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ALGEBRAS OF CORNER OPERATORS

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Abstract. We establish a new approach to algebras of corner operators in terms of a calculus consisting of corner-degenerate pseudo-differential operators. Those are connected with principal and complete symbol hierarchies together with quantizations in weighted Kegel- and edge- Sobolev spaces. The underlying configuration M is a specific stratified space, and the geometric background is translated to specific features for the algebras such as operator-valued symbols associated with certain rescaling groups acting in the involved weighted spaces. This presentation is another step to developing a framework for treating operators close to geometric singularities where M fails to be smooth.

1. Introduction

The analysis of differential- and pseudo-differential operators on spaces with conical singularities or edges has a long tradition, and it is important to understand natural pseudo-differential algebras which yield parametrices of elliptic elements within the calculus. The ellipticity itself is of specific structure, and it is an interesting experience to see how classical elliptic boundary value or transmission problems belong to the special cases. Also operators on infinite straight cones with smooth or singular base manifolds belong to the framework. We do not outline here all relevant details from the past development but illustate the involved techniques by keywords and references to the general background. Boundary value problems in connection with singularities on the boundary have been studied in [28] which is an exposition that prepared later on the edge pseudo-differential calculus of [32]. Boundary value problems on manifolds with conical singularities have been studied in [30] and [31] based on Boutet de Monvel's algebra from [1] on smooth manifolds with boundary, see also [28]. We will not quote all papers and monographs on operators on manifolds with conical or edge singularities, cf. the bibliographies in [34], [35] or [8]. The approach of [11], [12] plays a particularly important role here, and also the ideas of [2], [3]. Another source of information is the article [37]

2. Spaces with corner singularities

Let us first fix some notation on configurations with violated smoothness, here briefly called manifolds with singularities, see also [2], [3]. By $\mathfrak{M}_0$ we denote the category of C^∞ manifolds with differential maps as morphisms. On any such M we have the well-known spaces of pseudo-differential operators $L^\mu(M)$ of order $\mu \in \mathbb{R}$

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and the subspace $L^\mu_{\mathrm{cl}}(M)$ of classical pseudo-differential operators. While elements in $L^\mu(M)$ for $n = \dim M$ in local coordinatea $x \in \mathbb{R}^n$ modulo $L^{-\infty}(M)$ are defined by expressions

$$\mathrm{Op}_x(a) \quad \text{for symbols of Hörmander's classes} \quad a(x,\xi) \in S^\mu(\mathbb{R}^n \times \mathbb{R}^n) \tag{2.1}$$

where the class of smoothing operators $L^{-\infty}(M)$ is identified with the space of integral operators with kernels in $C^\infty(M \times M)$ via a Riemannian metric on M; for classical operators we assume $a(x,\xi) \in S^\mu_{\mathrm{cl}}(\mathbb{R}^n \times \mathbb{R}^n)$. Those symbols, also called classical, are defined in terms of asymptotic expansions $a(x,\xi) \sim \sum_{j=0}^\infty \chi(\xi) a_{(\mu-j)}(x,\xi)$ for homogeneous components $a_{(\mu-j)}(x,\xi) \in S^{(\mu-j)}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$, where

$$S^{(\nu)}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \quad \text{for any} \quad \nu \in \mathbb{R}, \tag{2.2}$$

and χ is any excision function. The space (2.2) is defined to be the space of all $f(x,\xi) \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ satisfying relation

$$f(x,\delta\xi) = \delta^\nu f(x,\xi) \tag{2.3}$$

for all $\delta \in \mathbb{R}_+$, and x and $\xi \neq 0$. Similar notation is valid when we replace covariables ξ by (ξ,λ) for an extra parameter $\lambda \in \mathbb{R}^d$. This gives us parameter-dependent families of pseudo-differential operators

$$L^\mu(M;\mathbb{R}^d_\lambda), \; L^\mu_{\mathrm{cl}}(M;\mathbb{R}^d_\lambda), \; \text{and} \; L^\mu_{(\mathrm{cl})}(M;\mathbb{R}^d_\lambda), \tag{2.4}$$

respectively, where subscript "(cl)" indicates spaces of classical or non-classical elements. Analogous notation will be used for respective spaces of symbols, where parameters λ are included insofar they are components of covariables, and covariables $(\xi,\lambda) \in \mathbb{R}^{n+d}$, are admitted in the above-mentioned definition of symbols anyway, since the dimension of variables may be independent of that of covariables. Note that the elements of corresponding families of operators like (2.4) have the form

$$A(\lambda) = \sum_{j=1}^N \varphi_j \, (\chi_j^{-1})_* \big(\mathrm{Op}_x(a_j)(\lambda)\big)\varphi' + C(\lambda) \tag{2.5}$$

defined in terms of local symbols of the class $S^\mu(\Omega_x \times \mathbb{R}^{n+d}_{\xi,\lambda})$ with respect to an open covering $\{\Omega_j\}_{j=1,\dots,N}$ of M by coordinate neighborhoods, and charts $\chi_j : \Omega_j \to \mathbb{R}^n$, a subordinate partition of unity $\{\varphi_j\}_{j=1,\dots,N}$ and localizing functions $\varphi'_j \succ \varphi_j$, $\varphi'_j \in C_0^\infty(\Omega_j)$, with remainders

$$C(\lambda) \in L^{-\infty}(M;\mathbb{R}^d_\lambda) := \mathcal{S}(\mathbb{R}^d_\lambda; L^{-\infty}(M)). \tag{2.6}$$

As we shall see, parameter-dependent operators of different kind will belong to the starting points of higher singular operator theories. Here we systematically employ structures from the calculus on manifolds with conical or edge singularities.

The notion of "non-smoothness" of a configuration is often formulated by means of very specific local models. Here we prefer a definition of some generality which covers piecewise smoothness. Corresponding definitions have been used already in [2], [3], but in order to fix some notation we recall here the idea in order to see the way of passing from a singularity order k to that of order $k+1$ for any $k \in \mathbb{N} := \{0, 1, 2, \dots\}$.

Definition 2.1. (i) By $\mathfrak{M}_0$ we denote the category of smooth manifolds with differentiable maps as morphisms.

(ii) A topological space M is said to belong to $\mathfrak{M}_k$ for $k \geq 1$ if there is fixed a subspace $s_k(M) \in \mathfrak{M}_0$ such that $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$ and if there is a neighborhood $W_k \subset M$ of $s_k(M)$ which is a locally trivial bundle over $s_k(M)$ with fibres

$$X^{\Delta}_{k-1} := (\overline{\mathbb{R}}_+ \times X_{k-1})/(\{0\} \times X_{k-1}) \tag{2.7}$$

for some compact $X_{k-1} \in \mathfrak{M}_{k-1}$.

Remark 2.2. Observe that between the open stretched singular cones

$$X^{\wedge}_{k-1} := \mathbb{R}_+ \times X_{k-1} \in \mathfrak{M}_{k-1}$$

we have natural transition maps in the category $\mathfrak{M}_{k-1}$, namely,

$$X^{\wedge}_{k-1} \to \tilde{X}^{\wedge}_{k-1}, \quad \text{as well as} \quad \mathbb{R} \times X_{k-1} \to \mathbb{R} \times \tilde{X}_{k-1} \tag{2.8}$$

and for the above-mentioned bundle W_k the transition maps between the respective fibers

$$X^{\Delta}_{k-1} \to \tilde{X}^{\Delta}_{k-1} \tag{2.9}$$

are asked to restrict to the transition maps as in the first relations of (2.8) which are then required to be extendible to maps as in the second relations of (2.8). In this process we inductively employ what we know about isomorphisms $X_{k-1} \to \tilde{X}_{k-1}$ in $\mathfrak{M}_{k-1}$ from the steps before. The latter mappings may be associated with a locally trivial X_{k-1}-bundle $V_{\mathbb{O}}$ over $s_k(M)$. This can be attached in an invariant way to $M \setminus s_k(M)$, and we obtain a space

$$\mathbb{M} = (M \setminus s_k(M)) \cup V_{\mathbb{O}}, \tag{2.10}$$

called the stretched space associated with M. Taking two copies $\mathbb{M}_{\pm}$ of (2.10) with the positive space being identified with $\mathbb{M}$ and gluing together $\mathbb{M}_+$ and $\mathbb{M}_-$ along the common $V_{\mathbb{O}}$ we obtain the double $2\mathbb{M}$ of $\mathbb{M}$ which belongs to $\mathfrak{M}_{k-1}$.

Definition 2.1 allows us to apply the construction of $s_{k-1}(M)$ to $M \setminus s_k(M) \in \mathfrak{M}_{k-1}$, and we obtain a next singular subspace $s_{k-1}(M) \in \mathfrak{M}_0$. Then successively we can look at $s_{k-2}(M) := s_{k-2}(M \setminus (s_k(M) \cup s_{k-1}(M))) \in \mathfrak{M}_0$ etc. After finitely many steps it follows a sequence

$$s(M) := (s_0(M), s_1(M), \dots, s_{k-1}(M), s_k(M)) \tag{2.11}$$

of disjoint smooth manifolds with

$$\dim M := \dim s_0(M) > \dim s_1(M) > \cdots > \dim s_{k-1}(M) > \dim s_k(M) \geq 0, \tag{2.12}$$

and

$$M = \bigcup_{j=0}^{k} s_j(M). \tag{2.13}$$

Remark 2.3. Prototypes of manifolds with singularities are cones $X^\Delta \in \mathfrak{M}_1$ for some closed manifolds X or wedges

$$W := X^\Delta \times \mathbb{R}^q \in \mathfrak{M}_1. \tag{2.14}$$

In particular, we may have $\dim X = 0$. Then we have $X^\Delta = \overline{\mathbb{R}}_+$, and the respective wedge is equal to the half space $\overline{\mathbb{R}}_+ \times \mathbb{R}^q$. More generally, a manifold M with smooth boundary ∂M may be interpreted as a manifold with edge which is in this case of codimension 1.

Remark 2.4. Note that any $M \in \mathfrak{M}_0$ may be equipped with different strata like (2.11). For instance, $M = \mathbb{R}^N$ turns to a space in $\mathfrak{M}_1$ by setting $s_1(M) := \{=\}$ (the origin which is a conical singularity) or $s_1(M) = \{x = \{x_1, \dots, x_N\} \in \mathbb{R}^N : x_N = 0\}$ (which is an edge).

Remark 2.5. For $M \in \mathfrak{M}_k$, $N \in \mathfrak{M}_l$ we have $M \times N \in \mathfrak{M}_{k+l}$.

3. Parameter-dependent edge calculus

We now outline some structures on the calculus on manifolds with conical singularities and edges. The following material will refer to this material, but it seems to be indispensable for the higher versions of pseudo-differential theories. Clearly we cannot recall here a complete introduction to this topic. Several monographs are devoted to these tools, cf. [34], [35], [8], and also the articles [11], [12], and we will adopt here notation from there. This presentation is aimed at illustrating new properties of operators on spaces in $\mathfrak{M}_k$ or some conical exit to infinity, altogether called manifolds with singularities. In particular, we discuss the role of conical exits to infinity and of involved dependence of operators on parameters and to what extent these ingredients are necessary for corresponding pseudo-differential algebras and their symbol structures.

Parameter-dependent operators $A(\lambda)$ in

$$L^\mu_{\mathrm{cl}}(X; \mathbb{R}^d_\lambda) \tag{3.1}$$

for a closed manifold $X \in \mathfrak{M}_0$ of dimension $n \in \mathbb{N}$ give rise to the ingredients of spaces of parameter-dependent operators

$$L^{\mu-m}(B, \boldsymbol{g}; \mathbb{R}^d_\lambda) \tag{3.2}$$

for a compact space $B \in \mathfrak{M}_1$ with q- dimensional edge $Y \in \mathfrak{M}_0$ and weight data $\boldsymbol{g} := (\gamma, \gamma - \mu)$ for any $m \in \mathbb{N}$. The dimension d in (3.1) is chosen independently of that in (3.2); otherwise we should employ indices for the different choices of λ, but the process of constructing (3.2) in terms of (3.1) consumes λ in (3.1) completely, and λ in (3.2) only plays a role for the next higher corner calculus on a manifold $M \in \mathfrak{M}_2$, etc. Before we discuss the step from B to M we first outline technicalities from X to B. We employ λ in (3.1) in the meaning of $\lambda = (\rho, \eta) \in \mathbb{R}_\rho \times \mathbb{R}^q_\eta$, where we assume that B is locally close to Y modeled on $X^\Delta \times \mathbb{R}^q$, and ρ is dual to the axial variable $r \in \mathbb{R}$ while η is the covariable to $y \in \mathbb{R}^q$. For convenience we will explain the procedure for $m = 0$. The arguments for arbitrary m are similar.

Starting point are edge-degenerate families of operators

$$\tilde{p}(r, y, \tilde{\rho}, \tilde{\eta}) \in C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q}_{\tilde{\rho},\tilde{\eta}})), \tag{3.3}$$

and

$$p(r, y, \rho, \eta) := \tilde{p}(r, y, r\rho, r\eta). \tag{3.4}$$

Mellin-edge quantization gives us an

$$\tilde{h}(r, y, w, \tilde{\eta}) \in C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^{\mu}_{\mathcal{O}_w}(X; \mathbb{R}^q_{\tilde{\eta}})), \tag{3.5}$$

with subscript $[0, R]$ indicating functions which are independent of r for $r > R$, such that

$$h(r, y, w, \eta) := \tilde{h}(r, y, w, r\eta) \tag{3.6}$$

satisfies relation

$$\mathrm{Op}_r(p)(y, \eta) - \mathrm{Op}^{\gamma}_M(h)(y, \eta) = \mathrm{Op}_r(q)(y, \eta) \tag{3.7}$$

for a regularizing operator family

$$q(r, r', y, \rho, \eta) = (1 - \varphi(r'/r))p(r, y, \rho, \eta), \quad r, r' \in \mathbb{R}_+, y \in \mathbb{R}^q, (\rho, \eta) \in \mathbb{R}^{1+q}. \tag{3.8}$$

The correspondence $p \mapsto h$ also works in converse direction. It induces an isomorphism

$$\begin{aligned} &C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}^{1+q})/C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, L^{-\infty}(X; \mathbb{R}^{1+q}) \\ &\quad \to C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^{\mu}_{\mathcal{O}}(X; \mathbb{R}^q)/C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^{-\infty}_{\mathcal{O}}(X; \mathbb{R}^q). \end{aligned} \tag{3.9}$$

The latter constructions also hold in the version with parameters λ, and those will be admitted now. Thus, similarly to [35] and [12] we consider two operator-valued symbol classes

$$R^{\mu}_{\mathrm{trad}}(\mathbb{R}^{2q}_{y,\eta}, \boldsymbol{g}; \mathbb{R}^d_{\lambda}) \text{ or } R^{\mu}_{\mathrm{new}}(\mathbb{R}^{2q}_{y,\eta}, \boldsymbol{g}; \mathbb{R}^d_{\lambda}) \tag{3.10}$$

which are essentially equivalent, see [12].

The space

$$R^{\mu}_{\mathrm{trad}}(\mathbb{R}^{2q}_{y,\eta}, \boldsymbol{g}; \mathbb{R}^d_{\lambda}) \quad \text{for weight data} \quad \boldsymbol{g} = (\gamma, \gamma - \mu) \tag{3.11}$$

of edge symbols in traditional Mellin-edge quantization from [32] is defined to be the set of all operator-functions of the form

$$\begin{aligned} a(y, \eta, \lambda) = {} & \sigma_1(r)\big(a_0(y, \eta, \lambda) + a_1(y, \eta, \lambda)\big)\sigma_0(r) \\ & + (1 - \sigma_1(r))a_{\mathrm{int}}(y, \eta, \lambda)(1 - \sigma_2(r)) + (m + g)(y, \eta, \lambda) \end{aligned} \tag{3.12}$$

for cut-off functions

$$\sigma_2(r) \prec \sigma_1(r) \prec \sigma_0(r),$$

and

$$a_0(y, \eta, \lambda) := \omega_1(r[\eta, \lambda])r^{-\mu}\mathrm{Op}^{\gamma - \frac{n}{2}}_M(h)(y, \eta, \lambda)\omega_0(r[\eta, \lambda]), \tag{3.13}$$

$$a_1(y, \eta, \lambda) := (1 - \omega_1(r[\eta, \lambda]))r^{-\mu}\mathrm{Op}_r(p)(y, \eta, \lambda)(1 - \omega_2(r[\eta, \lambda])), \tag{3.14}$$

where

$$\omega_2(r) \prec \omega_1(r) \prec \omega_0(r),$$

are cut-off functions, and

$$h(r,y,w,\eta,\lambda)\in C^\infty_{[0,\mathrm{R}]}(\overline{\mathbb{R}}_+\times\mathbb{R}^q, M^\mu_{\mathcal{O}}(X;\mathbb{R}^{q+d}_{\tilde\eta,\tilde\lambda})|_{(\tilde\eta,\tilde\lambda)=(r\eta,r\lambda)},\tag{3.15}$$

and

$$p(r,y,\rho,\eta,\lambda)\in C^\infty_{[0,\mathrm{R}]}(\overline{\mathbb{R}}_+\times\mathbb{R}^q, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q+d}_{\tilde\rho,\tilde\eta,\tilde\lambda}))|_{(\tilde\rho,\tilde\eta,\tilde\lambda)=(r\rho,r\eta,r\lambda)}.\tag{3.16}$$

Recall that subscript "$[0,R]$" for some $R>0$ indicates the subspace of $C^\infty(\overline{\mathbb{R}}_+\dots)$ the elements of which are independent of the first r-variable for $r>R$. Moreover, we assume

$$a_{\mathrm{int}}(y,\eta,\lambda)\in C^\infty(\mathbb{R}^q, L^\mu_{\mathrm{cl}}(X^\wedge;\mathbb{R}^{q+d}_{\eta,\lambda})_0)$$

for

$$\begin{aligned}L^\mu_{\mathrm{cl}}(X^\wedge;\mathbb{R}^{q+d}_{\eta,\lambda})_0 := \big\{a_{\mathrm{int}}(\eta,\lambda)\in L^\mu_{\mathrm{cl}}(X^\wedge;\mathbb{R}^{q+d}_{\eta,\lambda}) : \tilde\sigma a_{\mathrm{int}}(\eta,\lambda)\tilde{\tilde\sigma} = a_{\mathrm{int}}(\eta,\lambda)\\ \text{for some cut-off functions } \tilde\sigma,\tilde{\tilde\sigma}\big\}.\end{aligned}\tag{3.17}$$

The operators $A_{\mathrm{c}}(\lambda):=\mathrm{Op}_y(a)(\lambda)$ which are later on used in the edge calculus have symbols $a(y,\eta,\lambda)$ in

$$R^\mu_{\mathrm{trad}}(\mathbb{R}^{2q}_{y,\eta},\boldsymbol{g};\mathbb{R}^d_\lambda)\subset S^\mu(\mathbb{R}^{2q+d}_{y,\eta,\lambda};H,\tilde H)\tag{3.18}$$

for Kegel spaces

$$H=\mathcal{K}^{s,\gamma}(X^\wedge),\ \tilde H=\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge),\ s\in\mathbb{R}.$$

There is another choice of edge amplitude functions, introduced in [12], namely,

$$R^\mu_{\mathrm{new}}(\mathbb{R}^{2q}_{y,\eta},\boldsymbol{g};\mathbb{R}^d_\lambda)\quad\text{for weight data}\quad \boldsymbol{g}=(\gamma,\gamma-\mu)\tag{3.19}$$

consists of all operator functions

$$\begin{aligned}a(y,\eta,\lambda):=\sigma_1(r)r^{-\mu}\mathrm{Op}_M^{\gamma-n/2}(h)(y,\eta,\lambda)\sigma_0(t)\\ +(1-\sigma_1(r))a_{\mathrm{int}}(y,\eta,\lambda)(1-\sigma_2(r))+(m+g)(y,\eta,\lambda).\end{aligned}\tag{3.20}$$

The essential result of [12, page 236] is that

$$R^\mu_{\mathrm{trad}}(\mathbb{R}^{2q}_{y,\eta},\boldsymbol{g};\mathbb{R}^d_\lambda)\quad\text{and}\quad R^\mu_{\mathrm{new}}(\mathbb{R}^{2q}_{y,\eta},\boldsymbol{g};\mathbb{R}^d_\lambda)\quad\text{are equivalent.}\tag{3.21}$$

The Green symbols $g(y,\eta,\lambda)$ in (3.12) and (3.19) are described in [12] for the case without parameters, and smoothing Mellin symbols $m(y,\eta,\lambda)$, here also without parameters, will be ignored. They do not occur in the proof of (3.21).

4. Kegel- and edge- spaces with multiple weights

Let B be a compact manifold with edge Y of dimension $q>0$, locally close to the edge modeled on $X^\Delta\times\mathbb{R}^q$ for a closed smooth manifold X of dimension n. We will form Kegel spaces

$$\mathcal{K}^{s,\beta,\gamma;e}(B^\wedge)\ \text{ in variables }\ (t,b)\in B^\wedge=\mathbb{R}_+\times B\tag{4.1}$$

of smoothness $s\in\mathbb{R}$, a base weight $\beta\in\mathbb{R}$, a corner weight $\gamma\in\mathbb{R}$ and an exit weight $e\in\mathbb{R}$. Using local wedge spaces

$$\mathcal{W}^s(\mathbb{R}^{1+q}_{\tilde y},\mathcal{K}^{s,\beta}(X^\wedge_{r,x}))\tag{4.2}$$

with $\tilde{y} \in \mathbb{R}^{1+q}$ being treated as edge variables, and the group action on the space $\mathcal{K}^{s,\beta}(X^\wedge_{r,x})$, given by

$$\kappa = \{\kappa_\delta\}_{\delta\in\mathbb{R}_+} \text{ given by } \kappa_\delta u(r,x) = \delta^{n+1/2}u(\delta r, x).$$

For compact Y we first form the spaces

$$\mathcal{W}^s_{\varepsilon,\mathrm{cone}}(Y^\wedge, \mathcal{K}^{s,\beta}(X^\wedge_{r,x})) \tag{4.3}$$

as follows. Let $\{U_j\}_{j=1,\dots,N}$ be an open covering of Y by coordinate neighborhoods U_j and $\{\varphi_j\}_{j=1,\dots,N}$ a subordinate partition of unity. Moreover, we consider charts

$$\chi_j : U_j \to \{\tilde{y}' \in \mathbb{R}^q : |\tilde{y}'| < 1\}$$

with $\mathbb{R}^q$ being identified with

$$\tilde{y} \in \mathbb{R}^{1+q} : \tilde{y} = (\tilde{y}_0, \tilde{y}') : \tilde{y}_0 = 1, \tilde{y}' \in \mathbb{R}^q\}.$$

We then define diffeomorphisms

$$\chi^<_j : (\varepsilon, \infty) \times U_j \to C^<_j \tag{4.4}$$

by setting

$$\chi^<_j(t,y) := (t, [t]\chi_j(y)),$$

where t on the right-hand side is identified with $\tilde{y}_0 \in \mathbb{R}_+$. In other words, the variables $\tilde{y}$ in $C^<_j$ admit the splitting into $(\tilde{y}_0, \tilde{y}')$ and the inverse of the bijection (4.4) has the form

$$(\chi^<_j)^{-1}(\tilde{y}) = (t, [t]^{-1}\tilde{y}').$$

The space (4.3) is then defined as the subspace of all

$$f(t,y) \in \mathcal{W}^s_{\mathrm{loc}}(\mathbb{R}_+ \times Y, \mathcal{K}^{s,\beta}(X^\wedge))$$

supported by $[\varepsilon, \infty) \times Y$ such that

$$(\chi^<_j)_*(\varphi_j f|_{(\varepsilon,\infty)\times U_j}) \in \mathcal{W}^s(\mathbb{R}^{1+q}, \mathcal{K}^{s,\beta}(X^\wedge)),\ j = 1, \dots, N. \tag{4.5}$$

Definition 4.1. For a cut-off function $\sigma(t)$ we define the space

(4.6)
$$\mathcal{W}^s_{\mathrm{cone}}(Y^\wedge, \mathcal{K}^{s,\beta}(X^\wedge_{r,x})) := \sigma(t)\mathcal{W}^s_{\mathrm{loc}}(\mathbb{R} \times Y, \mathcal{K}^{s,\beta}(X^\wedge_{r,x}))_{\mathbb{R}_+\times Y} + (1-\sigma(t))\mathcal{W}^s_{\varepsilon,\mathrm{cone}}(Y^\wedge, \mathcal{K}^{s,\beta}(X^\wedge_{r,x})).$$

Moreover, we set

$$H^s_{\mathrm{int}}((B \setminus Y)^\wedge) := H^s_{\mathrm{cone}}((2\mathbb{B})^\wedge)|_{\mathbb{R}_{+,t}\times(B\setminus Y)}. \tag{4.7}$$

Definition 4.2. Let us define the space

$$H^{s,\beta}_{\mathrm{cone}}(B^\wedge) := \big\{v := \omega_{\mathrm{glob}} v_{\mathrm{edge}} + (1-\omega_{\mathrm{glob}})v_{\mathrm{int}} : v_{\mathrm{edge}} \in \mathcal{W}^s_{\mathrm{cone}}(Y^\wedge, \mathcal{K}^{s,\beta}(X^\wedge_{r,x})), v_{\mathrm{int}} \in H^s_{\mathrm{int}}((B\setminus Y)^\wedge)\big\}. \tag{4.8}$$

Let us also define corner Mellin spaces, i.e., spaces close to $t = 0$. First we consider charts $\lambda_l : U_l \to \mathbb{R}^q$ and set

(4.9)
$$\mathcal{H}^{s,\gamma}_{\mathrm{edge}}(\mathbb{R}_{+,t} \times U_l, \mathcal{K}^{s,\beta}(X^\wedge_{r,x})) := \Big\{(\mathrm{id}_{\mathbb{R}_+} \times \lambda_l)^{-1}_* u) : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^q, \mathcal{K}^{s,\beta}(X^\wedge))\Big\},$$

and we form

(4.10)
$$\mathcal{H}^{s,\gamma}_{\mathrm{edge}}(\mathbb{R}_{+,t}\times Y,\mathcal{K}^{s,\beta}(X^{\wedge}_{r,x})):=\Big\{\sum_{l=1}^{N}\varphi_l u_l : u_l\in\mathcal{H}^{s,\gamma}_{\mathrm{edge}}(\mathbb{R}_{+,t}\times U_l,\mathcal{K}^{s,\beta}(X^{\wedge}_{r,x}))\Big\}.$$

Moreover, let

(4.11)
$$\mathcal{H}^{s,\gamma}_{\mathrm{int}}((B\setminus Y)^{\wedge}):=\mathcal{H}^{s,\gamma}((2\mathbb{B})^{\wedge})|_{\mathbb{R}_{+,t}\times(B\setminus Y)}.$$

Definition 4.3.

(4.12)
$$\begin{aligned}\mathcal{H}^{s,\beta,\gamma}(B^{\wedge}):=\big\{&u:=\omega_{\mathrm{glob}}u_{\mathrm{edge}}+(1-\omega_{\mathrm{glob}})u_{\mathrm{int}}\\ &: u_{\mathrm{edge}}\in\mathcal{H}^{s,\gamma}_{\mathrm{edge}}(\mathbb{R}_{+,t}\times Y,\mathcal{K}^{s,\beta}(X^{\wedge}_{r,x}),u_{\mathrm{int}}\in\mathcal{H}^{s,\gamma}_{\mathrm{int}}((B\setminus Y)^{\wedge})\big\}.\end{aligned}$$

Definition 4.4. The Kegel space (4.1) over $B^{\wedge}\ni(t,b)$ for a compact manifold B with edge Y of dimension q and

smoothness s,

base weight β, associated with the base B,

corner weight γ, associated with the corner axis t, for $t\to 0$,

exit weight e, associated with the conical exit for $t\to\infty$,

is defined by

(4.13)
$$\mathcal{K}^{s,\beta,\gamma;e}(B^{\wedge}):=[t]^{-e}\mathcal{K}^{s,\beta,\gamma}(B^{\wedge})$$

for

(4.14)
$$\mathcal{K}^{s,\beta,\gamma}(B^{\wedge}):=\sigma\mathcal{H}^{s,\beta,\gamma}(B^{\wedge})+(1-\sigma)H^{s,\beta}_{\mathrm{cone}}(B^{\wedge})$$

for some cut-off function $\sigma(t)$ on the half-axis $\mathbb{R}_{+,t}$, with obvious notation.

Thus we have

(4.15)
$$\begin{aligned}\mathcal{K}^{s,\beta,\gamma}(B^{\wedge})=\sigma\Big(&\omega\mathcal{H}^{s,\gamma}_{\mathrm{edge}}(Y^{\wedge},\mathcal{K}^{s,\beta}(X^{\wedge}))+(1-\omega)\mathcal{H}^{s,\gamma}_{\mathrm{int}}((B\setminus Y)^{\wedge})\Big)\\ &+(1-\sigma)\Big(\omega\mathcal{W}^{s}_{\mathrm{cone}}(Y^{\wedge},\mathcal{K}^{s,\beta}(X^{\wedge}))+(1-\omega)H^{s}_{\mathrm{int}}((B\setminus Y)^{\wedge}))\Big),\end{aligned}$$

with notation from (4.10), (4.11), (4.6), (4.7), and $\omega(r):=\omega_{\mathrm{glob}}(r)$.

Remark 4.5. Both $\mathcal{K}^{s,\beta,\gamma}(B^{\wedge})$ and $\mathcal{H}^{s,\beta,\gamma}(B^{\wedge})$, defined by (4.12), are independent of the involved cut-off functions.

Remark 4.6. For elements $u(t,b)\in\mathcal{K}^{s,\beta,\gamma}(B^{\wedge}):=\mathcal{K}^{s,\beta,\gamma;0}(B^{\wedge})$ we define

(4.16)
$$(\kappa_\delta u)(t,b):=\delta^{(\dim B+1)/2}u(\delta t,b),\quad \delta\in\mathbb{R}_+.$$

This gives us a group action $\kappa=\{\kappa_\delta\}_{\in\mathbb{R}_+}$, on $\mathcal{K}^{s,\beta,\gamma}(B^{\wedge})$, and we have associated edge spaces

(4.17)
$$\mathcal{W}^{s}(\mathbb{R}^{q^{\mathbf{1}}}_{z},\mathcal{K}^{s,\beta,\gamma}(B^{\wedge})).$$

in edge variables $z\in\mathbb{R}^{q^{\mathbf{1}}}$.

Parameter-dependent Green symbols $g(z,z',\zeta,\lambda)$ will be employed in a similar meaning as in [12], though here with parameters and weight data

$$\boldsymbol{g} := (\beta,\beta'),\ \boldsymbol{g^1} := (\gamma,\gamma').$$

The corresponding spaces are denoted by

$$R^{\nu}_{\mathrm{G}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,z'} \times \mathbb{R}^{q^{\mathbf{1}}}_{\zeta}, \boldsymbol{g}, \boldsymbol{g^1}; \mathbb{R}^d_{\lambda}) \tag{4.18}$$

for any order $\nu \in \mathbb{R}$. Elements $g(z,z',\zeta,\lambda)$ in (4.18) are defined by the properties

$$g \in \bigcap_{s,s',e,e'\in\mathbb{R}} S^{\nu}_{\mathrm{cl}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,z'} \times \mathbb{R}^{q^{\mathbf{1}}+d}_{\zeta,\lambda}; \mathcal{K}^{s,\beta,\gamma;e}(B^{\wedge}), \mathcal{K}^{s',\beta'+\varepsilon,\gamma'+\varepsilon;e'}(B^{\wedge})), \tag{4.19}$$

$$g^* \in \bigcap_{s,s',e,e'\in\mathbb{R}} S^{\nu}_{\mathrm{cl}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,z'} \times \mathbb{R}^{q^{\mathbf{1}}+d}_{\zeta,\lambda}; \mathcal{K}^{s,-\beta',-\gamma';e'}(B^{\wedge}), \mathcal{K}^{s,-\beta+\varepsilon,-\gamma+\epsilon;e}(B^{\wedge})), \tag{4.20}$$

for some $\varepsilon = \varepsilon(g) > 0$, where * indicates the point wise formal adjoint with respect to the reference scalar product of $\mathcal{K}^{0,0,0;0}(B^{\wedge})$. In particular,

$$R^{\nu}_{\mathrm{G}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,\zeta}, (\beta,\beta'), (\gamma,\gamma'); \mathbb{R}^d_{\lambda})_{\infty} \tag{4.21}$$

denotes the set of all $g(z,\zeta,\lambda)$, satisfying (4.19) and (4.20) for all $\varepsilon > 0$.

5. Mellin quantization with respect to corner parameters

Considering a compact manifolds $B \in \mathfrak{M}_1$ with edge Y we have the space

$$L^{\mu}(B,\boldsymbol{g};\mathbb{R}^d_{\lambda}) \quad \text{for weight data} \quad \boldsymbol{g} := (\beta,\beta-\mu) \tag{5.1}$$

of parameter-dependent edge operator families with parameter $\lambda \in \mathbb{R}^d$. The space (5.1) is defined to be the set of all families of operators

$$A(\lambda) = \omega_{\mathrm{glob}} A_{\mathrm{c}}(\lambda)\omega'_{\mathrm{glob}} + (1-\omega_{\mathrm{glob}})A_{\mathrm{int}}(\lambda)(1-\omega''_{\mathrm{glob}}) + C(\lambda) \tag{5.2}$$

for the above-mentioned $A_{\mathrm{c}}(\lambda)$, moreover, $A_{\mathrm{int}}(\lambda) \in L^{\mu}_{\mathrm{cl}}(B \setminus Y; \mathbb{R}^d_{\lambda})$, and global cut-off functions

$$\omega''_{\mathrm{glob}} \prec \omega_{\mathrm{glob}} \prec \omega'_{\mathrm{glob}}$$

on B. Furthermore, $C(\lambda) \in L^{-\infty}(B,\boldsymbol{g};\mathbb{R}^d_{\lambda})) = \mathcal{S}(\mathbb{R}^d, L^{-\infty}(B,\boldsymbol{g}))$ is a smoothing family, defined by asking continuity

$$C : H^{s,\gamma}(B) \to H^{\infty,\gamma-\mu+\varepsilon}(B)$$

for every s and some $\varepsilon > 0$, and a similar condition for its formal adjoint $C^* : H^{s,-\gamma+\mu}(B) \to H^{\infty,-\gamma+\varepsilon}(B)$ with respect to the reference scalar product from $H^{0,0}(B)$. This definition refers to global weighted Sobolev spaces

$$H^{s,\gamma}(B), \quad s,\gamma \in \mathbb{R}. \tag{5.3}$$

They are locally near Y modeled on $\mathcal{W}^s(\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^{\wedge}))$ while $H^{s,\gamma}(B)|_{B\setminus Y} \subseteq H^s_{\mathrm{loc}}(B\setminus Y)$. Such a notation extends in a natural way to $H^{s,\gamma}_{\mathrm{loc}}(B)$ when B is not compact. In the corner theory of singularity order 2 we start with the case $d = 1 + q^{\mathbf{1}}$ where

$$(\hat{\tau},\hat{\zeta}) := \lambda \quad \text{for} \quad \hat{\tau} \in \mathbb{R}, \hat{\zeta} \in \mathbb{R}^{q^{\mathbf{1}}} \tag{5.4}$$

in the later meaning

$$\hat{\tau} = t\tau, \hat{\zeta} = t\zeta.$$

We form

$$p^{\mathbf{1}}(t,z,t',z',\hat{\tau},\hat{\zeta}) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \mathbb{R}^{q^{\mathbf{1}}} \times \overline{\mathbb{R}}_+ \times \mathbb{R}^{q^{\mathbf{1}}}, L^{\mu}(B,\boldsymbol{g};\mathbb{R}^{1+q^{\mathbf{1}}}_{\hat{\tau},\hat{\zeta}})). \tag{5.5}$$

In some computations for convenience we drop the variables z, z'; the edge-analogue is also considered in [12, page 11, Theorem 1.26] and in [12, page 233, Theorem 3.2]. The corner-analogue of the Mellin quantization now reads as follows.

Theorem 5.1. [4]
For every

$$\hat{p}^{\mathbf{1}}(t,\hat{\tau},\hat{\zeta}) \in C^{\infty}_{[0,T]}(\overline{\mathbb{R}}_+, L^{\mu}(B,\boldsymbol{g};\mathbb{R}^{1+q^{\mathbf{1}}})) \tag{5.6}$$

and any $\psi \in C_0^{\infty}\mathbb{R}_+)$, $\psi(\cdot) \equiv 1$ *close to* 1 *there exists an*

$$\hat{h}^{\mathbf{1}}(t,v,\hat{\zeta}) \in C^{\infty}_{[0,T]}(\overline{\mathbb{R}}_+, M^{\mu}_{\mathcal{O}}(B,\boldsymbol{g};\mathbb{R}^{q^{\mathbf{1}}})) \tag{5.7}$$

such that

$$p^{\mathbf{1}}(t,\tau,\zeta) := \hat{p}^{\mathbf{1}}(t,t\tau,t\zeta) \tag{5.8}$$

and

$$h^{\mathbf{1}}(t,v,\zeta) := \hat{h}^{\mathbf{1}}(t,v,t\zeta) \tag{5.9}$$

satisfy

$$\mathrm{Op}(p^{\mathbf{1}})(\zeta) - \mathrm{Op}_M^{\gamma-n^{\mathbf{1}}/2}(h^{\mathbf{1}})(\zeta) = \mathrm{Op}_t(q)(\zeta) \tag{5.10}$$

for

$$q(t,t',\tau,\zeta) = \big(1-\psi(t'/t)\big)p^{\mathbf{1}}(t,\tau,\zeta) \quad \text{for} \quad t,t' \in \mathbb{R}_+, (\tau,\zeta) \in \mathbb{R}^{1+q^{\mathbf{1}}}, \tag{5.11}$$

$n^{\mathbf{1}} = \dim B$, *for every* $\gamma \in \mathbb{R}$. *A similar result holds including variables* z *or* z'.

6. Traditional and new Mellin-edge quantization for corner singularities

Starting point are corner-degenerate families of operators

$$\hat{p}^{\mathbf{1}}(t,z,\tilde{\tau},\tilde{\zeta}) \in C^{\infty}_{[0,T]}(\overline{\mathbb{R}}_+, L^{\mu}(B,\boldsymbol{g};\mathbb{R}^{1+q^{\mathbf{1}}})), \tag{6.1}$$

$$\hat{h}^{\mathbf{1}}(t,v,\hat{\zeta}) \in C^{\infty}_{[0,T]}(\overline{\mathbb{R}}_+, M^{\mu}_{\mathcal{O}}(B,\boldsymbol{g};\mathbb{R}^{q^{\mathbf{1}}})) \tag{6.2}$$

and

$$p^{\mathbf{1}}(t,\tau,\zeta) := \hat{p}^{\mathbf{1}}(t,t\tau,t\zeta), \quad h^{\mathbf{1}}(t,v,\zeta) := \hat{h}^{\mathbf{1}}(t,v,t\zeta), \tag{6.3}$$

such that, according to Theorem 5.1, Mellin quantization gives us

$$\mathrm{Op}(p^{\mathbf{1}})(\zeta) - \mathrm{Op}_M^{\gamma-n^{\mathbf{1}}/2}(h^{\mathbf{1}})(\zeta) = \mathrm{Op}_t(q)(\zeta) \in L^{-\infty}(\mathbb{R}_+ \times B,\boldsymbol{g};\mathbb{R}^{q^{\mathbf{1}}})) \tag{6.4}$$

for

$$q(t,t',\tau,\zeta) = \big(1-\psi(t'/t)\big)p^{\mathbf{1}}(t,\tau,\zeta) \quad \text{for} \quad t,t' \in \mathbb{R}_+, (\tau,\zeta) \in \mathbb{R}^{1+q^{\mathbf{1}}}. \tag{6.5}$$

Let us study

$$R^{\mathbf{1},\mu}_{\mathrm{trad}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,\zeta}, \boldsymbol{g}, \boldsymbol{g}^{\mathbf{1}})) \text{ and } R^{\mu}_{\mathrm{new}}(\mathbb{R}^{2q}_{y,\eta}, \boldsymbol{g}; \mathbb{R}^{d}_{\lambda}). \tag{6.6}$$

The space

$$R^{\mathbf{1},\mu}_{\mathrm{trad}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,\zeta},\boldsymbol{g},\boldsymbol{g}^{\mathbf{1}})) \quad \text{for weight data} \quad \boldsymbol{g}=(\beta-\mu),\boldsymbol{g}^{\mathbf{1}}=(\gamma,\gamma-\mu) \tag{6.7}$$

of edge symbols from [37] in traditional Mellin-edge quantization is defined to be the set of all operator-functions of the form

$$\begin{aligned} a^{\mathbf{1}}(z,\zeta) &= \sigma_1(t)\big(a_0^{\mathbf{1}}(z,\zeta)+a_1^{\mathbf{1}}(y,\eta)\big)\sigma_0(t) \\ &\quad + (1-\sigma_1(t))a^{\mathbf{1}}_{\mathrm{int}}(z,\zeta)(1-\sigma_2(t)) + (m^{\mathbf{1}}+g^{\mathbf{1}})(z,\zeta) \end{aligned} \tag{6.8}$$

for cut-off functions

$$\sigma_2(t)\prec\sigma_1(t)\prec\sigma_0(t),$$

and

$$a_0^{\mathbf{1}}(z,\zeta) := \omega_1(t[\zeta])t^{-\mu}\mathrm{Op}_M^{\gamma-\frac{n^{\mathbf{1}}}{2}}(h^{\mathbf{1}})(z,\zeta)\omega_0(t[z\eta]), \tag{6.9}$$

$$a_1^{\mathbf{1}}(z,\zeta) := (1-\omega_1(t[\zeta]))t^{-\mu}\mathrm{Op}_t(p^{\mathbf{1}})(z,\zeta)(1-\omega_2(t[\zeta])), \tag{6.10}$$

where

$$\omega_2(t)\prec\omega_1(t)\prec\omega_0(t),$$

are cut-off functions, and

$$h^{\mathbf{1}}(t,z,v,\zeta)\in C^\infty_{[0,\mathrm{T}]}(\overline{\mathbb{R}}_+\times\mathbb{R}^{q^{\mathbf{1}}}, M^\mu_{\mathcal{O}}(B;\mathbb{R}^{q^{\mathbf{1}}}_{\tilde{\zeta}})|_{\tilde{\zeta}=t\zeta}, \tag{6.11}$$

and

$$p^{\mathbf{1}}(t,z,\tau,\eta,\lambda)\in C^\infty_{[0,\mathrm{R}]}(\overline{\mathbb{R}}_+\times\mathbb{R}^q, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^{1+q^{\mathbf{1}}}_{\tilde{\tau},\tilde{\zeta}}))|_{(\tilde{\tau},\tilde{\zeta})=(t\tau,t\zeta)}. \tag{6.12}$$

Here subscript "$[0,T]$" for some $T>0$ indicates the subspace of $C^\infty(\overline{\mathbb{R}}_+\dots)$ the elements of which are independent of t for $t>T$. Moreover,

$$a^{\mathbf{1}}_{\mathrm{int}}(z,\zeta)\in C^\infty(\mathbb{R}^{q^{\mathbf{1}}}, L^\mu(\mathbb{R}_+\times B;\mathbb{R}^{q^{\mathbf{1}}}_\zeta)_0)$$

for

$$L^\mu(\mathbb{R}_+\times B,\boldsymbol{g};\mathbb{R}^{q^{\mathbf{1}}}_\zeta)_0 := \big\{a(\zeta)\in L^\mu(\mathbb{R}_+\times B,\boldsymbol{g};\mathbb{R}^{q^{\mathbf{1}}}_\zeta) : \tilde{\sigma}a(\zeta)\tilde{\tilde{\sigma}} = a(\zeta) \text{ for some cut-off functions } \tilde{\sigma},\tilde{\tilde{\sigma}}\big\}. \tag{6.13}$$

The operators $A_{\mathrm{c}}(\lambda)=\mathrm{Op}(a)(\lambda)$ have symbols $a(y,\eta,\lambda)$ in

$$R^{\mathbf{1},\mu}_{\mathrm{trad}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,\zeta},\boldsymbol{g},\boldsymbol{g}^{\mathbf{1}})\subset S^\mu(\mathbb{R}^{2q^{\mathbf{1}}}_{z,\zeta};H,\tilde{H}) \tag{6.14}$$

for

$$H=\mathcal{K}^{s,\beta,\gamma;e}(B^\wedge),\ \tilde{H}=\mathcal{K}^{s-\mu,\beta-\mu,\gamma-\mu}(B^\wedge),\ s,e\in\mathbb{R}.$$

There is another choice of edge amplitude functions, namely,

$$R^{\mathbf{1},\mu}_{\mathrm{new}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,\zeta},\boldsymbol{g},\boldsymbol{g}^{\mathbf{1}}) \quad \text{for weight data} \quad \boldsymbol{g}=(\beta,\beta-\mu),\boldsymbol{g}^{\mathbf{1}}=(\gamma,\gamma-\mu) \tag{6.15}$$

consisting of all operator functions

$$\begin{aligned} a^{\mathbf{1}}(z,\zeta) &:= \sigma_1(t)^{-\mu}\mathrm{Op}_M^{\gamma-\frac{n^{\mathbf{1}}}{2}}(h^{\mathbf{1}})(z,\zeta)\sigma_0(t) \\ &\quad + (1-\sigma_1(t))a_{\mathrm{int}}(z,\zeta)(1-\sigma_2(t)) + (m^{\mathbf{1}}+g^{\mathbf{1}})(z,\zeta). \end{aligned} \tag{6.16}$$

Theorem 6.1.

$$R^{\mathbf{1},\mu}_{\mathrm{trad}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,\zeta},\boldsymbol{g},\boldsymbol{g}^{\mathbf{1}}) \quad \text{and} \quad R^{\mathbf{1},\mu}_{\mathrm{new}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,\zeta},\boldsymbol{g},\boldsymbol{g}^{\mathbf{1}}) \quad \text{are equivalent.}$$

The ideas for proving Theorem 6.1 are as follows:

For convenience we drop variables z. Also the smoothing Mellin part $m^{\mathbf{1}}$ will be ignored because it is not involved in the proof of Theorem 6.1. Set

$$a^{\mathbf{1}}_M(\zeta) := t^{-\mu}\mathrm{Op}_M^{\gamma-\frac{n^{\mathbf{1}}}{2}}(h^{\mathbf{1}})(\zeta) \tag{6.17}$$

for $n^{\mathbf{1}} = \dim B$ and

$$h^{\mathbf{1}}(t,v,\zeta) \in C^\infty_{[0,T]}(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}_v}(B,\boldsymbol{g};\mathbb{R}^{q^{\mathbf{1}}}_{\tilde{\zeta}}))|_{\tilde{\zeta}=t\zeta}. \tag{6.18}$$

The symbol $a^{\mathbf{1}}_{\mathrm{new}}(\zeta) \in R^{\mathbf{1},\mu}_{\mathrm{new}}(\mathbb{R}^{q^{\mathbf{1}}},\boldsymbol{g},\boldsymbol{g}^{\mathbf{1}})$ given by formula (6.16) under dropped variable z and Mellin operator $m^{\mathbf{1}}$ takes the form

$$a^{\mathbf{1}}_{\mathrm{new}}(\zeta) = \sigma_1(t)a^{\mathbf{1}}_M(\zeta)\sigma_0(t) + (1-\sigma_1(t))a^{\mathbf{1}}_{\mathrm{int}}(\zeta)(1-\sigma_2(t)) + g^{\mathbf{1}}(\zeta). \tag{6.19}$$

Let us write

$$\omega_{i,\zeta}(t) := \omega_i(t[\zeta]), \quad i = 0,1,2, \tag{6.20}$$

and set

$$p^{\mathbf{1}}_\psi(\zeta) := t^{-\mu}\mathrm{Op}_t(p^{\mathbf{1}})(\zeta). \tag{6.21}$$

We have

Proposition 6.2.

$$a^{\mathbf{1}}_M(\zeta) = \omega_{1,\zeta}a^{\mathbf{1}}_M(\zeta)\omega_{0,\zeta} + (1-\omega_{1,\zeta})a^{\mathbf{1}}_M(\zeta)(1-\omega_{2,\zeta}) + g^{\mathbf{1}}_\infty(\zeta) \tag{6.22}$$

for

$$g^{\mathbf{1}}_\infty(\zeta) = \omega_{1,\zeta}a^{\mathbf{1}}_M(\zeta)(1-\omega_{0,\zeta}) + (1-\omega_{1,\zeta})a^{\mathbf{1}}_M(\zeta)\omega_{2,\zeta}. \tag{6.23}$$

Then, according to notation in (6.8) and by virtue of formula (6.22) we have

$$\begin{aligned} &a^{\mathbf{1}}_{\mathrm{trad}}(\zeta) \\ &= \sigma_1(\omega_{1,\zeta}a^{\mathbf{1}}_M(\zeta)\omega_{0,\zeta}+(1-\omega_{1,\zeta})p^{\mathbf{1}}_\psi(\zeta)(1-\omega_{2,\zeta}))\sigma_0+(1-\sigma_1)a^{\mathbf{1}}_{\mathrm{int}}(\zeta)(1-\sigma_2)+g^{\mathbf{1}}(\zeta) \\ &= \sigma_1(\omega_{1,\zeta}a_M(\zeta)\omega_{0,\zeta} + (1-\omega_{1,\zeta})a^{\mathbf{1}}_M(\zeta)(1-\omega_{2,\zeta}) + g^{\mathbf{1}}_\infty(\zeta))\sigma_0 \\ &+ \sigma_1((1-\omega_{1,\zeta})[p^{\mathbf{1}}_\psi(\zeta) - a^{\mathbf{1}}_M(\zeta)](1-\omega_{2,\zeta}))\sigma_0 + (1-\sigma_1)a^{\mathbf{1}}_{\mathrm{int}}(\zeta)(1-\sigma_2) + g^{\mathbf{1}}(\zeta) \end{aligned} \tag{6.24}$$

Thus we have

$$\begin{aligned} &a^{\mathbf{1}}_{\mathrm{trad}}(\zeta) \\ &= \sigma_1(\omega_{1,\zeta}a^{\mathbf{1}}_M(\zeta)\omega_{0,\zeta}+(1-\omega_{1,\zeta})p^{\mathbf{1}}_\psi(\zeta)(1-\omega_{2,\zeta}))\sigma_0+(1-\sigma_1)a^{\mathbf{1}}_{\mathrm{int}}(\zeta)(1-\sigma_2) \\ &+ g^{\mathbf{1}}(\zeta) + g^{\mathbf{1}}_\infty(\zeta) = \sigma_1(a^{\mathbf{1}}_M(\zeta))\sigma_0 + (1-\sigma_1)a^{\mathbf{1}}_{\mathrm{int}}(\zeta)(1-\sigma_2) + g^{\mathbf{1}}(\zeta) + g^{\mathbf{1}}_\infty(\zeta) \\ &+ \sigma_1\big((1-\omega_{1,\zeta})[p^{\mathbf{1}}_\psi(\zeta) - a^{\mathbf{1}}_M(\zeta)](1-\omega_{2,\zeta})\big)\sigma_0 + g^{\mathbf{1}}(\zeta) + g^{\mathbf{1}}_\infty(\zeta), \end{aligned} \tag{6.25}$$

i.e.,

$$a^{\mathbf{1}}_{\mathrm{trad}}(\zeta) = a^{\mathbf{1}}_{\mathrm{new}}(\zeta) + \sigma_1\big((1-\omega_{1,\zeta})[p^{\mathbf{1}}_\psi(\zeta) - a^{\mathbf{1}}_M(\zeta)](1-\omega_{2,\zeta})\big)\sigma_0. \tag{6.26}$$

Note that the summands $a^{\mathbf{1}}_{\mathrm{int}}(\zeta)$ may be different in different formulas, but they all belong to $L^\mu(\mathbb{R}_+ \times B, \boldsymbol{g}; \mathbb{R}^{q^{\mathbf{1}}}_\zeta)_0$, cf. notation (6.13). The right-hand side of (6.26) belongs to the class of flat Green symbols (4.21) for $\nu = \mu$. Compared with notation so far it suffices to replace ζ by ζ, λ. Therefore, since such Green symbols are contained in $R^{\mathbf{1},\mu}_{\mathrm{new}}(\mathbb{R}^{2q^{\mathbf{1}}}_{z,\zeta}, \boldsymbol{g}, \boldsymbol{g}^{\mathbf{1}}; \mathbb{R}^d_\lambda)$ anyway, the formulation of spaces of parameter-dependent corner operators

$$L^\mu(M, \boldsymbol{g}, \boldsymbol{g}^{\mathbf{1}}; \mathbb{R}^d_\lambda). \tag{6.27}$$

may focus on the non-smoothing contributions close to Z.

7. Calculus of second singular order

We often ignore the parameter λ which corresponds to the case $d = 0$. Nevertheless, arbitrary d is meaningful when we want to pass to corner singularities of higher order. This is, of course, another voluminous program.

Let us introduce global weighted Sobolev spaces

$$H^{s,\beta,\gamma}(M), \quad s, \beta, \gamma \in \mathbb{R}, \tag{7.1}$$

locally near Z modeled on $\mathcal{W}^s(\mathbb{R}^{q^{\mathbf{1}}}; \mathcal{K}^{s,\beta,\gamma}(B^\wedge))$ where $H^{s,\beta,\gamma}(M)|_{M\setminus Z} \subseteq H^{s,\beta}_{\mathrm{loc}}(M \setminus Z)$. The space (6.27) is defined to be the set of all families of operators

$$A^{\mathbf{1}}(\lambda) = \omega_{\mathrm{glob}} A_{\mathrm{c}}(\lambda)\omega'_{\mathrm{glob}} + (1 - \omega_{\mathrm{glob}})A^{\mathbf{1}}_{\mathrm{int}}(\lambda)(1 - \omega''_{\mathrm{glob}}) + C^{\mathbf{1}}(\lambda) \tag{7.2}$$

for global cut-off functions on M

$$\omega''_{\mathrm{glob}} \prec \omega_{\mathrm{glob}} \prec \omega'_{\mathrm{glob}} \tag{7.3}$$

where $A^{\mathbf{1}}_{\mathrm{c}}(\lambda)$ is locally close to the edge Z defined by $\mathrm{Op}_z(a^{\mathbf{1}})(\lambda)$ for symbols $a^{\mathbf{1}}(z, \zeta, \lambda)$ of the form (6.16). Concerning $A^{\mathbf{1}}_{\mathrm{int}}(\lambda)$ we assume

$$A^{\mathbf{1}}_{\mathrm{int}}(\lambda)|_{M\setminus Z} \in L^\mu(2\mathbb{M}, \boldsymbol{g}; \mathbb{R}^d)|_{M\setminus Z}$$

for the double $2\mathbb{M} \in \mathfrak{M}_1$ of the stretched space $\mathbb{M}$. Moreover, we ask

$$C(\lambda) \in L^{-\infty}(M, \boldsymbol{g}, \boldsymbol{g}^{\mathbf{1}}; \mathbb{R}^d) := \mathcal{S}(\mathbb{R}^d, L^{-\infty}(M, \boldsymbol{g}, \boldsymbol{g}^{\mathbf{1}})) = \bigcap_{m\in\mathbb{N}} L^{\mu-m}(M, \boldsymbol{g}, \boldsymbol{g}^{\mathbf{1}}; \mathbb{R}^d).$$

Here $C \in L^{-\infty}(M, \boldsymbol{g}, \boldsymbol{g}^{\mathbf{1}})$ means continuity

$$C : H^{s,\beta,\gamma}(M) \to H^{\infty,\beta-\mu+\varepsilon,\gamma-\mu+\varepsilon}(M)$$

together with the condition for its formal adjoint $C^* : H^{s,-\beta+\mu,-\gamma+\mu}(M) \to H^{\infty,-\beta+\varepsilon,-\gamma+\varepsilon}(M)$ with respect to the reference scalar product from $H^{0,0,0}(M)$ for every s and some $\varepsilon > 0$.

Corner operators $A(\lambda)$ in (6.27) families of induce continuous maps

$$A(\lambda) : H^{s,\beta,\gamma}(M) \to H^{s-\mu,\beta-\mu,\gamma-\mu}(M) \tag{7.4}$$

for every $s \in \mathbb{R}$. According to the stratification (2.11) of M for $k = 2$ they have a hierarchy of parameter-dependent principal symbols

$$\sigma(A)(\lambda) = (\sigma_0(A)(\lambda), \sigma_1(A)(\lambda), \sigma_2(A)(\lambda)) \tag{7.5}$$

which is of analogous meaning as that in the edge calculus for the case of a manifold B with edge. More precisely, $\sigma_0(A)(\lambda)$ is the interor parameter-dependent homogeneous principal symbol of $A(\lambda)$ interpreted as an element

$$A(\lambda) \in L^{\mu}_{\mathrm{cl}}(s_0(M); \mathbb{R}^d). \tag{7.6}$$

Moreover, $A(\lambda)$ also represents an element

$$A(\lambda) \in L^{\mu}(s_1(M), \boldsymbol{g}; \mathbb{R}^d) \tag{7.7}$$

on the (non-compact) manifold $s_1(M)$ with edge of dimension $\boldsymbol{q} = 1 + q + q^{\mathbf{1}}$, and as such it has a parameter-dependent homogenoeous principal edge symbol which is operator-valued, as an element

$$\sigma_1(A(\lambda))(\boldsymbol{y}, \boldsymbol{\eta}) \in S^{(\mu)}(\mathbb{R}^{\boldsymbol{q}}_{\boldsymbol{y}} \times (\mathbb{R}^{\boldsymbol{q}+d}_{\boldsymbol{\eta},\lambda} \setminus \{0\}); \mathcal{K}^{s,\beta}(X^\wedge), \mathcal{K}^{s-\mu,\beta-\mu}(X^\wedge)), \tag{7.8}$$

expressed in local coordinates $\boldsymbol{y} \in \mathbb{R}^{\boldsymbol{q}}$ of the edge of $s_1(M)$ and 0 indicating $(\boldsymbol{\eta}, \lambda) = 0$. For the third component we have

$$\sigma_2(A(\lambda))(z, \zeta) \in S^{(\mu)}(\mathbb{R}^{q^{\mathbf{1}}}_z \times (\mathbb{R}^{q^{\mathbf{1}}+d}_{\zeta,\lambda} \setminus \{0\}); \mathcal{K}^{s,\beta,\gamma}(B^\wedge), \mathcal{K}^{s-\mu,\beta-\mu,\gamma-\mu}(B^\wedge)) \tag{7.9}$$

in local coordinates on Z, with 0 indicating $(\zeta, \lambda) = 0$.

Theorem 7.1. *Assume that $A(\lambda) \in L^{\mu}(M, \boldsymbol{a}, \boldsymbol{a}^{\mathbf{1}}; \mathbb{R}^d_\lambda)$ and $B(\lambda) \in L^{\nu}(M, \boldsymbol{b}, \boldsymbol{b}^{\mathbf{1}}; \mathbb{R}^d_\lambda)$ for weight data*

$$\boldsymbol{a} = (\beta - \nu, \beta - (\mu + \nu)), \boldsymbol{a}^{\mathbf{1}} = (\gamma - \nu, \gamma - (\mu + \nu)), \quad \boldsymbol{b} = (\beta, \beta - \nu), \boldsymbol{b}^{\mathbf{1}} = (\gamma, \gamma - \nu)$$

Then we have $A(\lambda)B(\lambda) \in L^{\mu+\nu}(M, \boldsymbol{a} \circ \boldsymbol{b}, \boldsymbol{a}^{\mathbf{1}} \circ \boldsymbol{b}^{\mathbf{1}}; \mathbb{R}^d_\lambda)$ with obvious meaning of "$\circ$" and $\sigma(A)(\lambda)\sigma(B)(\lambda) = \sigma\big((A)(\lambda)(B)(\lambda)\big)$ with componentwise composition, cf. notation (7.5).

The proof follows methods of analogous character as in [12].

Conclusions on ellipticity with respect to the symbol hierarchies, Fredhomness in weighted Sobolev spaces and the existence of parametrices within the calculus are of analogous structure as in the case of operators on manifolds with edge, see, for instance [32], [35], and the technique from [12].

Observe that operator theories including ellipticity on spaces in $\mathfrak{M}_k$ for $k > 2$ are of iterative character, see [2], [3]. In addition they may be enlarged to a calculus of operator matrices with extra edge conditions of trace and potential type along the various strata of M. It would be an interesting task to perform corresponding elliptic theories of complexes, see, in particular, the article [38], and the monograph [21], including the remarks there about Toeplitz-analogues of elliptic theories which are a natural extension of the present consideration for corner theories with extra interface conditions.

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