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A NUMERICAL METHOD FOR SOLVING NONLINEAR VOLTERRA–FREDHOLM INTEGRAL EQUATIONS

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Abstract. In this paper, we introduce a numerical method for solving nonlinear Volterra–Fredholm integral equations. Our method consists of two steps. First, we define a discretized form of the integral equation by quadrature methods. We then propose an iterative method, which is based on a hybrid of the method of contractive mapping and parameter continuation method, to solve the perturbed system of nonlinear equations obtained from discretization of the considered problem. Finally, some examples are given to demonstrate the validity and applicability of our method.

1. Introduction

Many problems which arise in mathematics, physics, biology, etc., lead to integral equations. Volterra–Fredholm integral equations play an important role in the theory of integral equations. Since Volterra–Fredholm integral equations are usually difficult to get their exact solution, therefore, many authors have worked on analytical methods and numerical methods for approximate solution of this kind of equations. For example, Adomian decomposition method was used to solve Volterra–Fredholm integral equations in [23]. H. Brunner [4] and P. J. Kauthen [8] have employed the Collocation method to solve mixed Volterra–Fredholm integral equations. The numerical solutions of the nonlinear Volterra–Fredholm integral equations by using Homotopy perturbation method was introduced in [7]. Y. Mahmoudi [11] and S. Yalçinbaş [25] developed the Taylor polynomial solutions for the nonlinear Volterra–Fredholm integral equations. In [16], Y. Ordokhani applied the rationalized Haar functions to solve nonlinear Volterra–Fredholm–Hammerstein integral equations. Recently, M. Zarebnia [26] used the sinc functions to solve the nonlinear Volterra–Fredholm integral equations. F. Mirzaee and A. A. Hoseini [13] obtained an approximate solution for the nonlinear Volterra–Fredholm integral equations by the hybrid of block–pulse function and Taylor series (HBT). An iterative scheme for extracting approximate solutions of the Volterra–Fredholm integral equations has been presented by A. H. Borzabadi and M. Heidari [3]. J. Xie et al. [24] developed the numerical technique based on block–pulse functions (BPFs) to approximate the solutions of nonlinear Volterra–Fredholm–Hammerstein integral equations in two–dimensional spaces.

In this paper, we intend to present a numerical method for approximating the

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solution of nonlinear Volterra–Fredholm integral equation as follows

$$x(t) + \int_a^t K_1(t,s,x(s))ds + \int_a^b K_2(t,s,x(s))ds = g(t), \ \ a \le t \le b, \tag{1.1}$$

where $x(t)$ is an unknown function that will be determined, $g(t), K_i(t,s,x)$, $i = 1,2$ are known functions and a, b are known constants. At first, we use one of frequently used quadrature methods to reduce the equation (1.1) into a perturbed system of nonlinear equations. For further information on quadrature methods in this respect, see [5, 9, 17]. Next, we propose an iterative method to solve the obtained perturbed system of nonlinear equations. The method is based on a hybrid of the method of contractive mapping and parameter continuation method. Parameter continuation method was suggested and developed by S. N. Bernstein [2] and J. Schauder [10]. Later on, V. A. Trenoghin [18–21] has developed a generalized variants of the parameter continuation method and used to prove the invertibility of nonlinear operators, which map a metric space or a weak metric space into a Banach space. Y. L. Gaponenko [6] proposed and justifed the parameter continuation method for solving operator equations of the second kind with a Lipschitz - continuous and monotone operator, which operates in an arbitrary Banach space. K. V. Ninh [14,15] has studied parameter continuation method for solving the operator equations of the second kind with a sum of two operators. In [22], V. G. Vetekha presented the application of parameter continuation method to solving the boundary value problem for the ordinary differential equations of second order. Parameter continuation method has some advantages that encourage us to use it. Firstly, the properties of contractions such as iteration, error estimates are used to find approximate solutions and estimate the errors of approximate solutions. Furthermore, this method is very simple to apply and to make an algorithm.

The paper is organized as follows. In Section 2, the parameter continuation method for solving operator equations of the second kind is briefly presented. In this section, we recall some definitions and results that will be useful in the sequel. In Section 3, we transform the equation (1.1) into a perturbed system of nonlinear equations. Then we discuss the existence and uniqueness of the solution of the obtained perturbed system of nonlinear equations and prove that its solution converges to the exact solution of the problem. We also study approximate solutions of the perturbed system of nonlinear equations and their error estimates. Some illustrative examples are given in Section 4 to illustrate the efficiency of the introduced method. Finally, Section 5 draws some conclusions from the paper.

2. Parameter continuation method for solving operator equations of the second kind

In this section, we recall some definitions and results which we will use in the sequel. For details, we refer to [6].

Let X be a Banach space and A be a mapping, which operates in the space X. Consider the operator equation of the second kind

$$x + A(x) = f. \tag{2.1}$$

Definition 2.1 ([6])**.** The mapping A, which operates in the Banach space X is called monotone if for any elements $x_1, x_2 \in X$ and any $\varepsilon > 0$ the following inequality holds

$$\|x_1 - x_2 + \varepsilon\,[A(x_1) - A(x_2)]\,\| \geq \|x_1 - x_2\|. \tag{2.2}$$

Remark 2.2 ([6])**.** If X is Hilbert space then the condition of monotonicity (2.2) is equivalent to the classical condition

$$\langle A(x_1) - A(x_2), x_1 - x_2\rangle \geq 0,\ \forall x_1, x_2 \in X,$$

where $\langle , \rangle$ is an inner product in the Hilbert space X.

Lemma 2.3 ([6])**.** *Assume that A is a monotone mapping which operates in the Banach space X. Then for any elements $x_1, x_2 \in X$ and any positive numbers $\varepsilon_1, \varepsilon_2, 0 < \varepsilon_1 \leq \varepsilon_2 \leq 1$, the following inequality holds*

$$\|x_1 - x_2 + \varepsilon_1\,[A(x_1) - A(x_2)]\,\| \leq \|x_1 - x_2 + \varepsilon_2\,[A(x_1) - A(x_2)]\,\|.$$

The basic idea of the parameter continuation method for solving operator equations of the second kind (2.1) is as follows. Consider a one-parametric family of equations

$$x + \varepsilon A(x) = f, 0 \leq \varepsilon \leq 1,$$

which when $\varepsilon = 0$ gives the trivial equation $x = f$ and when $\varepsilon = 1$ gives the initial equation (2.1). Dividing $[0, 1]$ into N equal parts with N is a natural number such that $q = L\varepsilon_0 < 1, \varepsilon_0 = \frac{1}{N}$, where L is Lipschitz coefficient of the operator A. After $N - 1$ changes of variables

$$u = x + \varepsilon_0 A(x) \equiv G_1(x), \tag{2.3a}$$

$$v = u + \varepsilon_0 A G_1^{-1}(u) \equiv G_2(u), \tag{2.3b}$$

$$\ldots, \tag{2.3c}$$

$$y = \omega + \varepsilon_0 A G_1^{-1} \cdots G_{N-2}^{-1}(\omega) \equiv G_{N-1}(\omega), \tag{2.3d}$$

we construct intermediate equations with contractive operators in new variables. By virtue of the monotonicity and Lipschitz continuity of the operator A, contraction coefficients of these contractive operators equal q. By shifting the parameter ε step by step ε_0 from 0 to 1 we can verify that the equation (2.1) has a unique solution.

Theorem 2.4 ([6])**.** *Suppose that the mapping A, which operates in the Banach space X is Lipschitz - continuous and monotone. Then the equation* (2.1) *has a unique solution for any element $f \in X$.*

To find approximate solutions of the equation (2.1), Y. L. Gaponenko has constructed the following iteration process

$$x_{k+1} = \underbrace{-\frac{1}{N}A(x_k) - \frac{1}{N}A(x_l) - \cdots - \frac{1}{N}A(x_p)}_{N\ terms} + f,\ k, l, \ldots, p = 0, 1, 2, \ldots. \tag{2.4}$$

The symbolic notation (2.4) should be understood as the following iteration processes, which consist of N iteration processes

$$x_{k+1} = -\varepsilon_0 A(x_k) + u_l,\ k = 0, 1, 2, \dots, \tag{2.5a}$$

$$u_{l+1} = -\varepsilon_0 A G_1^{-1}(u_l) + v_c,\ l = 0, 1, 2, \dots, \tag{2.5b}$$

$$\dots, \tag{2.5c}$$

$$y_{p+1} = -\varepsilon_0 A G_1^{-1} \cdots G_{N-1}^{-1}(y_p) + f,\ p = 0, 1, 2, \dots. \tag{2.5d}$$

For simplicity, assume that $A(0) = 0$ and the number of steps in each iteration scheme of the iteration process (2.4) is the same and equals n_0. Denote $x(n_0, N) \equiv x_{n_0}$ as the approximate solutions of the equation (2.1), which is constructed by the iteration process (2.4). In this case, Y. L. Gaponenko received the error estimations of approximate solutions of the equation (2.1), which are presented in the following theorem.

Theorem 2.5 ([6]). *Assume that the conditions of Theorem 2.4 are satisfied. Then the sequence of approximate solutions* $\{x(n_0, N)\}$, $n_0 = 1, 2, \dots$ *constructed by iteration process* (2.4) *converges to the exact solution* x^* *of the equation* (2.1). *Moreover, the following estimates hold*

$$\|x(n_0, N) - x^*\| \le \frac{q^{n_0+1}}{1-q} \frac{e^{qN} - 1}{e^q - 1} \|f\|, \tag{2.6}$$

where L *is Lipschitz coefficient of the operator* A, N *is the smallest natural number such that* $q = \frac{L}{N} < 1, n_0 = 1, 2, \dots$.

3. Main results

Now, the nonlinear Volterra–Fredholm integral equation (1.1) will be investigated under the assumptions:

(i) $K_1(t, s, x)$ satisfies a Lipschitz condition of the type

$$|K_1(t, s, x) - K_1(t, s, \overline{x})| \le |\psi(t, s)|\, |x - \overline{x}|,$$

for all $a \le t, s \le b$ and for all reals $x, \overline{x}$, where $\int_a^t |\psi(t, s)|^2 ds \le Q^2(t)$ and $\int_a^b Q^2(t) dt \le M^2 < +\infty$;

(ii) $K_2(t, s, x)$ satisfies a Lipschitz condition of the type

$$|K_2(t, s, x) - K_2(t, s, \overline{x})| \le |\phi(t, s)|\, |x - \overline{x}|,$$

for all $a \le t, s \le b$ and for all reals $x, \overline{x}$, where $\int_a^b \int_a^b |\phi(t, s)|^2\, ds dt = L^2 < +\infty$;

(iii) $K_2(t, s, x)$ satisfies the condition

$$\int_a^b \left\{ \int_a^b [K_2(t, s, x(s)) - K_2(t, s, \overline{x}(s))]\, ds \right\} [x(t) - \overline{x}(t)]\, dt > 0,$$

for all $x(t), \overline{x}(t) \in L^2[a, b]$ with $x(t) \ne \overline{x}(t)$.

At the beginning, we transform the equation (1.1) into a discretized form. Let $\Pi = \{a = t_0, t_1, \ldots, t_{n-1}, t_n = b\}$ be an equidistant partition of $[a, b]$ where $h = t_{i+1} - t_i,\ i = 0, 1, \ldots, n-1$ is the discretization parameter of the partition. Now, if $x^*(t)$ is an analytical solution of (1.1), then for the partition Π on $[a, b]$, we have

$$x^*(t_i) + \int_a^{t_i} K_1(t_i, s, x^*(s))ds + \int_a^b K_2(t_i, s, x^*(s))ds = g(t_i),\ i = 0, 1, \ldots, n. \tag{3.1}$$

In (3.1), the integral term can be estimated by a numerical method of integration, e.g. Newton–Cotes methods. Therefore, by taking equidistant partition Π, as above with $h = s_{i+1} - s_i,\ i = 0, 1, \ldots, n-1$ and also the known weights $w_{i_j},\ j = 0, 1, \ldots, i$ for interval $[a, t_i]$ and $w_r,\ r = 0, 1, \ldots, n$ for interval $[a, b]$, equality (3.1) can be written as

$$\begin{aligned} &x_i^* + \sum_{j=0}^{i} w_{i_j} K_1(t_i, s_j, x_j^*) + O(h^{\nu_1}) + \sum_{r=0}^{n} w_r K_2(t_i, s_r, x_r^*) + O(h^{\nu_2}) = g_i, \\ &i = 0, 1, \ldots, n, \end{aligned} \tag{3.2}$$

where $x_i^* = x^*(t_i), g_i = g(t_i),\ i = 0, 1, \ldots, n$ and ν_1, ν_2 depend upon the used method of Newton–Cotes for estimating the integrals in (3.1). From (3.2), we have

$$x_i^* + \sum_{j=0}^{i} w_{i_j} K_1(t_i, s_j, x_j^*) + \sum_{r=0}^{n} w_r K_2(t_i, s_r, x_r^*) + O(h^{\nu}) = g_i,\ i = 0, 1, \ldots, n, \tag{3.3}$$

where $\nu = \min\{\nu_1, \nu_2\}$.

For partition Π, we consider a perturbed system of nonlinear equations obtained by neglecting the truncation error of (3.1) as follows

$$\xi_i + \sum_{j=0}^{i} w_{i_j} K_1(t_i, s_j, \xi_j) + \sum_{r=0}^{n} w_r K_2(t_i, s_r, \xi_r) = g_i,\ i = 0, 1, \ldots, n. \tag{3.4}$$

The perturbed system of nonlinear equations (3.4) can be rewritten as

$$\xi + \Phi(\xi) + F(\xi) = g, \tag{3.5}$$

where $\xi = (\xi_0, \xi_1, \ldots, \xi_n)^T, g = (g_0, g_1, \ldots, g_n)^T, \Phi(\xi) = (\varphi_0(\xi), \varphi_1(\xi), \ldots,\ \varphi_n(\xi))^T$ and $F(\xi) = (f_0(\xi), f_1(\xi), \ldots, f_n(\xi))^T$ with

$$\varphi_i(\xi) = \sum_{j=0}^{i} w_{i_j} K_1(t_i, s_j, \xi_j),\ f_i(\xi) = \sum_{r=0}^{n} w_r K_2(t_i, s_r, \xi_r),\ i = 0, 1, \ldots, n.$$

For partition Π and the known weights $w_i,\ i = 0, 1, \ldots, n$, we define an inner product in $\mathbb{R}^{n+1}$ by

$$\langle \xi, \overline{\xi} \rangle = \sum_{i=0}^{n} w_i \xi_i \overline{\xi}_i,\ \forall \xi = (\xi_0, \xi_1, \ldots, \xi_n)^T, \overline{\xi} = (\overline{\xi}_0, \overline{\xi}_1, \ldots, \overline{\xi}_n)^T \in \mathbb{R}^{n+1}.$$

This inner product induces the norm

$$\|\xi\| = \sqrt{\langle \xi, \xi \rangle} = \left(\sum_{i=0}^{n} w_i |\xi_i|^2 \right)^{\frac{1}{2}}.$$

The following proposition gives us the property of the mapping Φ.

Proposition 3.1. *Let the assumption* (i) *be satisfied. Then*

$$\|\Phi^m(\xi) - \Phi^m(\overline{\xi})\| \leq \frac{M^m}{\sqrt{(m-1)!}} \|\xi - \overline{\xi}\|, \ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1} \tag{3.6}$$

for some positive integer m.

Proof. Define the operator V as

$$(Vx)(t) = \int_a^t K_1(t, s, x(s)) ds, \ \forall x(t) \in L^2[a, b].$$

First, we prove that (3.6) is hold for $m = 1$. From (i), we have

$$\begin{aligned} |(Vx)(t) - (V\overline{x})(t)| &= \left| \int_a^t [K_1(t, s, x(s)) - K_1(t, s, \overline{x}(s))] \, ds \right| \\ &\leq \int_a^t |K_1(t, s, x(s)) - K_1(t, s, \overline{x}(s))| \, ds \\ &\leq \int_a^t |\psi(t, s)| \, |x(s) - \overline{x}(s)| \, ds \end{aligned}$$

for all $x(t), \overline{x}(t) \in L^2[a, b]$. From this and Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} |(Vx)(t) - (V\overline{x})(t)|^2 &\leq \int_a^t |\psi(t, s)|^2 \, ds \int_a^t |x(s) - \overline{x}(s)|^2 \, ds \\ &\leq Q^2(t) \int_a^t |x(s) - \overline{x}(s)|^2 \, ds \qquad (3.7) \\ &= Q^2(t) \int_a^t |x(s_1) - \overline{x}(s_1)|^2 \, ds_1 \end{aligned}$$

and hence

$$|(Vx)(t) - (V\overline{x})(t)|^2 \leq Q^2(t) \int_a^b |x(s_1) - \overline{x}(s_1)|^2 \, ds_1.$$

Then we have

$$\int_a^b |(Vx)(t) - (V\overline{x})(t)|^2\, dt \le \int_a^b Q^2(t)dt \int_a^b |x(t) - \overline{x}(t)|^2\, dt$$
$$\le M^2 \int_a^b |x(t) - \overline{x}(t)|^2\, dt.$$

This implies that

$$\int_a^b \left| \int_a^t [K_1(t,s,x(s)) - K_1(t,s,\overline{x}(s))]ds \right|^2 dt - M^2 \int_a^b |x(t) - \overline{x}(t)|^2 dt \le 0$$

for all $x(t), \overline{x}(t) \in L^2[a,b]$. We may assume without loss of generality that

$$\int_a^b \left| \int_a^t [K_1(t,s,x(s)) - K_1(t,s,\overline{x}(s))]ds \right|^2 dt - M^2 \int_a^b |x(t) - \overline{x}(t)|^2 ds < 0 \tag{3.8}$$

for all $x(t), \overline{x}(t) \in L^2[a,b]$ with $x(t) \ne \overline{x}(t)$.

By taking equidistant partition Π, as above with $h = t_{i+1} - t_i,\ i = 0, 1, \ldots, n-1$ and also the known weights $w_i, w_e,\ i, e = 0, 1, \ldots, n$ for interval $[a,b]$, we have

$$\begin{aligned} &\sum_{i=0}^n w_i \left| \int_a^{t_i} [K_1(t_i,s,x(s)) - K_1(t_i,s,\overline{x}(s))]\, ds \right|^2 + O(h^{\eta_1}) \\ &- M^2 \sum_{e=0}^n w_e |x_e - \overline{x}_e|^2 - O(h^{\eta_2}) < 0, \end{aligned} \tag{3.9}$$

where $x_e = x(t_e), \overline{x}_e = \overline{x}(t_e),\ e = 0, 1, \ldots, n$ and η_1, η_2 depend upon the used method of Newton–Cotes for estimating the integrals in (3.8). From (3.9), we have

$$\begin{aligned} &\sum_{i=0}^n w_i \left| \int_a^{t_i} [K_1(t_i,s,x(s)) - K_1(t_i,s,\overline{x}(s))]\, ds \right|^2 \\ &- M^2 \sum_{e=0}^n w_e |x_e - \overline{x}_e|^2 + O(h^{\eta_3}) < 0, \end{aligned} \tag{3.10}$$

where $O(h^{\eta_3}) = O(h^{\eta_1}) - O(h^{\eta_2})$. Therefore, by taking equidistant partition Π, as above with $h = s_{i+1} - s_i,\ i = 0, 1, \ldots, n-1$ and also the known weights $w_{i_j},\ j = 0, 1, \ldots, i$ for interval $[a, t_i]$, the inequality (3.10) can be written as

$$\begin{aligned} &\sum_{i=0}^n w_i \left| \sum_{j=0}^i w_{i_j} [K_1(t_i,s_j,x_j) - K_1(t_i,s_j,\overline{x}_j)] + O(h^{\eta_4}) \right|^2 \\ &- M^2 \sum_{e=0}^n w_e |x_e - \overline{x}_e|^2 + O(h^{\eta_3}) < 0, \end{aligned}$$

where $x_j = x(s_j), \overline{x}_j = \overline{x}(s_j)$ and η_4 depend upon the used method of Newton–Cotes for estimating the integral in (3.10). Hence, for sufficiently large n, we have

$$\sum_{i=0}^{n} w_i \left| \sum_{j=0}^{i} w_{i_j} [K_1(t_i, s_j, \xi_j) - K_1(t_i, s_j, \overline{\xi}_j)] \right|^2 - M^2 \sum_{e=0}^{n} w_e |\xi_e - \overline{\xi}_e|^2 \leq 0$$

for all $\xi, \overline{\xi} \in \mathbb{R}^{n+1}$. That means

$$\|\Phi(\xi) - \Phi(\overline{\xi})\|^2 - M^2 \|\xi - \overline{\xi}\|^2 \leq 0, \ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1}.$$

Hence

$$\|\Phi(\xi) - \Phi(\overline{\xi})\| \leq M \|\xi - \overline{\xi}\|, \ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1}.$$

Consequently, (3.6) is hold for $m = 1$.

We now prove that (3.6) is hold for $m = 2$. From (3.7), we have

$$\begin{aligned} |(V^2 x)(t) - (V^2 \overline{x})(t)|^2 &\leq Q^2(t) \int_a^t |(Vx)(s_1) - (V\overline{x})(s_1)|^2 ds_1 \\ &\leq Q^2(t) \int_a^t Q^2(s_1) ds_1 \int_a^{s_1} |x(s_2) - \overline{x}(s_2)|^2 ds_2 \end{aligned} \tag{3.11}$$

and hence

$$|(V^2 x)(t) - (V^2 \overline{x})(t)|^2 \leq Q^2(t) \int_a^t Q^2(s_1) ds_1 \int_a^b |x(s_2) - \overline{x}(s_2)|^2 ds_2.$$

Then we have

$$\begin{aligned} \int_a^b |(V^2 x)(t) - (V^2 \overline{x})(t)|^2 dt &\leq \int_a^b Q^2(t) dt \int_a^b Q^2(s_1) ds_1 \int_a^b |x(t) - \overline{x}(t)|^2 dt \\ &\leq M^4 \int_a^b |x(t) - \overline{x}(t)|^2 dt. \end{aligned}$$

It follows that

$$\begin{aligned} &\int_a^b \left| \int_a^t [K_1(t, s, (Vx)(s)) - K_1(t, s, (V\overline{x})(s))] ds \right|^2 dt \\ &- M^4 \int_a^b |x(t) - \overline{x}(t)|^2 dt \leq 0 \end{aligned}$$

for all $x(t), \overline{x}(t) \in L^2[a,b]$. We may assume without loss of generality that

$$\begin{aligned} &\int_a^b \left| \int_a^t [K_1(t,s,(Vx)(s)) - K_1(t,s,(V\overline{x})(s))]ds \right|^2 dt \\ &- M^4 \int_a^b |x(t) - \overline{x}(t)|^2 dt < 0 \end{aligned} \tag{3.12}$$

for all $x(t), \overline{x}(t) \in L^2[a,b]$ with $x(t) \neq \overline{x}(t)$.

By taking equidistant partition Π, as above with $h = t_{i+1} - t_i,\ i = 0, 1, \ldots, n-1$ and also the known weights $w_i, w_e,\ i, e = 0, 1, \ldots, n$ for interval $[a,b]$, we have

$$\begin{aligned} &\sum_{i=0}^{n} w_i \left| \int_a^{t_i} [K_1(t_i,s,(Vx)(s)) - K_1(t_i,s,(V\overline{x})(s))]\, ds \right|^2 + O(h^{\eta_5}) \\ &- M^4 \sum_{e=0}^{n} w_e |x_e - \overline{x}_e|^2 - O(h^{\eta_6}) < 0, \end{aligned} \tag{3.13}$$

where $x_e = x(t_e), \overline{x}_e = \overline{x}(t_e),\ e = 0, 1, \ldots, n$ and η_5, η_6 depend upon the used method of Newton–Cotes for estimating the integrals in (3.12). From (3.13), we have

$$\begin{aligned} &\sum_{i=0}^{n} w_i \left| \int_a^{t_i} [K_1(t_i,s,(Vx)(s)) - K_1(t_i,s,(V\overline{x})(s))]\, ds \right|^2 \\ &- M^4 \sum_{e=0}^{n} w_e |x_e - \overline{x}_e|^2 + O(h^{\eta_7}) < 0, \end{aligned} \tag{3.14}$$

where $O(h^{\eta_7}) = O(h^{\eta_5}) - O(h^{\eta_6})$. Therefore, by taking equidistant partition Π, as above with $h = s_{i+1} - s_i,\ i = 0, 1, \ldots, n-1$ and also the known weights $w_{i_j},\ j = 0, 1, \ldots, i$ for interval $[a, t_i]$, the inequality (3.14) can be written as

$$\begin{aligned} &\sum_{i=0}^{n} w_i \left| \sum_{j=0}^{i} w_{i_j} [K_1(t_i, s_j, \int_a^{s_j} K_1(s_j, \varsigma, x(\varsigma)) d\varsigma) \right. \\ &\qquad \left. - K_1(t_i, s_j, \int_a^{s_j} K_1(s_j, \varsigma, \overline{x}(\varsigma)) d\varsigma)] + O(h^{\eta_8}) \right|^2 \\ &- M^2 \sum_{e=0}^{n} w_e |x_e - \overline{x}_e|^2 + O(h^{\eta_7}) < 0, \end{aligned} \tag{3.15}$$

where η_8 depend upon the used method of Newton–Cotes for estimating the integral in (3.14). By taking equidistant partition Π, as above with $h = \varsigma_{i+1} - \varsigma_i,\ i = 0, 1, \ldots, n-1$ and also the known weights $w_{j_\rho}, w_{j_{\rho'}},\ \rho, \rho' = 0, 1, \ldots, j$ for interval

$[a, s_j]$, the inequality (3.15) can be written as

$$\sum_{i=0}^{n} w_i \Bigg| \sum_{j=0}^{i} w_{i_j} [K_1(t_i, s_j, \sum_{\rho=0}^{j} w_{j_\rho} K_1(s_j, \varsigma_\rho, x_\rho) + O(h^{\eta_9}))$$
$$- K_1(t_i, s_j, \sum_{\rho'=0}^{j} w_{j_{\rho'}} K_1(s_j, \varsigma_{\rho'}, \overline{x}_{\rho'}) + O(h^{\eta_{10}}))] + O(h^{\eta_8}) \Bigg|^2$$
$$- M^2 \sum_{e=0}^{n} w_e |x_e - \overline{x}_e|^2 + O(h^{\eta_7}) < 0,$$

where $x_\rho = x(\varsigma_\rho), \overline{x}_{\rho'} = \overline{x}(\varsigma_{\rho'}),\ \rho, \rho' = 0, 1, \ldots, j$ and η_9, η_{10} depend upon the used method of Newton–Cotes for estimating the integrals in (3.15). Therefore, for sufficiently large n, we have

$$\sum_{i=0}^{n} w_i \Bigg| \sum_{j=0}^{i} w_{i_j} [K_1(t_i, s_j, \sum_{\rho=0}^{j} w_{j_\rho} K_1(s_j, \varsigma_\rho, \xi_\rho))$$
$$- K_1(t_i, s_j, \sum_{\rho'=0}^{j} w_{j_{\rho'}} K_1(s_j, \varsigma_{\rho'}, \overline{\xi}_{\rho'}))] \Bigg|^2 - M^2 \sum_{e=0}^{n} w_e |\xi_e - \overline{\xi}_e|^2 \leq 0$$

for all $\xi, \overline{\xi} \in \mathbb{R}^{n+1}$. That means

$$\|\Phi^2(\xi) - \Phi^2(\overline{\xi})\|^2 - M^4 \|\xi - \overline{\xi}\|^2 \leq 0,\ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1}.$$

Thus

$$\|\Phi^2(\xi) - \Phi^2(\overline{\xi})\| \leq M^2 \|\xi - \overline{\xi}\|,\ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1}.$$

Consequently, (3.6) is hold for $m = 2$.

Next, we shall prove that (3.6) is hold for some integer $m \geq 3$. Continuing the process from (3.7) and (3.11), we obtain

$$|(V^m x)(t) - (V^m \overline{x})(t)|^2 \leq Q^2(t) \int_a^t |(V^{m-1} x)(s_1) - (V^{m-1} \overline{x})(s_1)|^2 ds_1 \leq Q^2(t)$$
$$\times \int_a^t Q^2(s_1) ds_1 \int_a^{s_1} Q^2(s_2) ds_2 \cdots \int_a^{s_{m-2}} Q^2(s_{m-1}) ds_{m-1} \int_a^{s_{m-1}} |x(s_m) - \overline{x}(s_m)|^2 ds_m$$

and hence

(3.16)
$$|(V^m x)(t) - (V^m \overline{x})(t)|^2 \leq Q^2(t)$$
$$\times \int_a^t Q^2(s_1) ds_1 \int_a^{s_1} Q^2(s_2) ds_2 \cdots \int_a^{s_{m-2}} Q^2(s_{m-1}) ds_{m-1} \int_a^b |x(s_m) - \overline{x}(s_m)|^2 ds_m.$$

By induction, we can show that

$$\begin{aligned}&\int_a^t Q^2(s_1)ds_1 \int_a^{s_1} Q^2(s_2)ds_2 \cdots \int_a^{s_{m-2}} Q^2(s_{m-1})ds_{m-1}\\ &= \frac{1}{(m-1)!}\left(\int_a^t Q^2(s)ds\right)^{m-1}.\end{aligned} \tag{3.17}$$

Combining now (3.16) and (3.17), we get

$$\begin{aligned}&|(V^m x)(t) - (V^m \overline{x})(t)|^2\\ &\le Q^2(t)\frac{1}{(m-1)!}\left(\int_a^t Q^2(s)ds\right)^{m-1}\int_a^b |x(s_m)-\overline{x}(s_m)|^2 ds_m\\ &\le Q^2(t)\frac{1}{(m-1)!}\left(\int_a^b Q^2(s)ds\right)^{m-1}\int_a^b |x(s_m)-\overline{x}(s_m)|^2 ds_m\\ &\le Q^2(t)\frac{M^{2(m-1)}}{(m-1)!}\int_a^b |x(s_m)-\overline{x}(s_m)|^2 ds_m.\end{aligned}$$

Hence

$$\begin{aligned}\int_a^b |(V^m x)(t) - (V^m \overline{x})(t)|^2\, dt &\le \frac{M^{2(m-1)}}{(m-1)!}\int_a^b Q^2(t)dt \int_a^b |x(t)-\overline{x}(t)|^2\, dt\\ &\le \frac{M^{2m}}{(m-1)!}\int_a^b |x(t)-\overline{x}(t)|^2\, dt\end{aligned}$$

for all $x(t), \overline{x}(t) \in L^2[a,b]$. In a similar way as above, we can show that

$$\|\Phi^m(\xi) - \Phi^m(\overline{\xi})\|^2 - \frac{M^{2m}}{(m-1)!}\|\xi - \overline{\xi}\|^2 \le 0,\ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1}.$$

This implies that

$$\|\Phi^m(\xi) - \Phi^m(\overline{\xi})\| \le \frac{M^m}{\sqrt{(m-1)!}}\|\xi - \overline{\xi}\|,\ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1}.$$

Consequently, (3.6) is hold for some integer $m \ge 3$. This completes the proof of the proposition. □

We now give some properties of the mapping F in the following proposition.

Proposition 3.2. *Let the assumptions* (ii) *and* (iii) *be satisfied. Then*

$$\|F(\xi) - F(\overline{\xi})\| \le L\|\xi - \overline{\xi}\|,\ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1} \tag{3.18}$$

and

$$\left\langle F(\xi) - F(\overline{\xi}), \xi - \overline{\xi}\right\rangle > 0,\ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1},\ \xi \ne \overline{\xi}. \tag{3.19}$$

Proof. From (ii), we have

$$\left|\int_a^b [K_2(t,s,x(s)) - K_2(t,s,\overline{x}(s))]ds\right| \le \int_a^b |K_2(t,s,x(s)) - K_2(t,s,\overline{x}(s))|ds$$
$$\le \int_a^b |\phi(t,s)||x(s) - \overline{x}(s)|ds$$

for all $x(t), \overline{x}(t) \in L^2[a,b]$. From this and Cauchy–Schwarz inequality, we obtain

$$\int_a^b \left|\int_a^b [K_2(t,s,x(s)) - K_2(t,s,\overline{x}(s))]ds\right|^2 dt$$
$$\le \int_a^b \int_a^b |\phi(t,s)|^2 ds dt \int_a^b |x(s) - \overline{x}(s)|^2 ds$$
$$= L^2 \int_a^b |x(t) - \overline{x}(t)|^2 dt$$

and hence

$$\int_a^b \left|\int_a^b [K_2(t,s,x(s)) - K_2(t,s,\overline{x}(s))]\, ds\right|^2 dt - L^2 \int_a^b |x(t) - \overline{x}(t)|^2\, dt \le 0.$$

We may assume without loss of generality that

$$\int_a^b \left|\int_a^b [K_2(t,s,x(s)) - K_2(t,s,\overline{x}(s))]\, ds\right|^2 dt - L^2 \int_a^b |x(t) - \overline{x}(t)|^2\, dt < 0 \tag{3.20}$$

for all $x(t), \overline{x}(t) \in L^2[a,b]$ with $x(t) \ne \overline{x}(t)$.

By taking equidistant partition Π, as above with $h = t_{i+1} - t_i,\ i = 0, 1, \ldots, n-1$ and also the known weights $w_i, w_e,\ i, e = 0, 1, \ldots, n$ for interval $[a,b]$, we have

$$\begin{aligned} &\sum_{i=0}^n w_i \left|\int_a^b [K_2(t_i,s,x(s)) - K_2(t_i,s,\overline{x}(s))]\, ds\right|^2 + O(h^{\vartheta_1}) \\ &- L^2 \sum_{e=0}^n w_e |x_e - \overline{x}_e|^2 - O(h^{\vartheta_2}) < 0, \end{aligned} \tag{3.21}$$

where $x_e = x(t_e), \overline{x}_e = \overline{x}(t_e),\ e = 0, 1, \ldots, n$ and ϑ_1, ϑ_2 depend upon the used method of Newton–Cotes for estimating the integrals in (3.20). From (3.21), we

have

$$\sum_{i=0}^{n} w_i \left| \int_a^b [K_2(t_i, s, x(s)) - K_2(t_i, s, \overline{x}(s))]\, ds \right|^2 - L^2 \sum_{e=0}^{n} w_e |x_e - \overline{x}_e|^2 + O(h^{\vartheta_3}) < 0, \tag{3.22}$$

where $O(h^{\vartheta_3}) = O(h^{\vartheta_1}) - O(h^{\vartheta_2})$. Therefore, by taking equidistant partition Π, as above with $h = s_{i+1} - s_i,\ i = 0, 1, \ldots, n-1$ and also the known weights $w_r,\ r = 0, 1, \ldots, n$ for interval $[a, b]$, the inequality (3.22) can be written as

$$\sum_{i=0}^{n} w_i \left| \sum_{r=0}^{n} w_r \left[K_2(t_i, s_r, x_r) - K_2(t_i, s_r, \overline{x}_r)\right] + O(h^{\vartheta_4}) \right|^2 - L^2 \sum_{e=0}^{n} w_e |x_e - \overline{x}_e|^2 + O(h^{\vartheta_3}) < 0,$$

where $x_r = x(s_r), \overline{x}_r = \overline{x}(s_r),\ r = 0, 1, \ldots, n$ and ϑ_4 depend upon the used method of Newton–Cotes for estimating the integral in (3.22). Hence, for sufficiently large n, we have

$$\sum_{i=0}^{n} w_i \left| \sum_{r=0}^{n} w_r \left[K_2(t_i, s_r, \xi_r) - K_2(t_i, s_r, \overline{\xi}_r)\right] \right|^2 - L^2 \sum_{e=0}^{n} w_e |\xi_e - \overline{\xi}_e|^2 \le 0$$

for all $\xi, \overline{\xi} \in \mathbb{R}^{n+1}$. That means

$$\|F(\xi) - F(\overline{\xi})\|^2 - L^2 \|\xi - \overline{\xi}\|^2 \le 0,\ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1}.$$

Hence

$$\|F(\xi) - F(\overline{\xi})\| \le L \|\xi - \overline{\xi}\|, \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1},$$

which proves (3.18).

Let us prove (3.19). From (iii), by taking equidistant partition Π, as above with $h = t_{i+1} - t_i,\ i = 0, 1, \ldots, n-1$, and also the known weights $w_i,\ i = 0, 1, \ldots, n$ for interval $[a, b]$, we obtain

$$\sum_{i=0}^{n} w_i \int_a^b [K_2(t_i, s, x(s)) - K_2(t_i, s, \overline{x}(s))]\, ds\, [x_i - \overline{x}_i] + O(h^{\sigma_1}) > 0, \tag{3.23}$$

where $x_i = x(t_i), \overline{x}_i = \overline{x}(t_i),\ i = 0, 1, \ldots, n$ and σ_1 depend upon the used method of Newton–Cotes for estimating the integral. Therefore, by taking equidistant partition Π, as above with $h = s_{i+1} - s_i,\ i = 0, 1, \ldots, n-1$ and also the known weights $w_r,\ r = 0, 1, \ldots, n$ for interval $[a, b]$, the inequality (3.23) can be written as

$$\sum_{i=0}^{n} w_i \left\{ \sum_{r=0}^{n} w_r [K_2(t_i, s_r, x_r) - K_2(t_i, s_r, \overline{x}_r)] + O(h^{\sigma_2}) \right\} [x_i - \overline{x}_i] + O(h^{\sigma_1}) > 0,$$

where $x_r = x(s_r), \overline{x}_r = \overline{x}(s_r),\ r = 0, 1, \ldots, n$ and σ_2 depend upon the used method of Newton–Cotes for estimating the integral in (3.23). Hence, for sufficiently large n, we have

$$\sum_{i=0}^{n} w_i \left\{ \sum_{r=0}^{n} w_r \left[K_2(t_i, s_r, \xi_r) - K_2(t_i, s_r, \overline{\xi}_r) \right] \right\} [\xi_i - \overline{\xi}_i] > 0$$

for all $\xi, \overline{\xi} \in \mathbb{R}^{n+1}$ with $\xi \neq \overline{\xi}$. That means

$$\left\langle F(\xi) - F(\overline{\xi}), \xi - \overline{\xi} \right\rangle > 0,\ \forall \xi, \overline{\xi} \in \mathbb{R}^{n+1},\ \xi \neq \overline{\xi}.$$

This completes the proof of the proposition. □

In order to prove our main results, we need the following theorems.

Theorem 3.3. *Assume H is a nonempty closed set in a Banach space X and $T : H \to H$ is continuous. Suppose that T^m is a contractive operator for some positive integer m. Then T has a unique fixed point x^* in H. Moreover, the iteration process*

$$x_{k+1} = T(x_k), \quad k = 0, 1, 2, \ldots \tag{3.24}$$

converges to the fixed point x^*.

Proof. For proof see [1] or [12]. □

Theorem 3.4. *Let the assumptions of Theorem 3.3 be satisfied and let $\{x_k\}$, $k = 1, 2, \ldots$ be constructed by iteration process* (3.24)*. Then for $k \geq m$, the following estimates hold*

$$\|x_k - x^*\| \leq \frac{\alpha^{\frac{k-h_0}{m}}}{1-\alpha} \|x_{m+h_0} - x_{h_0}\|, \tag{3.25}$$

where α is the contraction coefficient of the operator $T^m, h_0 \in \{0, 1, \ldots, m-1\}$ is the residual of $\frac{k}{m}$.

Proof. For proof see [12]. □

Now, we shall give the existence and uniqueness of the solution of the perturbed system of nonlinear equations (3.5).

Theorem 3.5. *Let the assumptions* (i)-(iii) *be satisfied. Then the perturbed system of nonlinear equations* (3.5) *has a unique solution for any $g \in \mathbb{R}^{n+1}$.*

Proof. We shall carry out a change of variable

$$z = \xi + F(\xi) \equiv P(\xi). \tag{3.26}$$

By Proposition 3.2, the mapping F is monotone and Lipschitz - continuous with Lipschitz coefficient equal to L. Therefore, by Theorem 2.4, the system of equations (3.26) has a unique solution for any $z \in \mathbb{R}^{n+1}$, i.e., the mapping $P^{-1}(z)$ is determined in the whole space $\mathbb{R}^{n+1}$. By virtue of the monotonicity of the mapping F, the mapping P^{-1} is Lipschitz - continuous with Lipschitz coefficient equal to 1. Indeed, for all $z, \overline{z} \in \mathbb{R}^{n+1}$ we have

$$\|P^{-1}(z) - P^{-1}(\overline{z})\| = \|\xi - \overline{\xi}\| \leq \|\xi - \overline{\xi} + F(\xi) - F(\overline{\xi})]\| = \|z - \overline{z}\|.$$

After changing the variable (3.26), the perturbed system of nonlinear equations (3.5) will take the following form

$$z + \Phi P^{-1}(z) = g. \tag{3.27}$$

Define the mapping T as

$$T(z) = -\Phi P^{-1}(z) + g, \ \forall z \in \mathbb{R}^{n+1}. \tag{3.28}$$

Then the system of equations (3.27) can be rewritten as

$$z = T(z). \tag{3.29}$$

It follows from (3.28) that for all $z, \overline{z} \in \mathbb{R}^{n+1}$ and for some positive integer m,

$$\|T^m(z) - T^m(\overline{z})\| = \|(-\Phi P^{-1})^m(z) - (-\Phi P^{-1})^m(\overline{z})\|.$$

By virtue of Proposition 3.1 and Lipschitz continuity of the mapping P^{-1}, we have

$$\|T^m(z) - T^m(\overline{z})\| \leq \frac{M^m}{\sqrt{(m-1)!}}\|z - \overline{z}\|, \ \forall z, \overline{z} \in \mathbb{R}^{n+1}.$$

Since $\frac{M^m}{\sqrt{(m-1)!}} < 1$ when m is sufficiently large, we see that T^m is a contractive mapping with contraction coefficient equal to $\alpha = \frac{M^m}{\sqrt{(m-1)!}}$. By Theorem 3.3, the mapping T has a unique fixed point $z^* \in \mathbb{R}^{n+1}$, i.e., the system of equations (3.27) has a unique solution $z^* \in \mathbb{R}^{n+1}$ for any $g \in \mathbb{R}^{n+1}$. Consequently, the perturbed system of nonlinear equations (3.5) has a unique solution ξ^* for any $g \in \mathbb{R}^{n+1}$. This completes the proof of the theorem. □

In the following proposition, we shall estimate $\|x^* - \xi^*\|$, where $x^* = (x_0^*, x_1^*, \ldots, x_n^*)^T$ with $x_i^* = x^*(t_i), \ i = 0, 1, \ldots, n$ (note that $x^*(t)$ is an analytical solution of (1.1)) and $\xi^* = (\xi_0^*, \xi_1^*, \ldots, \xi_n^*)^T$ is the exact solution of the perturbed system of nonlinear equations (3.5).

Proposition 3.6. *Let the assumptions* (i)-(iii) *be satisfied. Then*

$$\|x^* - \xi^*\| \leq \frac{\sqrt{b-a}\,|O(h^\nu)|}{1-\alpha}, \tag{3.30}$$

where $\alpha = \frac{M^m}{\sqrt{(m-1)!}} < 1$ *when* m *is chosen sufficiently large.*

Proof. By (3.3) and (3.4), we have

$$x_i^* - \xi_i^* + \sum_{j=0}^{i} w_{i_j} K_1(t_i, s_j, x_j^*) - \sum_{j=0}^{i} w_{i_j} K_1(t_i, s_j, \xi_j^*) + \sum_{r=0}^{n} w_r K_2(t_i, s_r, x_r^*)$$
$$- \sum_{r=0}^{n} w_r K_2(t_i, s_r, \xi_r^*) = -O(h^\nu), \ i = 0, 1, \ldots, n,$$

which means

$$x_i^* - \xi_i^* + \varphi_i(x^*) - \varphi_i(\xi^*) + f_i(x^*) - f_i(\xi^*) = -O(h^\nu), \ i = 0, 1, \ldots, n.$$

Then we have

$$|x_i^* - \xi_i^* + \varphi_i(x^*) - \varphi_i(\xi^*) + f_i(x^*) - f_i(\xi^*)| = |O(h^\nu)|, \ i = 0, 1, \ldots, n,$$

and hence

$$w_i|x_i^* - \xi_i^* + \varphi_i(x^*) - \varphi_i(\xi^*) + f_i(x^*) - f_i(\xi^*)|^2 = w_i|O(h^\nu)|^2,\ i = 0, 1, \ldots, n.$$

It follows that

$$\begin{aligned}
&\|x^* - \xi^* + \Phi(x^*) - \Phi(\xi^*) + F(x^*) - F(\xi^*)\|^2 \\
&= \sum_{i=0}^{n} w_i|x_i^* - \xi_i^* + \varphi_i(x^*) - \varphi_i(\xi^*) + f_i(x^*) - f_i(\xi^*)|^2 = |O(h^\nu)|^2 \sum_{i=0}^{n} w_i.
\end{aligned}$$

Since in every Newton–Cotes formula $\sum\limits_{i=0}^{n} w_i = b - a$, we obtain

$$\|x^* - \xi^* + \Phi(x^*) - \Phi(\xi^*) + F(x^*) - F(\xi^*)\| = \sqrt{b-a}\ |O(h^\nu)|.$$

By virtue of the contraction of the mapping T^m and the monotonicity of the mapping F, we have

$$\begin{aligned}
\sqrt{b-a}\ |O(h^\nu)| &= \|x^* + F(x^*) - [\xi^* + F(\xi^*)] + \Phi(x^*) - \Phi(\xi^*)\| \\
&= \|z_x^* - z^* + \Phi P^{-1}(z_x^*) - \Phi P^{-1}(z^*)\| \\
&= \|z_x^* - z^* + T(z_x^*) - T(z^*)\| \\
&= \|z_x^* - z^* + T(T^m(z_x^*)) - T(T^m(z^*))\| \\
&= \|z_x^* - z^* + T^{m+1}(z_x^*) - T^{m+1}(z^*)\| \\
&= \|z_x^* - z^* + T^m(T(z_x^*)) - T^m(T(z^*))\| \\
&\geq \|z_x^* - z^*\| - \|T^m(T(z_x^*)) - T^m(T(z^*))\| \\
&\geq \|z_x^* - z^*\| - \alpha\|T(z_x^*) - T(z^*)\| \\
&= (1-\alpha)\|z_x^* - z^*\| \\
&= (1-\alpha)\|x^* - \xi^* + F(x^*) - F(\xi^*)\| \\
&\geq (1-\alpha)\|x^* - \xi^*\|,
\end{aligned}$$

where $z_x^* = x^* + F(x^*) \equiv P(x^*)$ and $z^* = \xi^* + F(\xi^*) \equiv P(\xi^*)$. Consequently,

$$\|x^* - \xi^*\| \leq \frac{\sqrt{b-a}\ |O(h^\nu)|}{1-\alpha}.$$

This completes the proof of the proposition. □

The inequality (3.30) leads to the following corollary.

Corollary 3.7. *$\|x^* - \xi^*\|$ vanishes when $h \to 0$.*

Next, we shall construct the iterative algorithm to find approximate solutions of the perturbed system of nonlinear equations (3.5). To solve the perturbed system of nonlinear equations (3.5), we first have to solve the system of equations (3.27) and after that we solve the system of equations (3.26). In the proof of Theorem 3.5, we have proved that the system of equations (3.27) has a unique solution by using the contraction mapping principle and the system of equations (3.26) has a unique solution for any $z \in \mathbb{R}^{n+1}$ by using parameter continuation method. The

approximate solutions of the system of equations (3.27) are obtained by using the standard iteration process

$$z^{(\tau+1)} = -\Phi P^{-1}(z^{(\tau)}) + g \equiv T(z^{(\tau)}),\ \tau = 0, 1, 2, \dots. \tag{3.31}$$

For the initial approximation we take $z^{(0)} = g$. At the same time at each step of above iteration process when calculating the value $P^{-1}(z^{(\tau)})$, we have to solve the system of equations of the form (3.26), as

$$\xi + F(\xi) = z^{(\tau)}. \tag{3.32}$$

Substituting F for A in the iteration processes (2.5a)–(2.5d), the approximate solutions of the system of equations (3.32) are obtained by using the following iteration processes

$$\xi^{(k+1)} = -\varepsilon_0 F(\xi^{(k)}) + u^{(l)},\ k = 0, 1, 2, \dots, \tag{3.33a}$$

$$u^{(l+1)} = -\varepsilon_0 F G_1^{-1}(u^{(l)}) + v^{(c)},\ l = 0, 1, 2, \dots, \tag{3.33b}$$

$$\dots, \tag{3.33c}$$

$$y^{(p+1)} = -\varepsilon_0 F G_1^{-1} \cdots G_{N-1}^{-1}(y^{(p)}) + z^{(\tau)},\ p = 0, 1, 2, \dots. \tag{3.33d}$$

Therefore the approximate solutions of the perturbed system of nonlinear equations (3.5) can be found by the following iteration processes

$$\xi^{(k+1)} = -\varepsilon_0 F(\xi^{(k)}) + u^{(l)},\ k = 0, 1, 2, \dots, \tag{3.34a}$$

$$u^{(l+1)} = -\varepsilon_0 F G_1^{-1}(u^{(l)}) + v^{(c)},\ l = 0, 1, 2, \dots, \tag{3.34b}$$

$$\dots, \tag{3.34c}$$

$$y^{(p+1)} = -\varepsilon_0 F G_1^{-1} \cdots G_{N-1}^{-1}(y^{(p)}) + z^{(\tau)},\ p = 0, 1, 2, \dots \tag{3.34d}$$

$$z^{(\tau+1)} = -\Phi P^{-1}(z^{(\tau)}) + g,\ \tau = 0, 1, 2, \dots, z^{(0)} = g. \tag{3.34e}$$

Now we estimate the error of approximate solutions of the perturbed system of nonlinear equations (3.5). Assume that the number of steps in each iteration scheme of the iteration processes (3.34a)–(3.34e) is the same and equals n_0. Let $\xi^{(n_0)}$ be approximate solutions of the perturbed system of nonlinear equations (3.5). Note that $\xi^{(n_0)}$ depends on N, hence we denote $\xi(n_0, N) \equiv \xi^{(n_0)}$. We have the following result.

Theorem 3.8. *Let the assumptions of Theorem 3.5 be satisfied. Then the sequence of approximate solutions* $\{\xi(n_0, N)\}$, $n_0 = 1, 2, \dots$ *constructed by iteration processes* (3.34a)–(3.34e) *converges to the exact solution* ξ^* *of the perturbed system of nonlinear equations* (3.5). *Moreover, the following estimates hold*

$$\|\xi(n_0, N) - \xi^*\| \le (1 + M)\frac{q^{n_0+1}}{1-q}\frac{e^{qN} - 1}{e^q - 1}C_{n'}\|g\| + \frac{\alpha^{\frac{n_0-h_0}{m}}}{1-\alpha}C_m\|g\|, \tag{3.35}$$

where $C_{n'} = \frac{\gamma}{1-\gamma}\frac{M^{n'}}{\sqrt{(n'-1)!}} + \frac{M^{n'}}{\sqrt{(n'-1)!}} + \dots + M + 1, C_m = \frac{M^m}{\sqrt{(m-1)!}} + \frac{M^{m-1}}{\sqrt{(m-2)!}} + \dots + M$, *$N$ is the smallest natural number such that* $q = \frac{L}{N} < 1$, *n' is a natural number such that* $\gamma = \frac{M}{\sqrt{n'}} < 1$ *and m is chosen sufficiently large such that* $\alpha = \frac{M^m}{\sqrt{(m-1)!}} < 1$, $h_0 \in \{0, 1, \dots, m-1\}$ *is the residual of* $\frac{n_0}{m}$, $n_0 > max\,\{m, n'\}$.

Proof. For simplicity, we assume that $\Phi(0) = 0$ and $F(0) = 0$, where $0 = (0, 0, \ldots, 0)^T$ denotes the zero element in $\mathbb{R}^{n+1}$. Indeed, if $\Phi(0) \neq 0$ or $F(0) \neq 0$, we can define two mappings $\Phi_1, F_1 : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\Phi_1(\xi) = \Phi(\xi) - \Phi(0), F_1(\xi) = F(\xi) - F(0), \tag{3.36}$$

then $\Phi_1(0) = F_1(0) = 0$ and the perturbed system of nonlinear equations (3.5) is equivalent to

$$\xi + \Phi_1(\xi) + F_1(\xi) = g_1, \tag{3.37}$$

where $g_1 = g - \Phi(0) - F(0)$. It follows from (3.36) that for all $\xi, \overline{\xi} \in \mathbb{R}^{n+1}$

$$\Phi_1(\xi) - \Phi_1(\overline{\xi}) = \Phi(\xi) - \Phi(\overline{\xi}), F_1(\xi) - F_1(\overline{\xi}) = F(\xi) - F(\overline{\xi}).$$

Therefore the Propositions 3.1 and 3.2 can be applied to the mappings Φ_1 and F_1, respectively. Consequently, Theorem 3.5 can be applied to the perturbed system of nonlinear equations (3.37).

We split the proof into two steps.

Step 1. We estimate the error of approximate solutions of the system of equations (3.27). Firstly, we estimate the errors in calculating the values $T(z^{(\tau)}) = -\Phi P^{-1}(z^{(\tau)}) + g$, $\tau = 1, 2, \ldots, n_0 - 1$. Since $F(0) = 0$, it follows that $P(0) = 0 + F(0) = 0$. Thus $T(0) = -\Phi P^{-1}(0) + g = g \equiv z^{(0)}$. At the same time at each step of the iteration process (3.31) when calculating the value $P^{-1}(z^{(\tau)})$, we will use the iteration processes (3.33a)–(3.33d). Let $\xi^{(n_0)}_{z^{(\tau)}}$ and $\xi^*_{z^{(\tau)}}$ be the approximate and exact values of $P^{-1}(z^{(\tau)})$, respectively. It follows from Theorem 2.5 that the values $P^{-1}(z^{(\tau)})$ are calculated with the error

$$\|\xi^{(n_0)}_{z^{(\tau)}} - \xi^*_{z^{(\tau)}}\| \leq \frac{q^{n_0+1}}{1-q} \frac{e^{qN} - 1}{e^q - 1} \|z^{(\tau)}\|, \tag{3.38}$$

where N is the smallest natural number such that $q = \frac{L}{N} < 1, n_0 = 1, 2, \ldots$.

Since $\left\{z^{(\tau)}\right\}$, $\tau = 1, 2, \ldots$ is a convergence sequence, it follows that $\|z^{(\tau)}\|$ is bounded for all positive integer τ. We now determine the supremum of $\|z^{(\tau)}\|$, $\tau \in \{1, 2, \ldots, n_0\}$ (n_0 is the number of steps in each iteration scheme). For any $\tau \in \{1, 2, \ldots, n_0\}$ we have

$$\begin{aligned} \|z^{(\tau)}\| &\leq \|z^{(\tau)} - z^{(\tau-1)}\| + \|z^{(\tau-1)} - z^{(\tau-2)}\| + \cdots + \|z^{(1)} - z^{(0)}\| + \|z^{(0)}\| \\ &\leq \|z^{(n_0)} - z^{(n_0-1)}\| + \|z^{(n_0-1)} - z^{(n_0-2)}\| + \cdots + \|z^{(\tau+1)} - z^{(\tau)}\| \\ &\quad + \|z^{(\tau)} - z^{(\tau-1)}\| + \|z^{(\tau-1)} - z^{(\tau-2)}\| + \cdots + \|z^{(1)} - z^{(0)}\| + \|z^{(0)}\|. \end{aligned} \tag{3.39}$$

On the other hand, we have

$$\begin{aligned}
&\|z^{(n_0)} - z^{(n_0-1)}\| + \|z^{(n_0-1)} - z^{(n_0-2)}\| + \cdots + \|z^{(1)} - z^{(0)}\| + \|z^{(0)}\| \\
=&\|T^{n_0}(g) - T^{n_0}(0)\| + \|T^{n_0-1}(g) - T^{n_0-1}(0)\| + \cdots + \|T(g) - T(0)\| + \|g\| \\
\leq&\left[\frac{M^{n_0}}{\sqrt{(n_0-1)!}} + \frac{M^{n_0-1}}{\sqrt{(n_0-2)!}} + \cdots + M\right]\|g\| + \|g\| \\
=&\left[\frac{M^{n_0}}{\sqrt{(n_0-1)!}} + \frac{M^{n_0-1}}{\sqrt{(n_0-2)!}} + \cdots + M + 1\right]\|g\|.
\end{aligned}$$

Let n' be natural number such that $\gamma = \frac{M}{\sqrt{n'}} < 1$. Then for any $n_0 > n'$, we have

$$\begin{aligned}
&\frac{M^{n_0}}{\sqrt{(n_0-1)!}} + \frac{M^{n_0-1}}{\sqrt{(n_0-2)!}} + \cdots + M + 1 \\
=&\frac{M^{n_0}}{\sqrt{(n_0-1)!}} + \frac{M^{n_0-1}}{\sqrt{(n_0-2)!}} + \cdots + \frac{M^{n'+1}}{\sqrt{n'!}} + \frac{M^{n'}}{\sqrt{(n'-1)!}} + \cdots + M + 1 \\
\leq&\frac{M^{n'}}{\sqrt{(n'-1)!}}\gamma^{n_0-n'} + \frac{M^{n'}}{\sqrt{(n'-1)!}}\gamma^{n_0-n'-1} + \cdots + \frac{M^{n'}}{\sqrt{(n'-1)!}}\gamma + \frac{M^{n'}}{\sqrt{(n'-1)!}} \\
&+ \cdots + M + 1 \\
=&(\gamma^{n_0-n'} + \gamma^{n_0-n'-1} + \cdots + \gamma)\frac{M^{n'}}{\sqrt{(n'-1)!}} + \frac{M^{n'}}{\sqrt{(n'-1)!}} + \cdots + M + 1 \\
=&\gamma\frac{1-\gamma^{n_0-n'}}{1-\gamma}\frac{M^{n'}}{\sqrt{(n'-1)!}} + \frac{M^{n'}}{\sqrt{(n'-1)!}} + \cdots + M + 1 \\
\leq&\frac{\gamma}{1-\gamma}\frac{M^{n'}}{\sqrt{(n'-1)!}} + \frac{M^{n'}}{\sqrt{(n'-1)!}} + \cdots + M + 1 \equiv C_{n'}.
\end{aligned}$$

Thus

$$\|z^{(n_0)} - z^{(n_0-1)}\| + \|z^{(n_0-1)} - z^{(n_0-2)}\| + \cdots + \|z^{(1)} - z^{(0)}\| + \|z^{(0)}\| \leq C_{n'}\|g\|. \tag{3.40}$$

It follows from (3.39) and (3.40) that

$$\|z^{(\tau)}\| \leq C_{n'}\|g\|$$

for any $\tau \in \{1, 2, \ldots, n_0\}$. Hence the values $P^{-1}(z^{(\tau)})$ are calculated with the error

$$\|\xi^{(n_0)}_{z^{(\tau)}} - \xi^*_{z^{(\tau)}}\| \leq \Delta(n_0),$$

where

$$\Delta(n_0) = \frac{q^{n_0+1}}{1-q}\frac{e^{qN}-1}{e^q-1}C_{n'}\|g\|. \tag{3.41}$$

By Proposition 3.1, we have

$$\|T(z) - T(\overline{z})\| = \|\Phi P^{-1}(z) - \Phi P^{-1}(\overline{z})\| \leq M\|P^{-1}(z) - P^{-1}(\overline{z})\|, \ \forall z, \overline{z} \in \mathbb{R}^{n+1}.$$

Therefore the values $T(z^{(\tau)}) = -\Phi P^{-1}(z^{(\tau)}) + g, \ \tau = 1, 2, \ldots, n_0 - 1$ are calculated with the error not more than $M\Delta(n_0)$.

Next, we shall estimate the error of an iteration process in the calculation of

z. By Theorem 3.4, the error of an iteration process in the calculation of z equals $\frac{\alpha^{\frac{n_0-h_0}{m}}}{1-\alpha}\|z^{(m+h_0)}-z^{(h_0)}\|$, where m is chosen sufficiently large such that $\alpha=\frac{M^m}{\sqrt{(m-1)!}}<1$, $h_0\in\{0,1,\dots,m-1\}$ is the residual of $\frac{n_0}{m}$. We have

(3.42)
$$\begin{aligned}
&\|z^{(m+h_0)}-z^{(h_0)}\|\\
\le&\|z^{(m+h_0)}-z^{(m+h_0-1)}\|+\|z^{(m+h_0-1)}-z^{(m+h_0-2)}\|+\cdots+\|z^{(h_0+1)}-z^{(h_0)}\|\\
=&\|T^{m+h_0}(g)-T^{m+h_0}(0)\|+\|T^{m+h_0-1}(g)-T^{m+h_0-1}(0)\|\\
&+\cdots+\|T^{h_0+1}(g)-T^{h_0+1}(0)\|\\
\le&\left[\frac{M^{m+h_0}}{\sqrt{(m+h_0-1)!}}+\frac{M^{m+h_0-1}}{\sqrt{(m+h_0-2)!}}+\cdots+\frac{M^{h_0+1}}{\sqrt{h_0!}}\right]\|g\|\equiv R_{m,h_0}\|g\|,
\end{aligned}$$

where $R_{m,h_0}=\frac{M^{m+h_0}}{\sqrt{(m+h_0-1)!}}+\frac{M^{m+h_0-1}}{\sqrt{(m+h_0-2)!}}+\cdots+\frac{M^{h_0+1}}{\sqrt{h_0!}}$. Let us prove that

$$\begin{aligned}
\max\{R_{m,h_0},h_0\in\{0,1,\dots,m-1\}\}&=\frac{M^m}{\sqrt{(m-1)!}}+\frac{M^{m-1}}{\sqrt{(m-2)!}}+\cdots+M\\
&\equiv C_m. \qquad (3.43)
\end{aligned}$$

Obviously, $R_{m,0}=C_m$. For some non-negative integer $h',0\le h'\le h_0$, we have

$$R_{m,h'}=\frac{M^{m+h'}}{\sqrt{(m+h'-1)!}}+\frac{M^{m+h'-1}}{\sqrt{(m+h'-2)!}}+\cdots+\frac{M^{h'+1}}{\sqrt{h'!}}$$

and

$$R_{m,h'+1}=\frac{M^{m+h'+1}}{\sqrt{(m+h')!}}+\frac{M^{m+h'}}{\sqrt{(m+h'-1)!}}+\frac{M^{m+h'-1}}{\sqrt{(m+h'-2)!}}+\cdots+\frac{M^{h'+2}}{\sqrt{(h'+1)!}}.$$

Hence

$$R_{m,h'+1}-R_{m,h'}=\frac{M^{m+h'+1}}{\sqrt{(m+h')!}}-\frac{M^{h'+1}}{\sqrt{h'!}}.$$

We have

$$\begin{aligned}
\frac{M^{m+h'+1}}{\sqrt{(m+h')!}}&=\frac{M^m}{\sqrt{(m-1)!}}\frac{M^{h'+1}}{\sqrt{m(m+1)\cdots(m+h')}}\\
&=\alpha\frac{M^{h'+1}}{\sqrt{m(m+1)\cdots(m+h')}}.
\end{aligned}$$

Since $\alpha\in[0,1)$, it follows that

$$\frac{M^{m+h'+1}}{\sqrt{(m+h')!}}\le\frac{M^{h'+1}}{\sqrt{m(m+1)\cdots(m+h')}}. \qquad (3.44)$$

On the other hand, we have

$$m(m+1)\cdots(m+h')\ge 1(1+1)\cdots(h'+1)\ge h'!$$

for any positive integer m. Hence

$$\frac{M^{h'+1}}{\sqrt{m(m+1)\cdots(m+h')}} \leq \frac{M^{h'+1}}{\sqrt{h'!}}. \tag{3.45}$$

Combining (3.44) and (3.45), we get

$$\frac{M^{m+h'+1}}{\sqrt{(m+h')!}} \leq \frac{M^{h'+1}}{\sqrt{h'!}}.$$

Thus

$$R_{m,h'+1} - R_{m,h'} \leq 0,$$

which implies that $R_{m,h'+1} \leq R_{m,h'}$ for some non-negative integer $h', 0 \leq h' \leq h_0$. Therefore (3.43) is proved. It follows from (3.42) and (3.43) that

$$\|z^{(m+h_0)} - z^{(h_0)}\| \leq C_m \|g\|$$

for every integer $h_0 \in \{0, 1, \ldots, m-1\}$. Hence the error of an iteration process in the calculation of z equals $\frac{\alpha^{\frac{n_0-h_0}{m}}}{1-\alpha} C_m \|g\|$.

Consequently, the error of approximate solutions $z^{(n_0)}$ of the system of equations (3.27) gives the estimate

$$\|z^{(n_0)} - z^*\| \leq M\Delta(n_0) + \frac{\alpha^{\frac{n_0-h_0}{m}}}{1-\alpha} C_m \|g\|.$$

Step 2. We estimate the error of approximate solutions of the system of equations (3.26)

$$P(\xi) \equiv \xi + F(\xi) = z.$$

Since the mapping P^{-1} is Lipschitz - continuous with Lipschitz coefficient equal to 1, the substitution of the error $M\Delta(n_0) + \frac{\alpha^{\frac{n_0-h_0}{m}}}{1-\alpha} C_m \|g\|$ into the right - hand side of the system of equations (3.26) causes an error of not more than $M\Delta(n_0) + \frac{\alpha^{\frac{n_0-h_0}{m}}}{1-\alpha} C_m \|g\|$ in the corresponding solution ξ. The error of an iteration process in the calculation of ξ equals $\Delta(n_0)$. Consequently,

$$\begin{aligned}\|\xi(n_0, N) - \xi^*\| &\leq M\Delta(n_0) + \frac{\alpha^{\frac{n_0-h_0}{m}}}{1-\alpha} C_m \|g\| + \Delta(n_0) \\ &= (1+M)\Delta(n_0) + \frac{\alpha^{\frac{n_0-h_0}{m}}}{1-\alpha} C_m \|g\|.\end{aligned}$$

By (3.41), we obtain (3.35). This completes the proof of the theorem. □

Remark 3.9. Let d be integer part of $\frac{n_0}{m}$, i.e., $n_0 = md + h_0$, $h_0 \in \{0, 1, \ldots, m-1\}$. We have $n_0 + 1 = md + h_0 + 1 \geq md + 1$ for every $h_0 \in \{0, 1, \ldots, m-1\}$. Since

$0 < q < 1$, it follows that $q^{n_0+1} \leq q^{md+1}$. From this and (3.35), we have

$$\begin{aligned} \|\xi(n_0, N) - \xi^*\| &\leq (1+M)\frac{q^{md+1}}{1-q}\frac{e^{qN}-1}{e^q-1}C_{n'}\|g\| + \frac{\alpha^{\frac{n_0-h_0}{m}}}{1-\alpha}C_m\|g\| \\ &= (1+M)\frac{q}{1-q}\frac{e^{qN}-1}{e^q-1}C_{n'}\|g\|q^{md} + \frac{C_m}{1-\alpha}\|g\|\alpha^d \\ &= C_1 q^{md} + C_2\alpha^d, \end{aligned} \tag{3.46}$$

where

$$C_1 = (1+M)\frac{q}{1-q}\frac{e^{qN}-1}{e^q-1}C_{n'}\|g\|, C_2 = \frac{C_m}{1-\alpha}\|g\|.$$

Let $\beta = \max\{q^m, \alpha\}$. From (3.46), we get

$$\|\xi(n_0, N) - \xi^*\| \leq (C_1 + C_2)\beta^d. \tag{3.47}$$

It follows from (3.47) that for any given $\epsilon > 0$, we can find the number of iteration such that $\|\xi(n_0, N) - \xi^*\| \leq \epsilon$.

Remark 3.10. We shall now estimate the complexity of the proposed iterative algorithm (3.34a)–(3.34e). The iteration processes (3.34a)–(3.34e) can be written as the following symbolic notation

$$\begin{aligned} \xi^{(k+1)} = \underbrace{-\frac{1}{N}F(\xi^{(k)}) - \frac{1}{N}F(\xi^{(l)}) - \cdots - \frac{1}{N}F(\xi^{(p)})}_{N\ terms} - \Phi(\xi^{(\tau)}) + g, \\ k, l, \ldots, \tau = 0, 1, \ldots, n_0. \end{aligned} \tag{3.48}$$

The procedure for calculating each value $F(\xi), \Phi(\xi)$ in the specified element ξ is called an elementary operation. We shall call the number of elementary operations necessary to implement algorithm (3.34a)–(3.34e) is the volume of the calculations $M(n_0, N)$. From the symbolic notation (3.48) it follows that $M(n_0, N) \leq (n_0 + 1)^{N+1}$.

4. Illustrative examples

In this section, to illustrate our above results two examples are presented. The computations associated with the examples were performed using Maple 12 on personal computer.

Example 4.1. Consider the linear Volterra–Fredholm integral equation

$$x(t) + \frac{4}{3}\int_0^t (t+s)x(s)ds + 4\int_0^1 ts\, x(s)ds = \frac{64}{45}t^2\sqrt{t} - 2t^2 - \frac{2}{5}t + \sqrt{t} - 1,\ 0 \leq t \leq 1, \tag{4.1}$$

This integral equation has analytical solution $x(t) = \sqrt{t} - 1$ on $[0, 1]$. It is easy to verify that the functions $K_1(t, s, x) = \frac{4}{3}(t+s)\ x(s)$ and $K_2(t, s, x) = 4ts\, x(s)$ satisfy the conditions (i)-(iii) of the Theorem 3.5 with $M^2 = \frac{28}{27}, L^2 = \frac{16}{9}$. For approximating the left-hand integrals, we use composite midpoint rule and take a partition with the discretization parameter $h = \frac{1}{20}$. We take $\epsilon = 10^{-3}, N = 2, m = 6, n' = 5$. It follows from (3.47) that $d \geq 6$. Taking $d = 6$. Since $n_0 = md + h_0 =$

$6d + h_0,\ h_0 \in \{0, 1, \ldots, 5\}$, we have $n_0 \geq 41$. Taking $n_0 = 41$, the number of iterations needed is 68921 (the number of steps in each iteration scheme is the same and equals $n_0 = 41$). Table 1 presents approximate solutions obtained by using the iteration processes (3.34a)–(3.34e) with $N = 2$ and $n_0 = 41$, also exact solutions are given for comparison.

TABLE 1. Comparison of the exact and approximate solutions for Example 4.1.

Nodes t	Exact solutions	Approximate solutions	Absolute error
0.025	−0.8403128058	−0.8402268451	1.6592719×10^{-3}
0.075	−0.7261387212	−0.7224677310	3.6709902×10^{-3}
0.125	−0.6464466094	−0.6412768969	5.1697125×10^{-3}
0.175	−0.5816699867	−0.5754272719	6.2427148×10^{-3}
0.225	−0.5256583510	−0.5187213892	6.9369618×10^{-3}
0.275	−0.4755955759	−0.4683076490	7.2879269×10^{-3}
0.325	−0.4299122875	−0.4225844350	7.3278525×10^{-3}
0.375	−0.3876275643	−0.3805388128	7.0887515×10^{-3}
0.425	−0.3480797595	−0.3414763266	6.6034329×10^{-3}
0.475	−0.3107975624	−0.3048919371	5.9056253×10^{-3}
0.525	−0.2754311627	−0.2704015138	5.0296489×10^{-3}
0.575	−0.2417124556	−0.2377025947	4.0098609×10^{-3}
0.625	−0.2094305850	−0.2065505810	2.8800040×10^{-3}
0.675	−0.1784161637	−0.1767436183	1.6725454×10^{-3}
0.725	−0.1485306817	−0.1481126240	4.1805770×10^{-4}
0.775	−0.1196591569	−0.1205144706	8.5531370×10^{-4}
0.825	−0.0917048938	−0.0938271777	2.1222839×10^{-3}
0.875	−0.0645856533	−0.0679463857	3.3607324×10^{-3}
0.925	−0.0382307969	−0.0427826948	4.5518979×10^{-3}
0.975	−0.0125791171	−0.0182595524	5.6804353×10^{-3}

Example 4.2. Consider the nonlinear Volterra–Fredholm integral equation

$$x(t) + 5\int_0^t ts\cos[x(s)]ds + \frac{11}{2}\int_0^1 t^2 s^2\, x(s)ds = \frac{11}{8}t^2 - 4t + 5t\cos(t) + 5t^2\sin(t),\ 0 \leq t \leq 1, \tag{4.2}$$

The analytical solution of this integral equation is $x(t) = t$ on $[0, 1]$. It is easy to verify that the functions $K_1(t, s, x) = 5ts\cos[x(s)]$ and $K_2(t, s, x) = \frac{11}{2}t^2 s^2\, x(s)$ satisfy the conditions (i)-(iii) of the Theorem 3.5 with $M^2 = \frac{25}{18}, L^2 = \frac{121}{100}$. For approximating the left-hand integrals, we use composite midpoint rule and take a partition with the discretization parameter $h = \frac{1}{50}$. We take $\epsilon = 10^{-3}, N = 2, m = 8, n' = 6$. It follows from (3.47) that $d \geq 6$. Taking $d = 6$. Since $n_0 = md + h_0 = 8d + h_0,\ h_0 \in \{0, 1, \ldots, 7\}$, we have $n_0 \geq 55$. Taking $n_0 = 55$, the number of iterations needed is 166375 (the number of steps in each iteration scheme is the same and equals $n_0 = 55$). Table 2 presents approximate solutions obtained by using the iteration processes (3.34a)–(3.34e) with $N = 2$ and $n_0 = 55$, also exact solutions are given for comparison.

TABLE 2. Comparison of the exact and approximate solutions for Example 4.2.

Nodes t	Exact solutions	Approximate solutions	Absolute error	Nodes t	Exact solutions	Approximate solutions	Absolute error
0.01	0.01	0.0099948368	5.1632000×10^{-6}	0.51	0.51	0.5041937466	$5.80625340 \times 10^{-3}$
0.03	0.03	0.0299685542	3.1445800×10^{-5}	0.53	0.53	0.5238057793	$6.19422070 \times 10^{-3}$
0.05	0.05	0.0499210957	7.8904300×10^{-5}	0.55	0.55	0.5434091045	$6.59089550 \times 10^{-3}$
0.07	0.07	0.0698526617	1.4733830×10^{-4}	0.57	0.57	0.5630033477	$6.99665230 \times 10^{-3}$
0.09	0.09	0.0897635423	2.3645770×10^{-4}	0.59	0.59	0.5825879388	$7.41206120 \times 10^{-3}$
0.11	0.11	0.1096541138	3.4588620×10^{-4}	0.61	0.61	0.6021621161	$7.83788390 \times 10^{-3}$
0.13	0.13	0.1295248320	4.7516800×10^{-4}	0.63	0.63	0.6217248525	$8.27514750 \times 10^{-3}$
0.15	0.15	0.1493762257	6.2377430×10^{-4}	0.65	0.65	0.6412748634	$8.72513660 \times 10^{-3}$
0.17	0.17	0.1692088902	7.9110980×10^{-4}	0.67	0.67	0.6608105470	$9.18945300 \times 10^{-3}$
0.19	0.19	0.1890234780	9.7652200×10^{-4}	0.69	0.69	0.6803299576	$9.67004240 \times 10^{-3}$
0.21	0.21	0.2088206886	1.1793114×10^{-3}	0.71	0.71	0.6998307414	$1.01692586 \times 10^{-2}$
0.23	0.23	0.2286012617	1.3987383×10^{-3}	0.73	0.73	0.7193100805	$1.06899195 \times 10^{-2}$
0.25	0.25	0.2483659629	1.6340371×10^{-3}	0.75	0.75	0.7387646198	$1.12353802 \times 10^{-2}$
0.27	0.27	0.2681155773	1.8844227×10^{-3}	0.77	0.77	0.7581903691	$1.18096309 \times 10^{-2}$
0.29	0.29	0.2878508938	2.1491062×10^{-3}	0.79	0.79	0.7775825937	$1.24174063 \times 10^{-2}$
0.31	0.31	0.3075726969	2.4273031×10^{-3}	0.81	0.81	0.7969356971	$1.30643029 \times 10^{-2}$
0.33	0.33	0.3272817556	2.7182444×10^{-3}	0.83	0.83	0.8162430208	$1.37569792 \times 10^{-2}$
0.35	0.35	0.3469788114	3.0211886×10^{-3}	0.85	0.85	0.8354966758	$1.45033242 \times 10^{-2}$
0.37	0.37	0.3666645658	3.3354342×10^{-3}	0.87	0.87	0.8546872738	$1.53127262 \times 10^{-2}$
0.39	0.39	0.3863396700	3.6603300×10^{-3}	0.89	0.89	0.8738036297	$1.61963703 \times 10^{-2}$
0.41	0.41	0.4060047106	3.9952894×10^{-3}	0.91	0.91	0.8928323854	$1.71676146 \times 10^{-2}$
0.43	0.43	0.4256602069	4.3397931×10^{-3}	0.93	0.93	0.9117575067	$1.82424933 \times 10^{-2}$
0.45	0.45	0.4453065831	4.6934169×10^{-3}	0.95	0.95	0.9305597280	$1.94402720 \times 10^{-2}$
0.47	0.47	0.4649441741	5.0558259×10^{-3}	0.97	0.97	0.9492157979	$2.07842021 \times 10^{-2}$
0.49	0.49	0.4845731957	5.4268043×10^{-3}	0.99	0.99	0.9676975410	$2.23024590 \times 10^{-2}$

5. CONCLUSIONS

In this paper, a new numerical method has been proposed to solve nonlinear Volterra–Fredholm integral equations. With this method, the perturbed system of nonlinear equations obtained by discretization is solved by an iterative method, which is based on a hybrid of the method of contractive mapping and parameter continuation method. Lastly, two illustrative examples are given to demonstrate the effectiveness and convenience of the proposed method.

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