Applied Analysis and Optimization
Volume 3, Number 1, 2019, 29–42

Yokohama Publishers
ISSN 2189-1664 Online Journal

COUPLED FIXED POINTS FOR HARDY-ROGERS TYPE OPERATORS IN ORDERED GENERALIZED KASAHARA SPACES

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Abstract. In this paper we present some coupled fixed point theorems for Hardy-Rogers type operators in an ordered generalized Kasahara space $(X, \to, d, \leq)$, where $d : X \times X \to \mathbb{R}^m_+$ is a functional. Some applications concerning the existence and uniqueness of solutions for systems of functional-integral equations are also given.

1. Introduction and preliminaries

Recently, we have presented in [6] some coupled fixed point theorems for Zamfirescu type operators in ordered generalized Kasahara spaces. In this paper, we will give some generalizations of these results by considering Hardy-Rogers type operators. The notions of Zamfirescu and Hardy-Rogers type operators are recalled bellow.

Definition 1.1 (Zamfirescu type operator, [22]). Let (X, d) be a metric space. Let $f : X \to X$ be an operator. Then f is a Zamfirescu type operator if at least one of the following conditions holds:
(i) there exists $a \in [0, 1)$ such that $d(f(x), f(y)) \leq ad(x, y)$, for all $x, y \in X$;
(ii) there exists $b \in [0, \frac{1}{2})$ such that $d(f(x), f(y)) \leq b[d(x, f(x)) + d(y, f(y))]$, for all $x, y \in X$;
(iii) there exists $c \in [0, \frac{1}{2})$ such that $d(f(x), f(y)) \leq c[d(x, f(y)) + d(y, f(x))]$, for all $x, y \in X$.

Definition 1.2 (Hardy-Rogers type operator, [8]). Let (X, d) be a metric space. Let $f : X \to X$ be an operator. Then f is a Hardy-Rogers type operator if there exists $a, b, c \in \mathbb{R}_+$ with $a + 2b + 2c \in (0, 1)$ such that $d(f(x), f(y)) \leq ad(x, y) + b[d(x, f(x)) + d(y, f(y))] + c[d(x, f(y)) + d(y, f(x))]$, for all $x, y \in X$.

Notice that the notion of Hardy-Rogers type operator is more general than the notion of Zamfirescu type operator. In order to establish our results concerning Hardy-Rogers type operators, let us recall first some notions and notations.

Definition 1.3 (L-space, [7]). Let X be a nonempty set. Let

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\}.$$

2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. coupled fixed point, ordered generalized Kasahara space, Zamfirescu type operator, Hardy-Rogers type operator, Picard operator, matrix convergent to zero, sequence of successive approximations.
This work was supported by the grant GTC-31790/23. 03.2016, offered by the Babeş-Bolyai University of Cluj-Napoca, Romania.

Let $c(X)$ be a subset of $s(X)$ and $Lim : c(X) \to X$ be an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (denoted by $(X, \to)$) if the following conditions are satisfied:

(*i*) if $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(*ii*) if $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

Example 1.4. Let (X, d) be a metric space. Let $\stackrel{d}{\to}$ be the convergence structure induced by d on X. Then $(X, \stackrel{d}{\to})$ is an L-space.

Example 1.5. Let (X, p) be a partial metric space. Let $\stackrel{p}{\to}$ be the convergence structure induced by p on X. Then $(X, \stackrel{p}{\to})$ is an L-space. (Concerning partial metric spaces, see [10])

In general, an L-space is any set endowed with a structure implying a notion of convergence for sequences. Other examples of L-spaces are: Hausdorff topological spaces, generalized metric spaces in Perov' sense (i.e. $d(x, y) \in \mathbb{R}^m_+$), generalized metric spaces in Luxemburg' sense (i.e. $d(x, y) \in \mathbb{R}_+ \cup \{+\infty\}$), K-metric spaces (i.e. $d(x, y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces.

In 1922, S. Banach in [2] and later, in 1930 R. Caccioppoli in [4] have given one of the most famous tool in the fixed point theory domain: the Contraction Principle for α-contractions defined on complete metric spaces (X, d), where $d : X \times X \to \mathbb{R}_+$ is a metric.

In 1976, S. Kasahara proved in [9] that the Banach-Caccioppoli's contraction principle works in a more general setting: d-complete L-space, where $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is a function, not necessarily a metric.

In 2010, starting from Kasahara's work, I.A. Rus introduced in [17] the notion of generalized Kasahara space.

Definition 1.6. Let $(X, \to)$ be an L-space, $(G, +, \leq, \stackrel{G}{\to})$ be an L-space ordered semigroup with unity, 0 be the least element in $(G, \leq)$ and $d_G : X \times X \to G$ be an operator. The triple $(X, \to, d_G)$ is called a generalized Kasahara space if and only if the following compatibility condition between $\to$ and d_G holds:

$$\text{for all } (x_n)_{n\in\mathbb{N}} \subset X \text{ with } \sum_{n\in\mathbb{N}} d_G(x_n, x_{n+1}) < +\infty$$
$$\Rightarrow (x_n)_{n\in\mathbb{N}} \text{ is convergent in } (X, \to). \tag{1.1}$$

Remark 1.7. By the inequality with the symbol $+\infty$ in the compatibility condition (1.1), we understand that the series $\sum_{n\in\mathbb{N}} d_G(x_n, x_{n+1})$ is bounded in $(G, \leq)$.

Remark 1.8. In the context of generalized Kasahara spaces, fixed point results for self generalized contractions were already given by S. Kasahara in [9], for the case when $G = \mathbb{R}_+ \cup \{+\infty\}$ and by I.A. Rus in [17], for the case when $G = \mathbb{R}^m_+$.

Some examples of generalized Kasahara space are the following ones:

Example 1.9. Let $\rho : X \times X \to \mathbb{R}^m_+$ be a generalized complete metric on a set X. Let $d : X \times X \to \mathbb{R}^m_+$ be a functional. Assume that there exists $c > 0$ such that $\rho(x, y) \le cd(x, y)$, for all $x, y \in X$. Then $(X, \overset{\rho}{\to}, d)$ is a generalized Kasahara space.

In the above example, by the inequality denoted by $\le$, we mean that each positive real component of ρ is less than or equal to the corresponding positive real component of d.

Example 1.10 (I.A. Rus, [17])**.** Let $\rho : X \times X \to \mathbb{R}^m_+$ be a generalized complete metric on a set X. Let $x_0 \in X$ and $\lambda \in \mathbb{R}^m_+$ with $\lambda \neq 0$. Let $d_\lambda : X \times X \to \mathbb{R}^m_+$ be defined by

$$d_\lambda(x, y) = \begin{cases} \rho(x, y) & , \text{ if } x \neq x_0 \text{ and } y \neq x_0, \\ \lambda & , \text{ if } x = x_0 \text{ or } y = x_0. \end{cases}$$

Then $(X, \overset{\rho}{\to}, d_\lambda)$ is a generalized Kasahara space.

We recall also a very useful tool which helps us to prove the uniqueness of the fixed point for operators defined on generalized Kasahara spaces.

Lemma 1.11 (Kasahara's lemma [9])**.** *Let* $(X, \to, d_G)$ *be a generalized Kasahara space. Then* $d_G(x, y) = d_G(y, x) = 0$ *implies* $x = y$*, for all* $x, y \in X$.

Proof. Let $x, y \in X$. Assume that $d_G(x, y) = d_G(y, x) = 0$. Define the subsequences $(x_{2n})_{n\in\mathbb{N}} \subset X$, by $x_{2n} := x$, for all $n \in \mathbb{N}$ and $(x_{2n+1})_{n\in\mathbb{N}} \subset X$, by $x_{2n+1} := y$, for all $n \in \mathbb{N}$.

We get that $\sum_{n\in\mathbb{N}} d_G(x_n, x_{n+1}) = 0$. Since $(X, \to, d_G)$ is a generalized Kasahara space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. So, there exists an element $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since $x_{2n} \to x$ as $n \to \infty$, it follows that $x = x^*$. Similarly, we get $y = x^*$. Thus, $x = y$. □

More considerations on Kasahara spaces can be found in [5] and [17].

Definition 1.12 (Ordered generalized Kasahara space, [6])**.** Let $(X, \to, d_G)$ be a generalized Kasahara space and let $\le$ be a partial order relation on X. Then $(X, \to, d_G, \le)$ is an ordered generalized Kasahara space.

Example 1.13. Let $X := C([a, b], \mathbb{R}^m) = \{x : [a, b] \to \mathbb{R}^m \mid x \text{ is continuous on } [a, b]\}$ be endowed with the partial order relation

$$x \le_C y \Leftrightarrow x(t) \le y(t) \Leftrightarrow x_i(t) \le y_i(t), \text{ for all } t \in [a, b], \ i = \overline{1, m}.$$

We consider $\overset{\rho}{\to}$, the convergence structure induced by the Cebîşev norm $\rho : C([a, b], \mathbb{R}^m) \times C([a, b], \mathbb{R}^m) \to \mathbb{R}^m_+$, defined by

$$\rho(x, y) = \|x - y\|_C = \max_{t\in[a,b]} |x(t) - y(t)| = \begin{pmatrix} \max_{t\in[a,b]}|x_1(t) - y_1(t)| \\ \vdots \\ \max_{t\in[a,b]}|x_m(t) - y_m(t)| \end{pmatrix}.$$

Let $d : C([a,b],\mathbb{R}^m) \times C([a,b],\mathbb{R}^m) \to \mathbb{R}^m_+$, defined by

$$\begin{aligned} d(x,y) &= \|x-y\|_C + \|(x-y)^p\|_C \\ &= \max_{t\in[a,b]} |x(t)-y(t)| + \max_{t\in[a,b]} \{|x(t)-y(t)|^p\} \\ &= \begin{pmatrix} \max_{t\in[a,b]} |x_1(t)-y_1(t)| + \max_{t\in[a,b]} \{|x_1(t)-y_1(t)|^p\} \\ \vdots \\ \max_{t\in[a,b]} |x_m(t)-y_m(t)| + \max_{t\in[a,b]} \{|x_m(t)-y_m(t)|^p\} \end{pmatrix}, \text{ where } p \in \mathbb{R}_+. \end{aligned}$$

Since $\rho(x,y) \le d(x,y)$, for all $x, y \in C([a,b],\mathbb{R}^m)$, it follows that the structure $(C([a,b],\mathbb{R}^m), \overset{\rho}{\to}, d, \le_C)$ is an ordered generalized Kasahara space.

Let $(X, \to, d_G, \le)$ be an ordered generalized Kasahara space.
We define

$$X_{\le} := \{(x_1,x_2) \in X \times X \mid x_1 \le x_2 \text{ or } x_2 \le x_1\}.$$

In the above setting, if $f : X \to X$ is an operator, then the Cartesian product of f with itself is $f \times f : X \times X \to X \times X$, given by

$$(f \times f)(x_1,x_2) := (f(x_1), f(x_2)).$$

All the notions presented above, were related to the generalized Kasahara space $(X, \to, d_G)$ or the ordered generalized Kasahara space $(X, \to, d_G, \le)$, where $d_G : X \times X \to G$ is an operator. In the sequel, we will consider the particular case, when $G = \mathbb{R}^m_+$. So, we will consider the ordered generalized Kasahara space $(X, \to, d, \le)$, where $d : X \times X \to \mathbb{R}^m_+$ is a functional.

We mention that if $\alpha, \beta \in \mathbb{R}^m$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_m)$ and $c \in \mathbb{R}$, then by $\alpha \le \beta$ (respectively $\alpha < \beta$), we mean that $\alpha_i \le \beta_i$ (respectively $\alpha_i < \beta_i$), for all $i = \overline{1,m}$ and by $\alpha \le c$ we mean that $\alpha_i \le c$, for all $i = \overline{1,m}$.

We denote by $\mathcal{M}_m(\mathbb{R}_+)$ the set of all square matrices of order m, having positive real elements, by O_m the zero matrix of order m and by I_m the identity matrix of order m. If $A = (a_{ij})_{i,j=\overline{1,m}}$, $B = (b_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}_m(\mathbb{R}_+)$, then by $A \le B$ we understand $a_{ij} \le b_{ij}$, for all $i, j = \overline{1,m}$. The symbol A^τ stands for the transpose of the matrix A. Notice also that, for the sake of simplicity, we will make an identification between row and column vectors in $\mathbb{R}^m$.

A matrix $A \in \mathcal{M}_m(\mathbb{R}_+)$ is said to be convergent to zero if and only if $A^n \to O_m$ as $n \to \infty$ (see [18]). Regarding this class of matrices we have the following classical result in matrix analysis (see [1](Lemma 3.3.1, page 55), [19], [16](page 37), [21](page 12).

Theorem 1.14. *Let $A \in \mathcal{M}_m(\mathbb{R}_+)$. The following statements are equivalent:*

(i) *A is convergent to zero;*

(ii) *$A^n \to O_m$ as $n \to \infty$;*

(iii) *the eigenvalues of A are in the open unit disc, i.e., $|\lambda| < 1$, for all $\lambda \in \mathbb{C}$ with $\det(A - \lambda I_m) = 0$;*

(iv) *the matrix $I_m - A$ is non-singular and*

$$(I_m - A)^{-1} = I_m + A + A^2 + \ldots + A^n + \ldots;$$

(v) *the matrix* $(I_m - A)$ *is non-singular and* $(I_m - A)^{-1}$ *has nonnegative elements;*

(vi) $A^n q \to 0 \in \mathbb{R}^m$ *and* $q^\tau A^n \to 0 \in \mathbb{R}^m$ *as* $n \to \infty$*, for all* $q \in \mathbb{R}^m$.

Remark 1.15. Some examples of matrices which converge to zero are:

a) any matrix $A := \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

b) any matrix $A := \begin{pmatrix} a & b \\ a & b \end{pmatrix}$, where $a, b \in \mathbb{R}_+$ and $a + b < 1$;

c) any matrix $A := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in \mathbb{R}_+$ and $\max\{a, c\} < 1$.

Remark 1.16. For more considerations on matrices which converge to zero, see [11], [16] and [20].

Let $(X, \to)$ be an L-space and $f : X \to X$ be an operator. The following notations and notions will be needed in the sequel of this paper:

- $Fix(f) := \{x \in X \mid x = f(x)\}$ the set of all fixed points for f.
- $I(f) := \{Y \subset X \mid f(Y) \subset Y\}$ - the set of all invariant subsets of X with respect to f.
- $Graph(f) := \{(x, y) \in X \times X \mid y = f(x)\}$ the graph of f. We say that f has closed graph with respect to $\to$ or $Graph(f)$ is closed in $X \times X$ with respect to $\to$ if and only if for any sequences $(x_n)_{n\in\mathbb{N}} \subset X$, $(y_n)_{n\in\mathbb{N}} \subset X$ with $y_n = f(x_n)$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$, $y_n \to y \in X$, as $n \to \infty$, we have that $y = f(x)$.
- A sequence $(x_n)_{n\in\mathbb{N}} \subset X$ is called sequence of successive approximations for f starting from a given point $x_0 \in X$ if $x_{n+1} = f(x_n)$, for all $n \in \mathbb{N}$. Notice that $x_n = f^n(x_0)$, for all $n \in \mathbb{N}$.
- If $Fix(f) := \{x^*\}$ and the sequence of successive approximations for f starting from any given point $x_0 \in X$ converges to x^*, then f is a Picard operator in $(X, \to)$.

2. Main results

We present here the principal result of this paper, generalizing the Theorem 2.1 stated in [6] for Zamfirescu type operators.

Theorem 2.1. *Let* $(X, \to, d, \leq)$ *be an ordered generalized Kasahara space, where* $d : X \times X \to \mathbb{R}^m_+$ *is a functional, satisfying* $d(x, x) = 0$ *and* $d(x, z) \leq d(x, y) + d(y, z)$*, for all* $x, y, z \in X$*. Let* $f : X \to X$ *be an operator. We assume that:*

(i) *for each* $(x, y) \in X_\leq$*, there exists* $z_{(x,y)} := z \in X$ *such that* (x, z)*,* $(y, z) \in X_\leq$*;*

(ii) $X_\leq \in I(f \times f)$*;*

(iii) $f : (X, \to) \to (X, \to)$ *has closed graph;*

(iv) *there exists* $x_0 \in X$ *such that* $(x_0, f(x_0)) \in X_\leq$*;*

(v) *there exist* $A, B, C \in \mathcal{M}_m(\mathbb{R}_+)$ *such that* $(I_m - B - C)^{-1}(A + B + C)$ *is convergent to zero and*

$$d(f(x), f(y)) \leq Ad(x,y) + B[d(x, f(x)) + d(y, f(y))] + \\ + C[d(x, f(y)) + d(y, f(x))], \text{ for each } (x, y) \in X_{\leq}.$$

Then $f : (X, \to) \to (X, \to)$ *is a Picard operator.*

Proof. Let $x \in X$ be arbitrary.
Since $(x_0, f(x_0)) \in X_{\leq}$, by (ii) we have $(f(x_0), f^2(x_0)) \in X_{\leq}$ and by (v),

$$\begin{aligned} d(f(x_0), f^2(x_0)) &\leq Ad(x_0, f(x_0)) + B[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))] + \\ &\quad + C[d(x_0, f^2(x_0)) + d(f(x_0), f(x_0))] \\ &\leq Ad(x_0, f(x_0)) + B[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))] + \\ &\quad + C[d(x_0, f(x_0)) + d(f(x_0), f^2(x_0))]. \end{aligned}$$

So, $d(f(x_0), f^2(x_0)) \leq (I_m - B - C)^{-1}(A + B + C)d(x_0, f(x_0))$.
Let $\Lambda := (I_m - B - C)^{-1}(A + B + C)$. Then

$$d(f(x_0), f^2(x_0)) \leq \Lambda d(x_0, f(x_0)).$$

Since $(f(x_0), f^2(x_0)) \in X_{\leq}$, by (ii) we have $(f^2(x_0), f^3(x_0)) \in X_{\leq}$ and by (v), we get that

$$\begin{aligned} d(f^2(x_0), f^3(x_0)) &\leq Ad(f(x_0), f^2(x_0)) + \\ &\quad + B[d(f(x_0), f^2(x_0)) + d(f^2(x_0), f^3(x_0))] + \\ &\quad + C[d(f(x_0), f^3(x_0)) + d(f^2(x_0), f^2(x_0))] \\ &\leq Ad(f(x_0), f^2(x_0)) + \\ &\quad + B[d(f(x_0), f^2(x_0)) + d(f^2(x_0), f^3(x_0))] + \\ &\quad + C[d(f(x_0), f^2(x_0)) + d(f^2(x_0), f^3(x_0))]. \end{aligned}$$

So,

$$d(f^2(x_0), f^3(x_0)) \leq \Lambda d(f(x_0), f^2(x_0)) \leq \Lambda^2 d(x_0, f(x_0)).$$

By induction, for $n \in \mathbb{N}$, we get:

$$d(f^n(x_0), f^{n+1}(x_0)) \leq \Lambda^n d(x_0, f(x_0))$$

and since Λ is a matrix that converges to zero, we have

$$\begin{aligned} \sum_{n \in \mathbb{N}} d(f^n(x_0), f^{n+1}(x_0)) &\leq \sum_{n \in \mathbb{N}} \Lambda^n d(x_0, f(x_0)) \\ &= (I_m - \Lambda)^{-1} d(x_0, f(x_0)) < +\infty. \end{aligned}$$

Since $(X, \to, d)$ is a generalized Kasahara space, we get that the sequence of successive approximations for f, starting from x_0, is convergent in $(X, \to)$. So, there exists $x^* \in X$ such that $f^n(x_0) \to x^*$ as $n \to \infty$. By (iii) we get that $x^* \in Fix(f)$.

Notice also that:

• If $(x, x_0) \in X_{\leq}$ then by (ii) we have $(f^n(x), f^n(x_0)) \in X_{\leq}$ and by (v), $0 \leq d(f^n(x), f^n(x_0)) \leq \Lambda^n d(x, x_0) \overset{\mathbb{R}^m_+}{\to} 0$ as $n \to \infty$. Similarly, we get $d(f^n(x_0), f^n(x)) = 0$. By Kasahara's lemma 1.11, it follows that $f^n(x) = f^n(x_0)$, for all $n \in \mathbb{N}$.

• If $(x, x_0) \notin X_{\leq}$, then by (i), there exists $z_{(x,x_0)} := z \in X$ such that (x, z), $(x_0, z) \in X_{\leq}$. Since $(x, z) \in X_{\leq}$, by (ii) we have $(f^n(x), f^n(z)) \in X_{\leq}$ and by (v), $0 \leq d(f^n(x), f^n(z)) \leq \Lambda^n d(x, z) \overset{\mathbb{R}^m_+}{\to} 0$. Similarly, we get $d(f^n(z), f^n(x)) = 0$. By Kasahara's lemma 1.11, it follows that $f^n(x) = f^n(z)$, for all $n \in \mathbb{N}$. Since $(x_0, z) \in X_{\leq}$ we get that $f^n(x_0) = f^n(z)$, for all $n \in \mathbb{N}$. Hence $f^n(x) = f^n(x_0) \to x^*$ as $n \to \infty$.

We show next the uniqueness of the fixed point x^*.

Let $y^* \in Fix(f)$ such that $y^* \neq x^*$.

If $(x^*, y^*) \in X_{\leq}$, then by (ii) we have $(f^n(x^*), f^n(y^*)) \in X_{\leq}$ and by (v), $0 \leq d(x^*, y^*) = d(f^n(x^*), f^n(y^*)) \leq \Lambda^n d(x^*, y^*) \overset{\mathbb{R}^m_+}{\to} 0$. So, $d(x^*, y^*) = 0$. By a similar way, we obtain $d(y^*, x^*) = 0$. By Kasahara's lemma 1.11, it follows that $x^* = y^*$.

If $(x^*, y^*) \notin X_{\leq}$, then by (i), there exists $z_{(x^*,y^*)} := z \in X$ such that (x^*, z), $(y^*, z) \in X_{\leq}$. Since $(x^*, z) \in X_{\leq}$, by (ii) we get $(f^n(x^*), f^n(z)) \in X_{\leq}$, for all $n \in \mathbb{N}$, and by (v), $0 \leq d(x^*, z) = d(f^n(x^*), f^n(z)) \leq \Lambda^n d(x^*, z) \overset{\mathbb{R}^m_+}{\to} 0$. So, $d(x^*, z) = 0$. Similarly, we obtain $d(z, x^*) = 0$. By Kasahara's lemma 1.11, we get $x^* = z$.

Since $(y^*, z) \in X_{\leq}$, by (ii) we have $(f^n(y^*), f^n(z)) \in X_{\leq}$, for all $n \in \mathbb{N}$, and by (v), $0 \leq d(y^*, z) = d(f^n(y^*), f^n(z)) \leq \Lambda^n d(y^*, z) \overset{\mathbb{R}^m_+}{\to} 0$. So, $d(y^*, z) = 0$. Similarly, we obtain $d(z, y^*) = 0$. By Kasahara's lemma 1.11, we have $y^* = z$.

Hence $x^* = y^*$. □

Remark 2.2. There can be found matrices $A, B, C \in \mathcal{M}_m(\mathbb{R}_+)$ such that $(I_m - B - C)^{-1}(A + B + C)$ is convergent to zero.
Let $\xi > 0$. We consider the following particular upper triangular matrix set:

$$\mathcal{M}^{\xi}_m(\mathbb{R}_+) := \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\ 0 & a_{22} & a_{23} & \dots & a_{2m} \\ 0 & 0 & a_{33} & \dots & a_{3m} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & a_{mm} \end{pmatrix} \in \mathcal{M}_m(\mathbb{R}_+) \;\middle|\; \max_{i=\overline{1,m}} a_{ii} < \xi \right\}.$$

For $\xi \in (0, 1]$, it is clear that any matrix $A \in \mathcal{M}^{\xi}_m(\mathbb{R}_+)$ is convergent to zero.

Now, let $a, b, c \in \mathbb{R}_+$ such that $a + 2b + 2c < 1$. Let $A \in \mathcal{M}^a_m(\mathbb{R}_+)$, $B \in \mathcal{M}^b_m(\mathbb{R}_+)$ and $C \in \mathcal{M}^c_m(\mathbb{R}_+)$. Then the matrix $(I_m - B - C)^{-1}(A + B + C)$ is convergent to zero.

Indeed, let $A = (a_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}^a_m(\mathbb{R}_+)$, $B = (b_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}^b_m(\mathbb{R}_+)$ and $C = (c_{ij})_{i,j=\overline{1,m}} \in \mathcal{M}^c_m(\mathbb{R}_+)$. Since $a + 2b + 2c < 1$, we have that $a_{ii} + 2b_{ii} + 2c_{ii} < 1$ and also that $0 < a_{ii} + b_{ii} + c_{ii} < 1 - b_{ii} - c_{ii}$, for all $i = \overline{1, m}$. Since $\det(I_m - B - C) = \prod_{i=1}^m (1 - b_{ii} - c_{ii}) \neq 0$, it follows that the matrix $(I_m - B - C)$ is non-singular. In addition, it can be shown that $(I_m - B - C)^{-1}$ is an upper triangular matrix and its diagonal elements are $\frac{1}{1-b_{ii}-c_{ii}}$, for all $i = \overline{1, m}$. On the other hand, $A + B + C$ is an upper triangular matrix. The product $(I_m - B - C)^{-1}(A + B + C)$ is also an upper triangular matrix, having its diagonal elements $\frac{a_{ii}+b_{ii}+c_{ii}}{1-b_{ii}-c_{ii}} < 1$, for all $i = \overline{1,m}$. Hence, the eigenvalues of $(I_m - B - C)^{-1}(A + B + C)$ are in the open unit disk. The conclusion follows from Theorem 1.14.

In the sequel, we will apply Theorem 2.1 to the coupled fixed point problem generated by an operator.

Let X be a nonempty set, endowed with a partial order relation denoted by $\leq$. If we consider two arbitrary elements $z := (x, y)$, $w = (u, v)$ of $X \times X$, then, we can introduce a partial ordering relation on $X \times X$, denoted by $\preceq$ and defined as follows:

$$z \preceq w \text{ if and only if } (x \geq u \text{ and } y \leq v).$$

Theorem 2.3. *Let $(X, \to, d, \leq)$ be an ordered Kasahara space, where $d : X \times X \to \mathbb{R}_+$ is a functional. Let $S : X \times X \to X$ be an operator. We suppose that:*

(i) for each $z = (x, y)$, $w = (u, v) \in X \times X$, which are not comparable with respect to the partial ordering $\preceq$ in $X \times X$, there exists $t := (t_1, t_2) \in X \times X$, which may depend on (x, y) and (u, v), such that t is comparable with respect to the partial ordering $\preceq$, with both z and w;

(ii) for all $(x \geq u$ and $y \leq v)$ or $(u \geq x$ and $v \leq y)$, we have

$$\begin{cases} S(x,y) \geq S(u,v) \\ S(y,x) \leq S(v,u) \end{cases} \quad or \quad \begin{cases} S(u,v) \geq S(x,y) \\ S(v,u) \leq S(y,x) \end{cases}$$

i.e., S has the generalized mixed monotone property;

(iii) $S : X \times X \to X$ has closed graph with respect to $\to$;

(iv) there exists $z_0 := (z_0^1, z_0^2) \in X \times X$ such that

$$\begin{cases} z_0^1 \geq S(z_0^1, z_0^2) \\ z_0^2 \leq S(z_0^2, z_0^1) \end{cases} \quad or \quad \begin{cases} S(z_0^1, z_0^2) \geq z_0^1 \\ S(z_0^2, z_0^1) \leq z_0^2 \end{cases} \quad ;$$

(v) there exist $k_i \in \mathbb{R}_+$, $i = \overline{1,6}$, with

$$k_1 + k_2 + 2\max\{k_3, k_4\} + 2\max\{k_5, k_6\} < 1,$$

such that

$$\begin{aligned} d(S(x,y), S(u,v)) \leq\ & k_1 d(x,u) + k_2 d(y,v) + \\ & + k_3 d(x, S(x,y)) + k_4 d(u, S(u,v)) + \\ & + k_5 d(x, S(u,v)) + k_6 d(u, S(x,y)), \end{aligned} \tag{2.1}$$

for all $(x \geq u$ and $y \leq v)$ or $(u \geq x$ and $v \leq y)$.

Then there exists a unique element $(x^*, y^*) \in X \times X$ such that $x^* = S(x^*, y^*)$ and $y^* = S(y^*, x^*)$. In addition, the sequence of successive approximations $(S^n(w_0^1, w_0^2), S^n(w_0^2, w_0^1))$ converges to (x^*, y^*) as $n \to \infty$, for all $w_0 = (w_0^1, w_0^2) \in X \times X$.

Proof. Let $Z := X \times X$ and consider $\preceq$, the partial order relation on Z, defined as follows: for all $z := (x, y)$, $w := (u, v) \in Z$, $z \preceq w$ if and only if $(x \geq u$ and $y \leq v)$.

Let $Z_{\preceq} := \{(z, w) := ((x, y), (u, v)) \in Z \times Z \mid z \preceq w \text{ or } w \preceq z\}$.

Let $F : Z \to Z$ be an operator defined by

$$F(x, y) := \begin{pmatrix} S(x,y) \\ S(y,x) \end{pmatrix} = (S(x, y), S(y, x)).$$

We show that all of the assumptions of Theorem 2.1 are satisfied.

By (iii), F has closed graph with respect to $\to$, so the assumption (iii) of Theorem 2.1 holds.

By (ii), we have $Z_{\preceq} \in I(F \times F)$.

Indeed, let $z = (x, y)$, $w = (u, v) \in Z_{\preceq}$ be two arbitrary elements, where ($x \geq u$ and $y \leq v$) or ($u \geq x$ and $v \leq y$) such that

$$(1) \begin{cases} S(x,y) \geq S(u,v) \\ S(y,x) \leq S(v,u) \end{cases} \quad \text{or} \quad (2) \begin{cases} S(u,v) \geq S(x,y) \\ S(v,u) \leq S(y,x) \end{cases}$$

From (1) and (2) we have that $(S(x,y), S(y,x)) \preceq (S(u,v), S(v,u))$, i.e., $F(x,y) \preceq F(u,v)$ or $F(z) \preceq F(w)$. Similarly, we get $F(w) \preceq F(z)$. Hence, $(F(z), F(w)) \in Z_{\preceq}$, for all $(z,w) \in Z_{\preceq}$. So, $(F \times F)(Z_{\preceq}) \subset Z_{\preceq}$, i.e., $Z_{\preceq} \in I(F \times F)$. Thus, the assumption (ii) holds.

By (iv), since $(z_0^1, z_0^2) \in X \times X$ such that

$$\begin{cases} z_0^1 \geq S(z_0^1, z_0^2) \\ z_0^2 \leq S(z_0^2, z_0^1) \end{cases} \quad \text{or} \quad \begin{cases} S(z_0^1, z_0^2) \geq z_0^1 \\ S(z_0^2, z_0^1) \leq z_0^2 \end{cases}$$

we get that $(z_0^1, z_0^2) \preceq (S(z_0^1, z_0^2), S(z_0^2, z_0^1))$ and thus, $z_0 \preceq F(z_0)$. By a similar approach we get $F(z_0) \preceq z_0$. Hence, there exists $z_0 \in Z$ such that $(z_0, F(z_0)) \in Z_{\preceq}$, so, the assumption (iv) of Theorem 2.1 holds.

Finally, we prove the assumption (v) of Theorem 2.1.

Let $\tilde{d} : Z \times Z \to \mathbb{R}^2_+$, defined by $\tilde{d}((x,y),(u,v)) := \begin{pmatrix} d(x,u) \\ d(y,v) \end{pmatrix}$.

Since $(X, \to, d, \leq)$ is an ordered Kasahara space, it follows that $(X, \to, \tilde{d}, \leq)$ is an ordered generalized Kasahara space.

For the sake of simplicity, we will use the contraction condition (2.1) with the following notations:

$$d_{Sxy,Suv} \leq k_1 d_{x,u} + k_2 d_{y,v} + k_3 d_{x,Sxy} + k_4 d_{u,Suv} + k_5 d_{x,Suv} + k_6 d_{u,Sxy}.$$

Hence, we have:

$$\begin{aligned}
&\tilde{d}(F(x,y), F(u,v)) = \tilde{d}((S(x,y), S(y,x)), (S(u,v), S(v,u))) = \\
&= \begin{pmatrix} d(S(x,y), S(u,v)) \\ d(S(y,x), S(v,u)) \end{pmatrix} = \begin{pmatrix} d_{Sxy,Suv} \\ d_{Syx,Svu} \end{pmatrix} \leq \\
&\leq \begin{pmatrix} k_1 d_{x,u} + k_2 d_{y,v} + k_3 d_{x,Sxy} + k_4 d_{u,Suv} + k_5 d_{x,Suv} + k_6 d_{u,Sxy} \\ k_1 d_{y,v} + k_2 d_{x,u} + k_3 d_{y,Syx} + k_4 d_{v,Svu} + k_5 d_{y,Svu} + k_6 d_{v,Syx} \end{pmatrix} = \\
&= \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix} \begin{pmatrix} d_{x,u} \\ d_{y,v} \end{pmatrix} + \begin{pmatrix} k_3 & 0 \\ 0 & k_3 \end{pmatrix} \begin{pmatrix} d_{x,Sxy} \\ d_{y,Syx} \end{pmatrix} + \begin{pmatrix} k_4 & 0 \\ 0 & k_4 \end{pmatrix} \begin{pmatrix} d_{u,Suv} \\ d_{v,Svu} \end{pmatrix} + \\
&\quad + \begin{pmatrix} k_5 & 0 \\ 0 & k_5 \end{pmatrix} \begin{pmatrix} d_{x,Suv} \\ d_{y,Svu} \end{pmatrix} + \begin{pmatrix} k_6 & 0 \\ 0 & k_6 \end{pmatrix} \begin{pmatrix} d_{u,Sxy} \\ d_{v,Syx} \end{pmatrix}.
\end{aligned}$$

Let $k' := \max\{k_3, k_4\}$ and $k'' := \max\{k_5, k_6\}$. Then

$$\begin{pmatrix} d_{Sxy,Suv} \\ d_{Syx,Svu} \end{pmatrix} \leq \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix} \begin{pmatrix} d_{x,u} \\ d_{y,v} \end{pmatrix} + \begin{pmatrix} k' & 0 \\ 0 & k' \end{pmatrix} \left[\begin{pmatrix} d_{x,Sxy} \\ d_{y,Syx} \end{pmatrix} + \begin{pmatrix} d_{u,Suv} \\ d_{v,Svu} \end{pmatrix} \right] + \\ + \begin{pmatrix} k'' & 0 \\ 0 & k'' \end{pmatrix} \left[\begin{pmatrix} d_{x,Suv} \\ d_{y,Svu} \end{pmatrix} + \begin{pmatrix} d_{u,Sxy} \\ d_{v,Syx} \end{pmatrix} \right]$$

which means that

$$\begin{aligned} \tilde{d}(F(x,y),F(u,v)) \leq & A\tilde{d}((x,u),(y,v)) + \\ & + B\big[\tilde{d}((x,y),F(x,y)) + \tilde{d}((u,v),F(u,v))\big] + \\ & + C\big[\tilde{d}((x,y),F(u,v)) + \tilde{d}((u,v),F(x,y))\big]. \end{aligned}$$

Since $k_1, k_2, k', k'' \in \mathbb{R}_+$ and $k_1 + k_2 + 2k' + 2k'' < 1$, we get that the matrices

$$A := \begin{pmatrix} k_1 & k_2 \\ k_2 & k_1 \end{pmatrix}, \ B := \begin{pmatrix} k' & 0 \\ 0 & k' \end{pmatrix} \text{ and } C := \begin{pmatrix} k'' & 0 \\ 0 & k'' \end{pmatrix}$$

satisfy the condition that the matrix $\Lambda := (I_2 - B - C)^{-1}(A + B + C)$ is convergent to zero. Indeed, the matrix Λ has the eigenvalues $\lambda_1 = \frac{k_1+k'+k''-k_2}{1-k'-k''}$ and $\lambda_2 = \frac{k_1+k'+k''+k_2}{1-k'-k''}$, both being in the open unit disk. By applying Theorem 2.1, the conclusion follows. □

Remark 2.4. For several coupled fixed pont results in the case of metric or b-metric spaces see [12]-[15] and the references therein.

3. Applications

Let us consider the following system of functional-integral equations

$$(\mathbb{S}) \quad \begin{cases} x(t) = f(t, x(t), \int_a^b \Phi(t, s, x(s), y(s))ds) \\ y(t) = f(t, y(t), \int_a^b \Phi(t, s, y(s), x(s))ds) \end{cases}, \text{ for all } t \in [a,b] \subset \mathbb{R}_+.$$

By a solution of the system $(\mathbb{S})$ we understand a couple $(x, y) \in C[a, b] \times C[a, b]$, which satisfies the system for all $t \in [a, b] \subset \mathbb{R}_+$.

Let $X = C[a, b]$ be endowed with the partial order relation

$$x \leq_C y \Leftrightarrow x(t) \leq y(t), \text{ for all } t \in [a,b].$$

We consider $\xrightarrow{\rho}$, the convergence structure induced by the Cebîşev norm $\rho : C[a, b] \times C[a, b] \to \mathbb{R}_+$, $\rho(x, y) = \|x - y\|_C = \max_{t\in[a,b]}|x(t) - y(t)|$.

Let $d : C[a, b] \times C[a, b] \to \mathbb{R}_+$, defined by

$$d(x,y) = \|(x-y)\|_C + \|(x-y)^2\|_C = \max_{t\in[a,b]}|x(t) - y(t)| + \max_{t\in[a,b]}\{(x(t) - y(t))^2\}.$$

Since $\rho(x, y) \leq d(x, y)$, for all $x, y \in C[a, b]$, we get that $(C[a, b], \xrightarrow{\rho}, d, \leq_C)$ is an ordered Kasahara space.

Theorem 3.1. *Let $\Phi : [a, b] \times [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ and $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ be two continuous mappings and consider the system $(\mathbb{S})$. We suppose that:*

(i) there exists $z_0 := (z_0^1, z_0^2) \in C[a,b] \times C[a,b]$ *such that*

$$\begin{cases} z_0^1(t) \geq f(t, z_0^1(t), \int_a^b \Phi(t,s,z_0^1(t), z_0^2(t))ds) \\ z_0^2(t) \leq f(t, z_0^2(t), \int_a^b \Phi(t,s,z_0^2(t), z_0^1(t))ds) \end{cases}$$

or $\begin{cases} z_0^1(t) \leq f(t, z_0^1(t), \int_a^b \Phi(t,s,z_0^1(t), z_0^2(t))ds) \\ z_0^2(t) \geq f(t, z_0^2(t), \int_a^b \Phi(t,s,z_0^2(t), z_0^1(t))ds) \end{cases}$;

(ii) $f(t,\cdot,z)$ *is increasing for all* $t \in [a,b]$, $z \in \mathbb{R}$ *and* $\Phi(t,s,\cdot,z)$ *is increasing,* $\Phi(t,s,w,\cdot)$ *is decreasing and* $f(t,w,\cdot)$ *is increasing for all* $t,s \in [a,b]$, $w,z \in \mathbb{R}$,

or, $f(t,\cdot,z)$ *is decreasing for all* $t \in [a,b]$, $z \in \mathbb{R}$ *and* $\Phi(t,s,\cdot,z)$ *is decreasing,* $\Phi(t,s,w,\cdot)$ *is increasing and* $f(t,w,\cdot)$ *is decreasing for all* $t,s \in [a,b]$, $w,z \in \mathbb{R}$

(iii) there exists $k_1, k_2 \in \mathbb{R}_+$ *such that*

$$|f(t,w_1,z_1) - f(t,w_2,z_2)| \leq k_1|w_1 - w_2| + k_2|z_1 - z_2|$$

for all $t \in [a,b]$ *and* $w_1, w_2, z_1, z_2 \in \mathbb{R}$;

(iv) there exists $\alpha, \beta \in \mathbb{R}_+$ *such that for all* $t,s \in [a,b]$ *and* $w_1, w_2, z_1, z_2 \in \mathbb{R}$ *we have*

$$|\Phi(t,s,w_1,z_1) - \Phi(t,s,w_2,z_2)| \leq \alpha|w_1 - w_2| + \beta|z_1 - z_2|;$$

(v) the following conditions hold:

$$\begin{cases} k_1 + k_2\alpha(b-a) + 3k_1^2 + 3k_2^2\alpha^2(b-a)^2 < \frac{1}{2} \\ k_2\beta(b-a) + 3k_2^2\beta^2(b-a)^2 < \frac{1}{2}. \end{cases}$$

Then there exists a unique solution (x^*, y^*) *for the system* $(\mathbb{S})$.

Proof. Let us consider the operator $S : C[a,b] \times C[a,b] \to C[a,b]$, defined by

$$S(x,y)(t) := f(t, x(t), \int_a^b \Phi(t,s,x(s),y(s))ds).$$

Then the system $(\mathbb{S})$ is equivalent with $\begin{cases} x = S(x,y) \\ y = S(y,x) \end{cases}$.

Since $S(x,y)$ is a continuous operator on $(C[a,b] \times C[a,b], \xrightarrow{\rho})$, it follows that $Graph(S)$ is closed with respect to $\xrightarrow{\rho}$.

For all $(x \geq u$ and $y \leq v)$ or $(u \geq x$ and $v \leq y)$ we have

$|S(x,y)(t) - S(u,v)(t)| =$

$= |f(t,x(t), \int_a^b \Phi(t,s,x(s),y(s))ds) - f(t,u(t), \int_a^b \Phi(t,s,u(s),v(s))ds)|$

$\overset{(iii)}{\leq} k_1|x(t) - u(t)| + k_2|\int_a^b \Phi(t,s,x(s),y(s))ds - \int_a^b \Phi(t,s,u(s),v(s))ds|$

$\leq k_1|x(t) - u(t)| + k_2 \int_a^b |\Phi(t,s,x(s),y(s)) - \Phi(t,s,u(s),v(s))|ds$

$\overset{(iv)}{\leq} k_1|x(t) - u(t)| + k_2 \int_a^b (\alpha|x(s) - u(s)| + \beta|y(s) - v(s)|)ds$

$\leq k_1(|x(t) - u(t)| + |x(t) - u(t)|^2)$

$+k_2 \int_a^b \alpha(|x(s) - u(s)| + |x(s) - u(s)|^2)ds$

$+k_2 \int_a^b \beta(|y(s) - v(s)| + |y(s) - v(s)|^2)ds$

$\le k_1 d(x,u) + k_2 \int_a^b \alpha d(x,u)ds + k_2 \int_a^b \beta d(y,v)ds$
$= [k_1 + k_2\alpha(b-a)]d(x,u) + [k_2\beta(b-a)]d(y,v).$

On the other hand, we have
$|S(x,y)(t) - S(u,v)(t)|^2 \le$
$\le [k_1|x(t)-u(t)| + k_2 \int_a^b (\alpha|x(s)-u(s)| + \beta|y(s)-v(s)|)ds]^2$
$\le [k_1|x(t)-u(t)| + k_2 \int_a^b (\alpha\sqrt{|x(s)-u(s)|^2} + \beta\sqrt{|y(s)-v(s)|^2})ds]^2$
$\le [k_1|x(t)-u(t)|+$
$+k_2 \int_a^b \alpha\sqrt{|x(s)-u(s)| + |x(s)-u(s)|^2}ds$
$+k_2 \int_a^b \beta\sqrt{|y(s)-v(s)| + |y(s)-v(s)|^2})ds]^2$
$\le [k_1|x(t)-u(t)| + k_2 \int_a^b \alpha\sqrt{d(x,u)}ds + k_2 \int_a^b \beta\sqrt{d(y,v)}ds]^2$
$= [k_1|x(t)-u(t)| + k_2\alpha(b-a)\sqrt{d(x,u)} + k_2\beta(b-a)\sqrt{d(y,v)}]^2$
$\le 3[k_1^2|x(t)-u(t)|^2 + k_2^2\alpha^2(b-a)^2 d(x,u) + k_2^2\beta^2(b-a)^2 d(y,v)]$
$\le 3[k_1^2(|x(t)-u(t)| + |x(t)-u(t)|^2)+$
$+k_2^2\alpha^2(b-a)^2 d(x,u) + k_2^2\beta^2(b-a)^2 d(y,v)]$
$= 3[k_1^2 d(x,u) + k_2^2\alpha^2(b-a)^2 d(x,u) + k_2^2\beta^2(b-a)^2 d(y,v)]$
$= [3k_1^2 + 3k_2^2\alpha^2(b-a)^2]d(x,u) + [3k_2^2\beta^2(b-a)^2]d(y,v).$

It follows that:

$$\begin{aligned} &|S(x,y)(t) - S(u,v)(t)| + |S(x,y)(t) - S(u,v)(t)|^2 \\ &\quad \le [k_1 + k_2\alpha(b-a) + 3k_1^2 + 3k_2^2\alpha^2(b-a)^2]d(x,u) \\ &\quad\quad + [k_2\beta(b-a) + 3k_2^2\beta^2(b-a)^2]d(y,v). \end{aligned}$$

Hence, by taking the maximum over $t \in [a,b]$ we get:

$$d(S(x,y), S(u,v)) \le \mathcal{K}_1 d(x,u) + \mathcal{K}_2 d(y,v),$$

for all ($x \ge u$ and $y \le v$) or ($u \ge x$ and $v \le y$), where

$$\mathcal{K}_1 := k_1 + k_2\alpha(b-a) + 3k_1^2 + 3k_2^2\alpha^2(b-a)^2$$
$$\mathcal{K}_2 := k_2\beta(b-a) + 3k_2^2\beta^2(b-a)^2.$$

By (v) we get that $\mathcal{K}_1 + \mathcal{K}_2 < 1$.

We see that all the assumptions of Theorem 2.3 are satisfied and, by applying it, the conclusion follows. □

As a particular case of the above application, we consider the following system of integral equations

$$(\mathcal{S}) \quad \begin{cases} x(t) = g(t) + \int_a^b G(s,t)f(s,x(s),y(s))ds \\ y(t) = g(t) + \int_a^b G(s,t)f(s,y(s),x(s))ds \end{cases}, t \in [a,b] \subset \mathbb{R}_+.$$

A solution of the system $(\mathcal{S})$ is a pair $(x,y) \in C[a,b] \times C[a,b]$, satisfying the above relations for all $t \in [a,b]$, $0 \le a < b$.

By considering the same ordered Kasahara space $(C[a,b], \xrightarrow{\rho}, d, \le_C)$ as in the previous application, we have the following result:

Theorem 3.2. *Consider the integral system $(\mathcal{S})$. We assume that:*

(*i*) *$g : [a, b] \to \mathbb{R}$ and $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous and $G : [a, b] \times [a, b] \to \mathbb{R}$, is integrable w.r.t. the first variable;*

(*ii*) *$f(s, \cdot, \cdot)$ has the generalized mixed monotone property w.r.t. the last two variables for all $s \in [a, b]$;*

(*iii*) *there exists $x_0, y_0 \in C[a, b]$ such that*

$$\begin{cases} x_0(t) \leq g(t) + \int_a^b G(s,t) f(s, x_0(s), y_0(s)) ds \\ y_0(t) \geq g(t) + \int_a^b G(s,t) f(s, y_0(s), x_0(s)) ds \end{cases}$$

or $$\begin{cases} x_0(t) \geq g(t) + \int_a^b G(s,t) f(s, x_0(s), y_0(s)) ds \\ y_0(t) \leq g(t) + \int_a^b G(s,t) f(s, y_0(s), x_0(s)) ds \end{cases}$$ *, for all $t \in [a, b]$;*

(*iv*) *there exists $\alpha, \beta : [a, b] \to \mathbb{R}_+$ in $L^1[a, b]$ such that, for each u_1, u_2, v_1, $v_2 \in \mathbb{R}$ with $u_1 \leq v_1$ and $u_2 \geq v_2$ (or reversely), we have*

$$|f(s, u_1, u_2) - f(s, v_1, v_2)| \leq \alpha(s)|u_1 - v_1| + \beta(s)|u_2 - v_2|$$

for each $s \in [a, b]$;

(*v*) *the following conditions are satisfied:*

$$\begin{cases} \max_{t \in [a,b]} (\int_a^b G(s,t)\alpha(s) ds) + 2 \max_{t \in [a,b]} (\int_a^b G(s,t)\alpha(s) ds)^2 < \frac{1}{2} \\ \max_{t \in [a,b]} (\int_a^b G(s,t)\beta(s) ds) + 2 \max_{t \in [a,b]} (\int_a^b G(s,t)\beta(s) ds)^2 < \frac{1}{2} \end{cases}$$

Then there exists a unique solution (x^*, y^*) of the system

$$(\mathcal{S}) \quad \begin{cases} x(t) = g(t) + \int_a^b G(s,t) f(s, x(s), y(s)) ds \\ y(t) = g(t) + \int_a^b G(s,t) f(s, y(s), x(s)) ds \end{cases} , \ t \in [a, b] \subset \mathbb{R}_+.$$

Remark 3.3. Similar applications were given in [3] and [12].

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Manuscript received March 6 2018
revised September 21 2018

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