Applied Analysis and Optimization
Volume 3, Number 1, 2019, 43–56

Yokohama Publishers
ISSN 2189-1664 Online Journal

STRONG CONVERGENCE THEOREMS BY HYBRID METHODS FOR NONCOMMUTATIVE NORMALLY 2-GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

WATARU TAKAHASHI, CHING-FENG WEN*, AND JEN-CHIH YAO†

Abstract. In this paper, using the hybrid method defined by Nakajo and Takahashi [15], we first obtain a strong convergence theorem for noncommutative two normally 2-generalized hybrid mappings in a Hilbert space. Next, using the shrinking projection method defined by Takahashi, Takeuchi and Kubota [19], we prove another strong convergence for the mappings in a Hilbert space. Using these results, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. We denote by $F(T)$ the set of fixed points of T, i.e., $F(T) = \{z \in C : Tz = z\}$. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well-known that if C is a bounded, closed and convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty. Furthermore, from Baillon [2] we know the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space. Let C be a nonempty, closed and convex subset of H and let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

In 2010, Kocourek, Takahashi and Yao [8] defined a broad class of nonlinear mappings in a Hilbert space: Let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [8] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. Such a mapping T is called $(\alpha,\ \beta)$-*generalized hybrid*. We also know the following: For $\lambda \in \mathbb{R}$, a mapping $U : C \to H$ is called λ-*hybrid* [1] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + 2(1 - \lambda)\langle x - Ux, y - Uy\rangle \tag{1.2}$$

2010 *Mathematics Subject Classification.* 47H05, 47H09.

Key words and phrases. Fixed point, generalized hybrid mapping, nonexpansive mapping, hybrid method, shrinking projection method.

*† The second and the third authors were partially supported by the grant MOST 106-2115-M-037-001 and the grant MOST 105-2221-E-039-009-MY3, respectively.

for all $x, y \in C$. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is *nonspreading* [10, 11] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [18] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [6]. The nonlinear ergodic theorem by Baillon [2] for nonexpansive mappings has been extended to generalized hybrid mappings in a Hilbert space by Kocourek, Takahashi and Yao [8]. The generalized hybrid mappings were extended by Maruyama, Takahashi and Yao [13] as follows: A mapping $T : C \to C$ is called 2-*generalized hybrid* [13] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_2 \left\|T^2x - Ty\right\|^2 + \alpha_1 \left\|Tx - Ty\right\|^2 + (1 - \alpha_1 - \alpha_2) \left\|x - Ty\right\|^2 \\ \leq \beta_2 \left\|T^2x - y\right\|^2 + \beta_1 \left\|Tx - y\right\|^2 + (1 - \beta_1 - \beta_2) \left\|x - y\right\|^2\end{aligned}$$

for all $x, y \in C$. Very recently, the concept of 2-generalized hybrid mappings was further extended by Kondo and Takahashi [12]. A mapping $T : C \to C$ is called *normally* 2-*generalized hybrid* [12] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_2\|T^2x - Ty\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ + \beta_2\|T^2x - y\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \leq 0\end{aligned} \tag{1.3}$$

for all $x, y \in C$, where $\sum_{n=0}^{2} (\alpha_n + \beta_n) \geq 0$ and $\alpha_2 + \alpha_1 + \alpha_0 > 0$. On the other hand, we know the hybrid method by Nakajo and Takahashi [15] and the shrinking projection methd by Takahashi, Takeuchi and Kubota [19]. By using these methods, Hojo, Kondo and Takahashi [3] proved the following theorems for normally 2-generalized hybrid mappings in a Hilbert space; see also [4].

Theorem 1.1 ([3]). *Let H be a Hilbert space, let C be a nonempty, convex and closed subset of H. Let S and T be commutative normally 2-generalized hybrid mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset [0, 1]$ satisfies $0 \leq \alpha_n \leq a < 1$ for some $a \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S) \cap F(T)} x$, where $P_{F(S) \cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Theorem 1.2 ([3]). *Let H be a Hilbert space and let C be a nonempty, convex and closed subset of H. Let S and T be commutative normally 2-generalized hybrid mappings of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in*

C such that $u_n \to u$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $\{\alpha_n\} \subset [0,1]$ is a sequence such that $\liminf_{n\to\infty} \alpha_n < 1$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)}u$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

In this paper, using the hybrid method defined, we first obtain a strong convergence theorem for noncommutative two normally 2-generalized hybrid mappings in a Hilbert space. Next, using the shrinking projection method, we prove another strong convergence for the mappings in a Hilbert space. Using these results, we get well-known and new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. In a Hilbert space, it is known that

$$2\langle x - y, y\rangle \le \|x\|^2 - \|y\|^2 \le 2\langle x - y, x\rangle \tag{2.1}$$

for all $x, y \in H$ and

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [17]. Furthermore, in a Hilbert space, we have that

$$2\langle x - y, z - w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.3}$$

for all $x, y, z, w \in H$. We also have the following result from [13].

Lemma 2.1 ([13])**.** *Let $x, y, z \in H$ and $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$. Then,*

$$\begin{aligned} &\|ax + by + cz\|^2 \\ &\quad = a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2. \end{aligned}$$

Additionally, if $a, b, c \in [0,1]$, then

$$\|ax + by + cz\|^2 \le a\|x\|^2 + b\|y\|^2 + c\|z\|^2.$$

Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx - u\| \le \|x - u\|, \quad \forall x \in C,\ u \in F(T).$$

If C is closed and convex and $T : C \to H$ with $F(T) \neq \emptyset$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [7]. For a nonempty, closed and convex subset D of H, the nearest point projection of H onto D is denoted by

P_D, that is, $\|x - P_D x\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such a mapping P_D is called the metric projection of H onto D. We know that the metric projection P_D is firmly nonexpansive; $\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, x - y\rangle$ for all $x, y \in H$. Furthermore, $\langle x - P_D x, y - P_D x\rangle \leq 0$ holds for all $x \in H$ and $y \in D$; see [16, 17]. Using this inequality and (2.3), we have that

$$\|P_D x - y\|^2 + \|P_D x - x\|^2 \leq \|x - y\|^2, \quad \forall x \in H,\ y \in D. \tag{2.4}$$

Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to C$ is called normally 2-*generalized hybrid* [12] if it satisfies (1.3). We also call such a mapping $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$-*normally* 2-*generalized hybrid.* If $x = Tx$ in (1.3), then for any $y \in C$,

$$\alpha_2\|x-Ty\|^2 + \alpha_1\|x - Ty\|^2 + \alpha_0\|x - Ty\|^2 + \beta_2\|x - y\|^2 + \beta_1\|x - y\|^2 + \beta_0\|x - y\|^2 \leq 0$$

and hence

$$(\alpha_2 + \alpha_1 + \alpha_0)\|x - Ty\|^2 \leq -(\beta_2 + \beta_1 + \beta_0)\|x - y\|^2.$$

From $\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0$, we have that

$$(\alpha_2 + \alpha_1 + \alpha_0)\|x - Ty\|^2 \leq -(\beta_2 + \beta_1 + \beta_0)\|x - y\|^2 \leq (\alpha_2 + \alpha_1 + \alpha_0)\|x - y\|^2.$$

Since $\alpha_2 + \alpha_1 + \alpha_0 > 0$, it follows that

$$\|x - Ty\| \leq \|x - y\|, \quad \forall x \in F(T),\ y \in C. \tag{2.5}$$

Thus if T is a normally 2-generalized hybrid mapping and $F(T) \neq \emptyset$, then it is quasi-nonexpansive; see also [12]. Furthermore, we have the following result for normally 2-generalized hybrid mappings in a Hilbert space.

Lemma 2.2 ([12])**.** *Let C be a nonempty, closed and convex subset of H, let $T : C \to C$ be a normally 2-generalized hybrid mapping, and let $\{x_n\}$ be a sequence in C satisfying $x_n - Tx_n \to 0$, $T^2x_n - x_n \to 0$ and $x_n \rightharpoonup v$. Then, $v \in F(T)$.*

For a sequence $\{C_n\}$ of nonempty closed convex subsets of a Hilbert space H, define s-Li$_n C_n$ and w-Ls$_n C_n$ as follows: $x \in$s-Li$_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$w-Ls$_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.6}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [14] and we write $C_0 =$M-$\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [14]. Tsukada [21] proved the following theorem.

Theorem 2.3 ([21])**.** *Let H be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 =M$-$\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in H$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of H onto C_n and C_0, respectively.*

3. Strong convergence theorems by hybrid methods

In this section, using the hybrid method by Nakajo and Takahashi [15], we first prove a strong convergence theorem for noncommutative normally 2-generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be normally 2-generalized hybrid mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\gamma_n S + (1-\gamma_n)T\big)x_n + c_n\big(\delta_n S^2 + (1-\delta_n)T^2\big)x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $a, b, c, d, e, f \in \mathbb{R}$ and $\{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1]$ satisfy the following:

$$0 < a \leq \gamma_n \leq b < 1,\ 0 < c \leq \delta_n \leq d < 1,$$
$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \leq a_n, b_n, c_n \leq f < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} x_1$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Setting $S_n = \gamma_n S + (1-\gamma_n)T$ and $T_n = \delta_n S^2 + (1-\delta_n)T^2$, we have that

$$y_n = a_n x_n + b_n S_n x_n + c_n T_n x_n$$

for all $n \in \mathbb{N}$. Since

$$\|y_n - z\|^2 \leq \|x_n - z\|^2$$
$$\Longleftrightarrow \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \leq 0,$$

we have that C_n, Q_n and $C_n \cap Q_n$ are closed and convex for all $n \in \mathbb{N}$. We next show that $C_n \cap Q_n$ is nonempty. Since S and T are quasi-nonexpansive, we have that, for any $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$,

$$\begin{aligned} \|S_n x_n - q\| &= \|\gamma_n S x_n + (1-\gamma_n)T x_n - q\| \\ &\leq \gamma_n \|S x_n - q\| + (1-\gamma_n)\|T x_n - q\| \\ &\leq \gamma_n \|x_n - q\| + (1-\gamma_n)\|x_n - q\| \\ &= \|x_n - q\| \end{aligned}$$

and

$$\begin{aligned} \|T_n x_n - q\| &= \|\delta_n S^2 x_n + (1-\delta_n)T^2 x_n - q\| \\ &\leq \delta_n \|S^2 x_n - q\| + (1-\delta_n)\|T^2 x_n - q\| \\ &\leq \delta_n \|x_n - q\| + (1-\delta_n)\|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

Then we have that

$$\|y_n - q\| \quad = \quad \|a_n (x_n - q) + b_n (S_n x_n - q) + c_n (T_n x_n - q)\|$$

$$\begin{aligned} &\leq a_n \|x_n - q\| + b_n \|S_n x_n - q\| + c_n \|T_n x_n - q\| \\ &\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\|. \end{aligned}$$

Thus we have $q \in C_n$ and hence $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show by induction that $F(S) \cap F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From $F(S) \cap F(T) \subset Q_1$, it follows that $F(S) \cap F(T) \subset C_1 \cap Q_1$. Suppose that $F(S) \cap F(T) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. From $x_{k+1} = P_{C_k \cap Q_k} x_1$, we have that

$$\langle x_{k+1} - z, x_1 - x_{k+1} \rangle \geq 0, \quad \forall z \in C_k \cap Q_k.$$

Since $F(S) \cap F(T) \subset C_k \cap Q_k$, we also have

$$\langle x_{k+1} - q, x_1 - x_{k+1} \rangle \geq 0, \quad \forall q \in F(S) \cap F(T).$$

This implies $F(S) \cap F(T) \subset Q_{k+1}$. So, we have $F(S) \cap F(T) \subset C_{k+1} \cap Q_{k+1}$. By induction, we have $F(S) \cap F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This means that $\{x_n\}$ is well-defined. Furthermore, since S and T are quasi-nonexpansive, we have from [7] that $F(S) \cap F(T)$ is closed and convex. So, there exists the mertic projection $P_{F(S) \cap F(T)}$ of H onto $F(S) \cap F(T)$.

Since $x_n = P_{Q_n} x_1$ and $x_{n+1} = P_{C_n \cap Q_n} x_1 \in Q_n$, we have from (2.3) that

$$\begin{aligned} 0 &\leq 2\langle x_1 - x_n, x_n - x_{n+1} \rangle \\ &= \|x_1 - x_{n+1}\|^2 - \|x_1 - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\ &\leq \|x_1 - x_{n+1}\|^2 - \|x_1 - x_n\|^2. \end{aligned} \tag{3.1}$$

Thus we get that

$$\|x_1 - x_n\|^2 \leq \|x_1 - x_{n+1}\|^2. \tag{3.2}$$

Furthermore, since $x_n = P_{Q_n} x_1$ and $q \in F(S) \cap F(T) \subset Q_n$, we have

$$\|x_1 - x_n\| \leq \|x_1 - q\|. \tag{3.3}$$

We have from (3.2) and (3.3) that $\lim_{n \to \infty} \|x_1 - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{S_n x_n\}$ and $\{T_n x_n\}$ are also bounded. From (3.1), we have

$$\|x_n - x_{n+1}\|^2 \leq \|x_1 - x_{n+1}\|^2 - \|x_1 - x_n\|^2$$

and hence

$$\|x_n - x_{n+1}\| \to 0. \tag{3.4}$$

From $x_{n+1} \in C_n$, we have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. From (3.4), we have $\|y_n - x_{n+1}\| \to 0$. So, we get that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.5}$$

Next, we verify that $x_n - S_n x_n \to 0$ and $T_n x_n - x_n \to 0$. We obtain from Lemma 2.1 that, for any $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$,

$$\begin{aligned} \|y_n - q\|^2 &= \|a_n (x_n - q) + b_n (S_n x_n - q) + c_n (T_n x_n - q)\|^2 \\ &= a_n \|x_n - q\|^2 + b_n \|S_n x_n - q\|^2 + c_n \|T_n x_n - q\|^2 \\ &- a_n b_n \|x_n - S_n x_n\|^2 - b_n c_n \|S_n x_n - T_n x_n\|^2 - c_n a_n \|T_n x_n - x_n\|^2 \end{aligned}$$

$$\leq \|x_n - q\|^2 - a_n b_n \|x_n - S_n x_n\|^2 - b_n c_n \|S_n x_n - T_n x_n\|^2 - c_n a_n \|T_n x_n - x_n\|^2$$

and hence

$$\begin{aligned} a_n b_n \|x_n - S_n x_n\|^2 + b_n c_n \|S_n x_n - T_n x_n\|^2 + c_n a_n \|T_n x_n - x_n\|^2 \\ \leq \|x_n - q\|^2 - \|y_n - q\|^2 \\ = (\|x_n - q\| + \|y_n - q\|)(\|x_n - q\| - \|y_n - q\|). \end{aligned} \tag{3.6}$$

Since $\|x_n - q\| - \|y_n - q\| \leq \|x_n - y_n\| \to 0$ and $0 < e \leq a_n, b_n, c_n \leq f < 1$, we have from (3.6) that

$$x_n - S_n x_n \to 0 \quad \text{and} \quad T_n x_n - x_n \to 0. \tag{3.7}$$

We also have from (2.1) that, for any $q \in F(S) \cap F(T)$,

$$\begin{aligned} \|x_n - q\|^2 &= \|x_n - S_n x_n + S_n x_n - q\|^2 \\ &\leq \|S_n x_n - q\|^2 + 2\langle x_n - S_n x_n, x_n - q\rangle \\ &= \|\gamma_n S x_n + (1-\gamma_n) T x_n - q\|^2 + 2\langle x_n - S_n x_n, x_n - q\rangle \\ &= \gamma_n \|S x_n - q\|^2 + (1-\gamma_n)\|T x_n - q\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\|S x_n - T x_n\|^2 + 2\langle x_n - S_n x_n, x_n - q\rangle \\ &\leq \gamma_n \|x_n - z\|^2 + (1-\gamma_n)\|x_n - z\|^2 \\ &\quad - \gamma_n(1-\gamma_n)\|S x_n - T x_n\|^2 + 2\langle x_n - S_n x_n, x_n - q\rangle \\ &= \|x_n - z\|^2 - \gamma_n(1-\gamma_n)\|S x_n - T x_n\|^2 + 2\langle x_n - S_n x_n, x_n - q\rangle \end{aligned}$$

and hence

$$\gamma_n(1-\gamma_n)\|S x_n - T x_n\|^2 \leq 2\langle x_n - S_n x_n, x_n - q\rangle.$$

Since $x_n - S_n x_n \to 0$, we have that $S x_n - T x_n \to 0$. Then we have that

$$\begin{aligned} \|x_n - S x_n\| &= \|x_n - S_n x_n + S_n x_n - S x_n\| \\ &\leq \|x_n - S_n x_n\| + \|S_n x_n - S x_n\| \\ &= \|x_n - S_n x_n\| + (1-\gamma_n)\|T x_n - S x_n\| \\ &\to 0. \end{aligned}$$

We also have that $\|x_n - T x_n\| \to 0$. Similarly, we have that

$$\begin{aligned} \|x_n - q\|^2 &= \|x_n - T_n x_n + T_n x_n - q\|^2 \\ &\leq \|T_n x_n - q\|^2 + 2\langle x_n - T_n x_n, x_n - q\rangle \\ &= \|\delta_n S^2 x_n + (1-\delta_n) T^2 x_n - q\|^2 + 2\langle x_n - T_n x_n, x_n - q\rangle \\ &= \delta_n \|S^2 x_n - q\|^2 + (1-\delta_n)\|T^2 x_n - q\|^2 \\ &\quad - \delta_n(1-\delta_n)\|S^2 x_n - T^2 x_n\|^2 + 2\langle x_n - T_n x_n, x_n - q\rangle \\ &\leq \delta_n \|x_n - z\|^2 + (1-\delta_n)\|x_n - z\|^2 \\ &\quad - \delta_n(1-\delta_n)\|S^2 x_n - T^2 x_n\|^2 + 2\langle x_n - T_n x_n, x_n - q\rangle \\ &= \|x_n - z\|^2 - \delta_n(1-\delta_n)\|S^2 x_n - T^2 x_n\|^2 + 2\langle x_n - T_n x_n, x_n - q\rangle \end{aligned}$$

and hence

$$\delta_n(1-\delta_n)\|S^2x_n - T^2x_n\|^2 \le 2\langle x_n - T_nx_n, x_n - q\rangle.$$

Since $x_n - T_nx_n \to 0$, we have that $S^2x_n - T^2x_n \to 0$. Then we have that

$$\begin{aligned}\|x_n - S^2x_n\| &= \|x_n - T_nx_n + T_nx_n - S^2x_n\| \\ &\le \|x_n - T_nx_n\| + \|T_nx_n - S^2x_n\| \\ &= \|x_n - T_nx_n\| + (1-\delta_n)\|T^2x_n - S^2x_n\| \\ &\to 0.\end{aligned}$$

We also have that $\|x_n - T^2x_n\| \to 0$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup z^*$. From Lemma 2.2, we have $z^* \in F(S) \cap F(T)$.

Put $z_0 = P_{F(S)\cap F(T)}x_1$. Since $z_0 = P_{F(S)\cap F(T)}x_1 \in C_n \cap Q_n$ and $x_{n+1} = P_{C_n\cap Q_n}x_1$, we have that

$$\|x_1 - x_{n+1}\| \le \|x_1 - z_0\|. \tag{3.8}$$

Since $\|\cdot\|$ is weakly lower semicontinuous, from $x_{n_i} \rightharpoonup z^*$ we have that

$$\|x_1 - z^*\| \le \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \le \|x_1 - z_0\|.$$

From the definition of z_0, we have $z^* = z_0$. So, we obtain $x_n \rightharpoonup z_0$. We finally show that $x_n \to z_0$. We have that

$$\|z_0 - x_n\|^2 = \|z_0 - x_1\|^2 + \|x_1 - x_n\|^2 + 2\langle z_0 - x_1, x_1 - x_n\rangle, \quad \forall n \in \mathbb{N}.$$

So, we have from (3.8) that

$$\begin{aligned}\limsup_{n\to\infty} \|z_0 - x_n\|^2 &= \limsup_{n\to\infty}(\|z_0 - x_1\|^2 + \|x_1 - x_n\|^2 + 2\langle z_0 - x_1, x_1 - x_n\rangle) \\ &\le \limsup_{n\to\infty}(\|z_0 - x_1\|^2 + \|x_1 - z_0\|^2 + 2\langle z_0 - x_1, x_1 - x_n\rangle) \\ &= \|z_0 - x_1\|^2 + \|x_1 - z_0\|^2 + 2\langle z_0 - x_1, x_1 - z_0\rangle \\ &= 0.\end{aligned}$$

Thus we obtain $\lim_{n\to\infty} \|z_0 - x_n\| = 0$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

Next, we prove a strong convergence theorem by the shrinking projection method [19] for noncommutative normally 2-generalized hybrid mappings in a Hilbert space.

Theorem 3.2. *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be normally 2-generalized hybrid mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_nx_n + b_n\big(\gamma_nS + (1-\gamma_n)T\big)x_n + c_n\big(\delta_nS^2 + (1-\delta_n)T^2\big)x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $a, b, c, d, e, f \in \mathbb{R}$ and $\{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1]$ satisfy the following:

$$0 < a \le \gamma_n \le b < 1,\ 0 < c \le \delta_n \le d < 1,$$
$$a_n + b_n + c_n = 1 \quad and \quad 0 < e \le a_n, b_n, c_n \le f < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)}u$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Setting $S_n = \gamma_n S + (1 - \gamma_n)T$ and $T_n = \delta_n S^2 + (1 - \delta_n)T^2$, we have that

$$y_n = a_n x_n + b_n S_n x_n + c_n T_n x_n$$

for all $n \in \mathbb{N}$. We shall show that C_n is closed and convex, and $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from assumption that $C_1 = C$ is closed and convex, and $F(S) \cap F(T) \subset C_1$. Suppose that C_k is closed and convex, and $F(S) \cap F(T) \subset C_k$ for some $k \in \mathbb{N}$. We know that, for $z \in C_k$,

$$\begin{aligned} &\|y_k - z\|^2 \le \|x_k - z\|^2 \\ \Longleftrightarrow &\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, z\rangle \le 0. \end{aligned}$$

So, C_{k+1} is closed and convex. By induction, C_n are closed and convex for all $n \in \mathbb{N}$. Furthermore, since S and T are quasi-nonexpansive, we have that, for any $q \in F(S) \cap F(T)$,

$$\begin{aligned} \|y_k - q\| &= \|a_k (x_k - q) + b_k (S_k x_k - q) + c_k (T_k x_k - q)\| \\ &\le a_k \|x_k - q\| + b_k \|S_k x_k - q\| + c_k \|T_k x_k - q\| \\ &\le a_k \|x_k - q\| + b_k \|x_k - q\| + c_k \|x_k - q\| \\ &= \|x_k - q\|. \end{aligned}$$

Hence, we have $q \in C_{k+1}$. By induction, we have that $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$. Since C_n is nonempty, closed and convex, there exists the metric projection P_{C_n} of H onto C_n. Thus, $\{x_n\}$ is well-defined.

Define $z_0 = P_{F(S)\cap F(T)}u$. Putting $w_n = P_{C_n}u$, we have that

$$\|u - w_n\| \le \|u - y\|$$

for all $y \in C_n$. Since $z_0 \in F(S) \cap F(T) \subset C_n$, we have that

$$\|u - w_n\| \le \|u - z_0\|. \tag{3.9}$$

This means that $\{w_n\}$ is bounded. From $w_n = P_{C_n}u$ and $w_{n+1} \in C_{n+1} \subset C_n$, we have that

$$\|u - w_n\| \le \|u - w_{n+1}\|.$$

Thus $\{\|u-w_n\|\}$ is bounded and nondecreasing. Then the limit of $\{\|u-w_n\|\}$ exists. Put $\lim_{n\to\infty} \|w_n - u\| = c$. For any $m, n \in \mathbb{N}$ with $m \ge n$, we have $C_m \subset C_n$. From $w_m = P_{C_m}u \in C_m \subset C_n$ and (2.4), we have that

$$\|w_m - P_{C_n}u\|^2 + \|P_{C_n}u - u\|^2 \le \|u - w_m\|^2.$$

This implies that

$$\|w_m - w_n\|^2 \le \|u - w_m\|^2 - \|w_n - u\|^2 \le c^2 - \|w_n - u\|^2. \tag{3.10}$$

Since $c^2 - \|w_n - u\|^2 \to 0$ as $n \to \infty$, we have that $\{w_n\}$ is a Caushy sequence. By the completeness of C, there exists a point $w_0 \in C$ such that $w_n \to w_0$.

Using Theorem 2.3, we can also prove that $w_n \to w_0$. In fact, since $\{C_n\}$ is a nonincreasing sequence of nonempty, closed and convex subsets of H with respect to inclusion, it follows that

$$\emptyset \neq F(S) \cap F(T) \subset \text{M-}\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n. \tag{3.11}$$

Put $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then, by Theorem 2.3 we have that $\{w_n\} = \{P_{C_n}u\}$ converges strongly to $w_0 = P_{C_0}u$, i.e., $w_n = P_{C_n}u \to w_0$.

Since the metric projection P_{C_n} is nonexpansive, it follows that

$$\begin{aligned}\|x_n - w_0\| &\leq \|x_n - w_n\| + \|w_n - w_0\| \\ &= \|P_{C_n}u_n - P_{C_n}u\| + \|w_n - w_0\| \\ &\leq \|u_n - u\| + \|w_n - w_0\|\end{aligned}$$

and hence

$$x_n \to w_0. \tag{3.12}$$

To complete the proof, it is sufficient to show that $z_0 = P_{F(S)\cap F(T)}u = w_0$.

From (3.12), we have that

$$\|x_n - x_{n+1}\| \to 0. \tag{3.13}$$

From $x_{n+1} \in C_{n+1}$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. So, we get that $\|y_n - x_{n+1}\| \to 0$. Using this, we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.14}$$

As in the proof of Theorem 3.1, we have that $Sx_n - x_n \to 0$, $Tx_n - x_n \to 0$, $S^2x_n - x_n \to 0$ and $T^2x_n - x_n \to 0$. From (3.12), we also get that $x_n \rightharpoonup w_0$. From Lemma 2.2, we have $w_0 \in F(S) \cap F(T)$. Since $z_0 = P_{F(S)\cap F(T)}u \in C_{n+1}$ and $x_{n+1} = P_{C_{n+1}}u_{n+1}$, we have that

$$\|u_{n+1} - x_{n+1}\| \leq \|u_{n+1} - z_0\|. \tag{3.15}$$

Thus we have that

$$\|u - w_0\| \leq \|u - z_0\|$$

and hence $z_0 = w_0$. Therefore, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

4. Applications

In this section, using Theorems 3.1 and 3.2, we get new strong convergence theorems by the hybrid method and the shrinking projection method in a Hilbert space.

Theorem 4.1. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let*

$\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and

$$\begin{cases} y_n = a_n x_n + b_n T x_n + c_n T^2 x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $\{a_n\}, \{b_n\}, \{c_n\} \subset [0,1]$ and $e, f \in \mathbb{N}$ satisfy

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \le a_n, b_n, c_n \le f < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(T)} x_1$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Proof. Put $S = T$ and $\gamma_n = \delta_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ in Theorem 3.1, Furthermore, since the class of nonexpansive mappings is contained in the class of normally 2-generalized hybrid mappings, we obtain the desired result from Theorem 3.1. □

Theorem 4.2. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be nonexpansive and nonspreading mappings, respectively, such that $F(S) \cap F(T) \ne \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n \Big(\gamma_n S x_n + (1-\gamma_n) T x_n\Big) + c_n \Big(\delta_n S^2 x_n + (1-\delta_n) T^2 x_n\Big), \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $a, b, c, d, e, f \in \mathbb{R}$ and $\{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0,1]$ satisfy the following:

$$0 < a \le \gamma_n \le b < 1, \; 0 < c \le \delta_n \le d < 1,$$
$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \le a_n, b_n, c_n \le f < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} x_1$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Since the classes of nonexpansive mappings and nonspreading mappings are contained in the class of normally 2-generalized hybrid mappings, we obtain the desired result from Theorem 3.1. □

Using Theorem 3.1, we have the following strong convergence theorem for 2-generalized hybrid mappings in a Hilbert space.

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be 2-generalized hybrid mappings such that $F(S) \cap F(T) \ne \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n \big(\gamma_n S + (1-\gamma_n) T\big) x_n + c_n \big(\delta_n S^2 + (1-\delta_n) T^2\big) x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n \cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $a, b, c, d, e, f \in \mathbb{R}$ and $\{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1]$ satisfy the following:

$$0 < a \leq \gamma_n \leq b < 1, \; 0 < c \leq \delta_n \leq d < 1,$$
$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \leq a_n, b_n, c_n \leq f < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} x_1$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Since the class of 2-generalized hybrid mappings is contained in the class of normally 2-generalized hybrid mappings, we obtain the desired result from Theorem 3.1. □

Similarly, using Theorem 3.2, we have the following results.

Theorem 4.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n T x_n + c_n T^2 x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1]$ and $e, f \in \mathbb{N}$ satisfy

$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \leq a_n, b_n, c_n \leq f < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(T)} x_1$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Theorem 4.5. *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let S and T be nonexpansive and hybrid mappings of C into itselt, respectively, such that $F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n \big(\gamma_n S x_n + (1 - \gamma_n) T x_n\big) + c_n \big(\delta_n S^2 x_n + (1 - \delta_n) T^2 x_n\big), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $a, b, c, d, e, f \in \mathbb{R}$ and $\{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1]$ satisfy the following:

$$0 < a \leq \gamma_n \leq b < 1, \; 0 < c \leq \delta_n \leq d < 1,$$
$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \leq a_n, b_n, c_n \leq f < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} u$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Proof. Since the classes of nonexpansive mappings and hybrid mappings are contained in the class of generalized hybrid mappings, we obtain the desired result from Theorem 3.2. □

Theorem 4.6. *Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be 2-generalized hybrid mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\gamma_n S + (1-\gamma_n)T\big)x_n + c_n\big(\delta_n S^2 + (1-\delta_n)T^2\big)x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $a, b, c, d, e, f \in \mathbb{R}$ and $\{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1]$ satisfy the following:

$$0 < a \leq \gamma_n \leq b < 1, \ 0 < c \leq \delta_n \leq d < 1,$$
$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \leq a_n, b_n, c_n \leq f < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)}u$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

References

[1] K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, *Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 335–343.

[2] J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Ser. A-B **280** (1975), 1511–1514.

[3] M. Hojo, A. Kondo and W. Takahashi, *Weak and strong convergence theorems for commutative normally 2-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **4** (2018), 145–156.

[4] M. Hojo and W. Takahashi, *Weak and strong convergence theorems for two commutative nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **18** (2017), 1519–1533.

[5] M. Hojo, W. Takahashi and J.-C.Yao, *Weak and strong convergence theorems for supper hybrid mappings in Hilbert spaces*, Fixed Point Theory, **12** (2011), 113–126.

[6] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2008, pp. 75–85.

[7] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings*, Pacific J. Math. **79** (1978), 493–508.

[8] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[9] F. Kohsaka, *Existence and approximation of common fixed points of two hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **16** (2015), 2193–2205.

[10] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.

[11] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166–177.

[12] A. Kondo and W. Takahashi, *Attractive point and weak convergence theorems for normally N-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **3** (2017), 297–310.

[13] T. Maruyama, W. Takahashi and M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 185–197.

[14] U. Mosco, *convergence of convex sets and of solutions of variational inequalities*, Adv. Math. **3** (1969), 510–585.

[15] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–378.

[16] W. Takahashi, *Nonlinear Functional Analysis. Fixed Point Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.

[17] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.
[18] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.
[19] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.
[20] W. Takahashi, J.-C. Yao and P. Kocourek, *Weak and strong convergence theorems for generalized hybrid nonself-mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 567–586.
[21] M. Tsukada, *Convergence of best approximation in a smooth Banach space*, J. Approx. Theory **40** (1984), 301–309.

Manuscript received June 15 2018
revised October 20 2018

Wataru Takahashi
Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40447, Taiwan; Keio Research and Education Center for Natural Sciences, Keio University, Kouhoku-ku, Yokohama 223-8521, Japan; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: `wataru@is.titech.ac.jp; wataru@a00.itscom.net`

Ching-Feng Wen
Center for Fundamental Science, and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung 80708, Taiwan
E-mail address: `cfwen@kmu.edu.tw`

Jen-Chih Yao
Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40447, Taiwan
E-mail address: `yaojc@mail.cmu.edu.tw`