leq kd(z_n, p) \leq kd(x_n, p),\ n \geq 0.$$

The rest of the proof is similar to those of Theorem 3.2, and Theorem 3.3, therefore we omit it. □

4. Data dependency analysis

The objective of this section is to develop data dependency issues related to the proposed schemes. We shall mainly concentrate on the approximate mappings study. Following [13], we give the following definition.

Definition 4.1. Consider that $(M, \langle\cdot,\cdot\rangle)$ is a Hadamard manifold, d its geodesic distance, and C a nonempty, closed and convex subset of M. Let T, $\bar{T}\colon C \to C$ be two mappings. The mapping $\bar{T}$ is called an approximate mapping of T if there exists $\varepsilon > 0$ such that $d(Tx, \bar{T}x) \leq \varepsilon$, for all $x \in C$.

Consider that T has a fixed point p, and $\bar{T}$ has a fixed point $\bar{p}$. A natural question arises: does $\bar{p}$ approximates p and, if it does, how good the approximation is? Also, what means do we have in order to provide an estimate for $d(p, \bar{p})$?

In order to establish data dependency results, we need the next lemma.

Lemma 4.2 ([19])**.** *Consider that $\{t_n\}$ is a sequence in $(0,1)$ so that $\sum_{n=0}^{\infty} t_n = \infty$, and $\{s_n\}$, $\{u_n\}$ are nonnegative real number sequences, so that*

$$s_{n+1} \leq (1-t_n)s_n + t_n u_n,\ n \geq 0.$$

Then $0 \leq \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} u_n$.

We start this part of our study by referring to the iterative method (2.1).

Theorem 4.3. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, d be the geodesic distance, and C a nonempty, closed and convex subset of M. Let $T\colon C \to C$ be a contraction mapping of constant $k \in (0,1)$ with the fixed point p, and $\bar{T}\colon C \to C$ be an approximate mapping of T, corresponding to $\varepsilon > 0$. Denote by $\{x_n\}$, and $\{\bar{x}_n\}$ the sequences obtained by applying process (2.1) with respect to mappings T, and $\bar{T}$ respectively. Also, consider the control sequences $\{a_n\}$, $\{c_n\} \subset (0,1)$, and $b_n \in [A,B] \subset (0,1)$, $n \geq 0$. Then*

$$d(p, \bar{p}) \leq \frac{\varepsilon}{1-k}.$$

Proof. The following relations hold true

$$\begin{aligned} d(Tx_n, \bar{T}\bar{x}_n) &\leq d(Tx_n, T\bar{x}_n) + d(T\bar{x}_n, \bar{T}\bar{x}_n) \\ &\leq kd(x_n, \bar{x}_n) + \varepsilon, \ n \geq 0. \end{aligned}$$

Analogously, it can be proved that

$$d(Ty_n, \bar{T}\bar{y}_n) \leq kd(y_n, \bar{y}_n) + \varepsilon, \ n \geq 0, \tag{4.1}$$

$$d(Tz_n, \bar{T}\bar{z}_n) \leq kd(z_n, \bar{z}_n) + \varepsilon, \ n \geq 0. \tag{4.2}$$

Let $\gamma_1 \colon [0,1] \to M$ be the geodesic which joins x_n with Tx_n, for a fixed $n \geq 0$, and $\gamma_2 \colon [0,1] \to M$ the geodesic which connects $\bar{x}_n$ with $\bar{T}\bar{x}_n$.

Moving on to the iteration scheme, we obtain

$$\begin{aligned} d(y_n, \bar{y}_n) &= d(\gamma_1(b_n), \gamma_2(b_n)) \leq (1-b_n)d(\gamma_1(0), \gamma_2(0)) + b_n d(\gamma_1(1), \gamma_2(1)) \\ &\leq (1-b_n)d(x_n, \bar{x}_n) + b_n d(Tx_n, \bar{T}\bar{x}_n) \\ &\leq (1-b_n)d(x_n, \bar{x}_n) + b_n(kd(x_n, \bar{x}_n) + \varepsilon) \\ &= (1-(1-k)b_n)d(x_n, \bar{x}_n) + \varepsilon b_n, \ n \geq 0. \end{aligned} \tag{4.3}$$

Consider now γ_3 as the geodesic which connects Tx_n with Ty_n, and γ_4 as the geodesic joining $\bar{\bar{x}}_n$ with $\bar{T}\bar{y}_n$, for $n \geq 0$ fixed. It follows

$$\begin{aligned} d(z_n, \bar{z}_n) &= d(\gamma_3(c_n), \gamma_4(c_n)) \\ &\leq (1-c_n)d(\gamma_3(0), \gamma_4(0)) + c_n d(\gamma_3(1), \gamma_4(1)) \\ &= (1-c_n)d(Tx_n, \bar{T}\bar{x}_n) + c_n d(Ty_n, \bar{T}\bar{y}_n) \\ &\leq (1-c_n)\big(kd(x_n, \bar{x}_n) + \varepsilon\big) + c_n\big(kd(y_n, \bar{y}_n) + \varepsilon\big) \\ &\leq (1-c_n)\big(kd(x_n, \bar{x}_n) + \varepsilon\big) \\ &\qquad + c_n\big(k(1-(1-k)b_n)d(x_n, \bar{x}_n) + k\varepsilon b_n + \varepsilon\big) \\ &= k(1-(1-k)b_n c_n)d(x_n, \bar{x}_n) + \varepsilon(1 + kb_n c_n), \ n \geq 0. \end{aligned} \tag{4.4}$$

Furthermore, denote by γ_5, $\gamma_6 \colon [0,1] \to M$ the geodesics connecting z_n with Tz_n, and $\bar{z}_n$ with $\bar{T}\bar{z}_n$, respectively. By using relations (4.2), and (4.4), we obtain

$$\begin{aligned} & d(x_{n+1}, \bar{x}_{n+1}) = d(\gamma_5(a_n), \gamma_6(c_n)) \\ \leq\ & (1-a_n)d(\gamma_5(0), \gamma_6(0)) + a_n d(\gamma_5(1), \gamma_6(1)) \\ =\ & (1-a_n)d(z_n, \bar{z}_n) + a_n d(Tz_n, \bar{T}\bar{z}_n) \\ \leq\ & (1-a_n)d(z_n, \bar{z}_n) + ka_n d(z_n, \bar{z}_n) + \varepsilon a_n \\ =\ & \big(1-(1-k)a_n\big)d(z_n, \bar{z}_n) + \varepsilon a_n \\ \leq\ & k\big(1-(1-k)a_n\big)\big(1-(1-k)b_n c_n\big)d(x_n, \bar{x}_n) \\ & + \varepsilon\Big(\big(1-(1-k)a_n\big)(1 + kb_n c_n) + a_n\Big) \\ \leq\ & \Big(1-(1-k)\big(1 + k(a_n + b_n c_n) - k(1-k)a_n b_n c_n\big)\Big)d(x_n, \bar{x}_n) \\ & + \varepsilon\big(1 + k(a_n + b_n c_n) - k(1-k)a_n b_n c_n\big), \ n \geq 0. \end{aligned}$$

Applying Lemma 4.2 for $t_n = (1-k)\big(1 + k(a_n + b_n c_n) - k(1-k)a_n b_n c_n\big)$, and $u_n = \frac{\varepsilon}{1-k}$, it follows that $d(p, \bar{p}) \leq \frac{\varepsilon}{1-k}$. □

With respect to iteration (2.7), consider first the case $Ax_n = Tx_n$, for all $n \geq 0$, which corresponds to iteration introduced in [9] for Banach spaces.

Theorem 4.4. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, d be the geodesic distance, and C a nonempty, closed and convex subset of M. Let $T\colon C \to C$ be a contraction mapping of constant $k \in (0,1)$ with the fixed point p, and $\bar{T}\colon C \to C$ be an approximate mapping of T, corresponding to $\varepsilon > 0$. Denote by $\{x_n\}$, and $\{\bar{x}_n\}$ the sequences obtained by applying process (2.7) for $A_n x_n = Tx_n$, for all $n \geq 0$, with respect to mappings T, and $\bar{T}$ respectively. Also, consider the control sequences $\{a_n\}$, $\{c_n\} \subset (0,1)$, and $b_n \in [A,B] \subset (0,1)$, $n \geq 0$. Then*

$$d(p,\bar{p}) \leq \frac{\varepsilon}{1-k}.$$

Proof. Let $\gamma_5\colon [0,1] \to M$, and $\gamma_6\colon [0,1] \to M$ be the geodesics which join Tx_n with Tz_n, an $\bar{T}\bar{x}_n$ with $\bar{T}\bar{z}_n$, respectively. In this case relations (4.1), (4.2), (4.3), and (4.4) still hold true. Taking advantage of them, it follows

$$\begin{aligned} d(x_{n+1},\bar{x}_{n+1}) &= d(\gamma_5(a_n),\gamma_6(a_n)) \\ &\leq (1-a_n)d(\gamma_5(0),\gamma_6(0)) + a_n d(\gamma_5(1),\gamma_6(1)) \\ &= (1-a_n)d(Tx_n,\bar{T}\bar{x}_n) + a_n d(Tz_n,\bar{T}\bar{z}_n) \\ &\leq k(1-(1-k)a_n - k(1-k)a_n b_n c_n)d(x_n,\bar{x}_n) \\ &\quad +\varepsilon(1+ka_n+k^2 a_n b_n c_n) \\ &= (1-(1-k)(1+ka_n+k^2 a_n b_n c_n))d(x_n,\bar{x}_n) \\ &\quad +\varepsilon(1+ka_n+k^2 a_n b_n c_n),\ n \geq 0. \end{aligned}$$

Taking $t_n = (1-k)\big(1+ka_n+k^2 a_n b_n c_n\big)$, and $u_n = \frac{\varepsilon}{1-k}$, $n \geq 0$ in Lemma 4.2, after considering $n \to \infty$, we get that $d(p,\bar{p}) \leq \frac{\varepsilon}{1-k}$, which completes the proof. □

We analyze now the case $Ax_n = x_n$, $n \geq 0$, in process (2.7), corresponding with process in [21], as the next theorem shows.

Theorem 4.5. *Consider the Hadamard manifold $(M,\langle\cdot,\cdot\rangle)$, d the geodesic distance, and C a nonempty, closed and convex subset of M. Let $T\colon C \to C$ be a contraction mapping of constant $k \in (0,1)$ with the fixed point p, and $\bar{T}\colon C \to C$ be an approximate mapping of T, corresponding to $\varepsilon > 0$. Denote by $\{x_n\}$, and $\{\bar{x}_n\}$ the sequences obtained by applying process (2.7) for $A_n = Tx_n$, for all $n \geq 0$, with respect to mappings T, and $\bar{T}$ respectively. Also, consider the control sequences $\{a_n\}$, $\{c_n\} \subset (0,1)$, and $b_n \in [A,B] \subset (0,1)$, $n \geq 0$. Then*

$$d(p,\bar{p}) \leq \frac{\varepsilon}{1-k}.$$

Proof. Relations (4.1), (4.2), and (4.3) from Theorem 4.3 are still valid.

Let γ_3 be the geodesic connecting x_n with Ty_n, and γ_4 the geodesic connecting $\bar{x}_n$ with $\bar{T}\bar{y}_n$, $n \geq 0$.

Keeping in mind inequalities (4.1) and (4.3), it follows

$$\begin{aligned} d(z_n,\bar{z}_n) &= d(\gamma_3(c_n),\gamma_4(c_n)) \\ &\leq (1-c_n)d(\gamma_3(0),\gamma_4(0)) + c_n d(\gamma_3(1),\gamma_4(1)) \\ &= (1-c_n)d(x_n,\bar{x}_n) + c_n d(Ty_n,\bar{T}\bar{y}_n) \end{aligned}$$

$$\begin{aligned} &\leq \big(1-(1-k)c_n - k(1-k)b_nc_n\big)d(x_n,\bar{x}_n) \\ (4.5)\qquad &\quad +\varepsilon c_n(kb_n+1),\ n\geq 0. \end{aligned}$$

Now consider γ_5 and γ_6 the geodesics joining Tx_n with Tz_n, and $\bar{T}\bar{x}_n$ with $\bar{T}\bar{z}_n$, for $n \geq 0$ taken. Taking advantage of (4.2) and (4.5), it follows

$$\begin{aligned} d(x_{n+1},\bar{x}_{n+1}) &= d(\gamma_5(a_n),\gamma_6(a_n)) \\ &\leq (1-a_n)d(Tx_n,\bar{T}\bar{x}_n) + a_n d(Tz_n,\bar{T}\bar{z}_n) \\ &\leq (1-a_n)\big(kd(x_n,\bar{x}_n)+\varepsilon\big) + a_n\big(kd(z_n,\bar{z}_n)+\varepsilon\big) \\ &\leq k\big(1-(1-k)a_nc_n - k(1-k)a_nb_nc_n\big)d(x_n,\bar{x}_n) \\ &\quad +\varepsilon(1+ka_nc_n+k^2a_nb_nc_n) \\ &\leq \Big(1-(1-k)\big(1+ka_nc_n+k^2a_nb_nc_n\big)\Big) \\ &\quad +\varepsilon(1+ka_nc_n+k^2a_nb_nc_n),\ n\geq 0. \end{aligned}$$

Considering $t_n = (1-k)(1+ka_nc_n+k^2a_nb_nc_n)$, and $u_n = \frac{\varepsilon}{1-k}$, Lemma 4.2 implies the conclusion of the theorem. □

Looking now at iteration (2.13), in the case when $A_n = Ty_n$, $n \geq 0$, we are in the position to state the next theorem.

Theorem 4.6. *Assume that $(M,\langle\cdot,\cdot\rangle)$ is a Hadamard manifold, d is its geodesic distance, and C a nonempty, closed and convex subset of M. Consider $T\colon C\to C$ a contraction mapping of constant $k\in(0,1)$ with the fixed point p, and $\bar{T}\colon C\to C$ an approximate mapping of T, corresponding to $\varepsilon>0$. $\{x_n\}$, and $\{\bar{x}_n\}$ are the sequences obtained by applying iteration (2.13) for $A_n = y_n$, for all $n\geq 0$, with respect to mappings T, and $\bar{T}$ respectively. Also, consider the control sequences $\{a_n\}$, $\{c_n\}\subset(0,1)$, and $b_n\in[A,B]\subset(0,1)$, $n\geq 0$. Then*

$$d(p,\bar{p}) \leq \frac{\varepsilon}{1-k}.$$

Proof. Inequalities (4.1), (4.2), and (4.3) from Theorem 4.3 still hold true.

Let γ_3 be the geodesic joining y_n with Ty_n, and by γ_4 the geodesic connecting $\bar{y}_n$ with $\bar{T}\bar{y}_n$, $n\geq 0$.

Using (4.3) and (4.1) from Theorem 4.3, it follows

$$\begin{aligned} d(z_n,\bar{z}_n) &= d(\gamma_3(c_n),\gamma_4(c_n)) \\ &\leq (1-c_n)d(\gamma_3(0),\gamma_4(0)) + c_n d(\gamma_3(1),\gamma_4(1)) \\ &= (1-c_n)d(y_n,\bar{y}_n) + c_n d(Ty_n,\bar{T}\bar{y}_n) \\ (4.6)\qquad &= (1-c_n)\Big(\big(1-(1-k)b_n\big)d(x_n,\bar{x}_n)+\varepsilon b_n\Big) \\ &\quad +c_n\Big(k\big(1-(1-k)b_n\big)d(x_n,\bar{x}_n)+\varepsilon(1+kb_n)\Big) \\ &\leq \big(1-(1-k)b_n\big)\big(1-(1-k)c_n\big)d(x_n,\bar{x}_n) \\ &\quad +\varepsilon(b_n+c_n-(1-k)b_nc_n),\ n\geq 0. \end{aligned}$$

Having also in mind (4.2) from Theorem 4.3, we obtain

$$d(Tz_n,\bar{T}\bar{z}_n) \leq kd(z_n,\bar{z}_n)+\varepsilon$$

$$\begin{aligned} &\leq k\big(1-(1-k)b_n\big)\big(1-(1-k)c_n\big)d(x_n,\bar{x}_n) \\ &\quad +\varepsilon(1+kb_n+kc_n-k(1-k)b_nc_n),\ n\geq 0. \end{aligned} \tag{4.7}$$

Let γ_5 and γ_6 be the geodesics which connect Tx_n with Tz_n, and $\bar{T}\bar{x}_n$ with $\bar{T}\bar{z}_n$, for $n \geq 0$ given. The use of (4.1) from Theorem 4.3, and (4.7) implies

$$\begin{aligned} d(x_{n+1},\bar{x}_{n+1}) &= d(\gamma_5(a_n),\gamma_6(a_n)) \\ &\leq (1-a_n)d(Ty_n,\bar{T}\bar{y}_n)+a_nd(Tz_n,\bar{T}\bar{z}_n) \\ &\leq k\big(1-(1-k)b_n\big)\big(1-(1-k)a_nc_n\big)d(x_n,\bar{x}_n) \\ &\quad +\varepsilon(1+kb_n+ka_nc_n-k(1-k)a_nb_nc_n) \\ &= \Big(1-(1-k)\big(1+kb_n+ka_nc_n \\ &\quad -k(1-k)a_nb_nc_n\big)\Big)d(x_n,\bar{x}_n) \\ &\quad +\varepsilon\big(1+kb_n+ka_nc_n-k(1-k)a_nb_nc_n\big),\ n\geq 0. \end{aligned}$$

Considering $t_n = (1-k)\big(1+kb_n+ka_nc_n-k(1-k)a_nb_nc_n\big)$, and $u_n = \frac{\varepsilon}{1-k}$, Lemma 4.2 implies the conclusion of the theorem. □

We move forward by analyzing the case $A_n = x_n$, $n \geq 0$, in the iteration process (2.13). For the sake of computation, here $(1-c_n)$ and c_n changed places.

Theorem 4.7. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, d be its geodesic distance, and C a nonempty, closed and convex subset of M. Consider $T\colon C \to C$ a contraction mapping of constant $k \in (0,1)$ with the fixed point p, and $\bar{T}\colon C \to C$ an approximate mapping of T, corresponding to $\varepsilon > 0$. $\{x_n\}$, and $\{\bar{x}_n\}$ are the sequences obtained by applying iteration (2.13) for $Ax_n = y_n$, for all $n \geq 0$, with respect to mappings T, and $\bar{T}$ respectively. Also, consider the control sequences $\{a_n\}$, $\{c_n\} \subset (0,1)$, and $b_n \in [A,B] \subset (0,1)$, $n \geq 0$. Then*

$$d(p,\bar{p}) \leq \frac{\varepsilon}{1-k}.$$

Proof. In the proof we will use inequalities (4.1), (4.2), and (4.3) from Theorem 4.3, which are still valid.

Denote by γ_3 the geodesic connecting x_n with y_n, and by γ_4 the geodesic connecting $\bar{x}_n$ with $\bar{y}_n$, $n \geq 0$.

Using (4.3), we obtain

$$\begin{aligned} d(z_n,\bar{z}_n) &= d(\gamma_3(c_n),\gamma_4(c_n)) \\ &\leq (1-c_n)d(\gamma_3(0),\gamma_4(0))+c_nd(\gamma_3(1),\gamma_4(1)) \\ &= (1-c_n)d(x_n,\bar{x}_n)+c_nd(y_n,\bar{y}_n) \\ &= (1-c_n)d(x_n,\bar{x}_n)+c_n\Big(\big(1-(1-k)b_n\big)d(x_n,\bar{x}_n)+\varepsilon b_n\Big) \\ &= \big(1-(1-k)b_nc_n\big)d(x_n,\bar{x}_n)+\varepsilon b_nc_n,\ n\geq 0. \end{aligned}$$

Combining this inequality with (4.2) from Theorem 4.3, we get

$$\begin{aligned} d(Tz_n,\bar{T}\bar{z}_n) &\leq kd(z_n,\bar{z}_n)+\varepsilon \\ &\leq k\big(1-(1-k)b_nc_n\big)d(x_n,\bar{x}_n)+\varepsilon(1+kb_nc_n),\ n\geq 0. \end{aligned} \tag{4.8}$$

Let γ_5 and γ_6 be the geodesics which connect Tx_n with Tz_n, and $\bar{T}\bar{x}_n$ with $\bar{T}\bar{z}_n$, for $n \geq 0$ given. The use of (4.1) from Theorem 4.3, and (4.8) implies

$$\begin{aligned}
& d(x_{n+1}, \bar{x}_{n+1}) = d(\gamma_5(a_n), \gamma_6(a_n)) \\
\leq\ & (1-a_n)d(Ty_n, \bar{T}\bar{y}_n) + a_n d(Tz_n, \bar{T}\bar{z}_n) \\
\leq\ & (1-a_n)\Big(k\big(1-(1-k)b_n\big)d(x_n, \bar{x}_n) + \varepsilon(1+kb_n)\Big) \\
& + a_n\Big(k\big(1-(1-k)b_n c_n\big)d(x_n, \bar{x}_n) + \varepsilon(1+kb_n c_n)\Big) \\
\leq\ & k\Big(1-(1-k)b_n + (1-k)a_n b_n - (1-k)a_n b_n c_n\Big)d(x_n, \bar{x}_n) \\
& + \varepsilon(1+kb_n - ka_n b_n + ka_n b_n c_n) \\
=\ & \Big(1-(1-k)\big(1+kb_n - ka_n b_n + ka_n b_n c_n\big)\Big)d(x_n, \bar{x}_n) \\
& + \varepsilon(1+kb_n - ka_n b_n + ka_n b_n c_n),\ n \geq 0.
\end{aligned}$$

Considering $t_n = (1-k)(1+kb_n - ka_n b_n + ka_n b_n c_n)$, and $u_n = \frac{\varepsilon}{1-k}$, Lemma 4.2 implies the conclusion of the theorem. □

We respect to the iterative method (2.27), we state the next theorem.

Theorem 4.8. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, d be its geodesic distance, and C a nonempty, closed and convex subset of M. Consider $T\colon C \to C$ a contraction mapping of constant $k \in (0,1)$ with the fixed point p, and $\bar{T}\colon C \to C$ an approximate mapping of T, corresponding to $\varepsilon > 0$. $\{x_n\}$, and $\{\bar{x}_n\}$ are the sequences obtained by applying iteration (2.27), with respect to mappings T, and $\bar{T}$ respectively. Also, consider the control sequences $\{a_n\} \subset (0,1)$, and $b_n \in [A,B] \subset (0,1)$, $n \geq 0$. Then*

$$d(p, \bar{p}) \leq \frac{\varepsilon}{1-k}.$$

Proof. Considering that the steps of the proof are similar to those in the previous theorems, we give only the main computational elements.

The following relations hold true

$$d(y_n, \bar{y}_n) \quad \leq \quad k\big(1-(1-k)b_n\big)d(x_n, \bar{x}_n) + \varepsilon(1+kb_n),\ n \geq 0.$$

Furthermore,

$$\begin{aligned}
& d(x_{n+1}, \bar{x}_{n+1}) \leq k^2\big(1-(1-k)a_n\big)\big(1-(1-k)b_n\big)d(x_n, \bar{x}_n) \\
& + \varepsilon\big(1+k-ka_n + k^2 a_n + k^2 b_n - k^2 a_n b_n + k^3 a_n b_n\big) \\
\leq\ & \Big(1-(1-k)\big(1+k+k^2 a_n + k^2 b_n - k^2(1-k)a_n b_n\big)\Big)d(x_n, \bar{x}_n) \\
& + \varepsilon\big(1+k+k^2 a_n + k^2 b_n - k^2(1-k)a_n b_n\big),\ n \geq 0.
\end{aligned}$$

Considering $t_n = (1-k)(1+k+k^2 a_n + k^2 b_n - k^2(1-k)a_n b_n)$, and $u_n = \frac{\varepsilon}{1-k}$, Lemma 4.2 implies the conclusion of the theorem. □

We end up with an analysis of the data dependency regarding scheme (2.30), as follows.

Theorem 4.9. *Let $(M, \langle\cdot,\cdot\rangle)$ be a Hadamard manifold, d be its geodesic distance, and C a nonempty, closed and convex subset of M. Consider $T\colon C \to C$ a contraction mapping of constant $k \in (0,1)$ with the fixed point p, and $\bar{T}\colon C \to C$ an approximate mapping of T, corresponding to $\varepsilon > 0$. $\{x_n\}$, and $\{\bar{x}_n\}$ are the sequences obtained by applying iteration (2.30), with respect to mappings T, and $\bar{T}$ respectively. Also, consider the control sequences $\{a_n\} \subset (0,1)$, and $b_n \in [A,B] \subset (0,1)$, $n \geq 0$. Then*

$$d(p,\bar{p}) \leq \frac{\varepsilon}{1-k}.$$

Proof. Considering that the steps of the proof are similar to those in the previous theorems, we give only the main computational elements.

The following relations hold true

$$d(z_n,\bar{z}_n) \leq k\big(1-(1-k)b_nc_n\big)d(x_n,\bar{x}_n)+\varepsilon(1+kb_nc_n),\ n \geq 0.$$

Furthermore,

$$\begin{aligned} d(x_{n+1},\bar{x}_{n+1}) &\leq k^2\big(1-(1-k)b_nc_n\big)d(x_n,\bar{x}_n)+\varepsilon\big(1+k+k^2b_nc_n\big) \\ &\leq \Big(1-(1-k)\big(1+k+k^2b_nc_n\big)\Big)d(x_n,\bar{x}_n) \\ &\quad +\varepsilon\big(1+k+k^2b_nc_n\big),\ n \geq 0. \end{aligned}$$

Considering $t_n = (1-k)(1+k+k^2b_nc_n)$, and $u_n = \frac{\varepsilon}{1-k}$, Lemma 4.2 implies the conclusion of the theorem. □

5. Conclusions

In this work, in the setting of Hadamard manifolds, we have studied some three-step iteration schemes for the determination of fixed points for mappings with adequate properties. Convergence properties have been stated and proved for the nine processes studied. Also, stability results in the same framework of manifolds are presented. A data dependency study is also performed.

References

[1] V. Berinde, *Iterative Approximation of Fixed Points* 2nd ed., Springer 2007.

[2] M. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature*, Springer, 1999.

[3] R. Chugh, V. Kumar, and S. Kumar, *Strong convergence of a new three step iterative scheme in Banach spaces*, Am. J. Comput. Math. **2** (2012), 345–357.

[4] K. Gdawiec and W. Kotarski, *Polynomiography for the polynomial infinity norm via Kalantaris formula and nonstandard iterations*, Appl. Math. Comput. **307** (2017), 17–30.

[5] A. M. Harder and T. L. Hicks, *A stable iteration procedure for nonexpansive mappings*, Math. Japon. **33** (1988), 687–692.

[6] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Am. Math. Soc. **44** (1974), 147–150.

[7] V. Karakaya, N. Bouzara, K. Doğan and Y. Atalan, *On different results for a new two-step iteration method under weak-contraction mappings in Banach spaces*, 2015, arXiv: 1507.00200.

[8] V. Karakaya, K. Doğan, F. Gürsoy and M. Ertürk, *Fixed point of a new three-step iteration algorithm under contractive-like operators over normed spaces*, Abstr. Appl. Anal. 2013 (2013), Article ID 560258.

[9] I. Karahan and M.Özdemir, *A general iterative method for approximation of fixed points and their applications*, Adv. Fixed Point Theory **3** (2013), 510–526.

[10] C. Li, G. López and V. Martín-Márquez, *Iterative algorithms for nonexpansive mappings in Hadamard manifolds*, Taiwan. J. Math. **14** (2010), 541–559.

[11] W. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc. **4** (1953), 506–510.
[12] E. Picard, *Mémoire sur la théorie des équations aux dérivées partielles et la méthode des approximations successives*, J. Math. Pures Appl. **6** (1890), 145–210.
[13] I. A.Rus and S. Mureşan, *Data dependency of fixed points set of weakly Picard operators*, Stud. Univ. Babes-Bolyai Math. **43** (1998), 79–83.
[14] P. Sainuan, *Rate of convergence of P-iteration and S-iteration for continuous functions on closed intervals*, Thai J. Math. **13** (2015), 451–459.
[15] T. Sakai, *Riemannian Geometry*, Translations of Mathematical Monographs, 149, American Mathematical Society, 1992.
[16] W. Shatanawi and M. Postolache, *Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces*, Fixed Point Theory Appl., 2013 (2013), Article ID 54.
[17] W. Sintunavarat and A. Pitea, *On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis*, J. Nonlinear Sci. Appl. **9** (2016), 2553–2562.
[18] S. Suantai, *Weak and strong convergence criteria of Noor iteration for asymptotically non-expansive mappings*, J. Math. Anal. Appl. **311** (2005), 506–517.
[19] Ş. M. Şoltuz and T. Groşan, *Data dependence for Ishikawa iteration when dealing with contractive like operators*, Fixed Point Theory and Appl. 2008 (2008), Article ID 242916.
[20] B.S. Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzukis generalized nonexpansive mappings*, Appl. Math. Comput. **275** (2016), 147–155.
[21] D. Thakur, B. S. Thakur and M. Postolache, *New iteration scheme for numerical reckoning fixed points of nonexpansive mappings*, J. Inequal. Appl. 2014 (2014) Article ID 328.
[22] C. Udrişte, *Functions and Optimization Methods on Riemannian Manifolds*, Kluwer Academic, 1994.
[23] E. Zeidler, *Nonlinear Functional Analysis and its Applications I, Fixed-Point Theorems*, Springer-Verlag, 1986.

Manuscript received July 7 2010
revised January 20 2019

A. Pitea
Department of Mathematics and Informatics, University "Politehnica" of Bucharest, Romania
E-mail address: `arianapitea@yahoo.com`