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THE SPLIT COMMON FIXED POINT PROBLEM FOR FAMILIES OF GENERALIZED DEMIMETRIC MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we consider the split common fixed point problem for families of generalized demimetric mappings in Banach spaces. Using the idea of Halpern iteration, we first prove a strong convergence theorem of finding a solution of the split common fixed point problem for families of generalized demimetric mappings in Banach spaces. Furthermore, using the idea of Mann iteration, we obtain a weak convergence theorem of finding a solution of the problem for the families of mappings in Banach spaces. Using these results, we obtain well-known and new strong and weak convergence theorems for the split common fixed point problem in Hilbert spaces and Banach spaces.

1. Introduction

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [21] if

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$, where $F(U)$ is the set of fixed points of U and J is the duality mapping on E. We have from [21] that $F(U)$ is closed and convex. Using this property, we proved strong and weak convergence theorems for demimetric mappings in Hilbert spaces and Banach spaces; see [12, 20, 21, 23, 26]. Very recently, Kawasaki and Takahashi [8] generalized the concept of demimetric mappings as follows: Let θ be a real number with $\theta \neq 0$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called generalized demimetric [8] if

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2 \tag{1.1}$$

for all $x \in C$ and $q \in F(U)$. This mapping U is called θ-generalized demimetric. The set $F(U)$ of fixed points of such a mapping U is also closed and convex; see [8].

On the other hand, in 1967, Halpern [6] introduced the following iteration process. Let C be a nonempty, closed and convex subset of a Banach space E. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Take $x_0, x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

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where $\{\alpha_n\}$ is a sequence in $[0,1]$.

In 1953, Mann [14] introduced the following iteration process. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. There are many investigations of Halpern and Mann iterative processes for finding fixed points of nonexpansive mappings in Hilbert spaces and Banach spaces.

Recently, Takahashi, Wen and Yao [27] proved strong and weak convergence theorems of Halpern type iteration and Mann type iteration for the split common fixed point problem by using families of demimetric mappings in Banach spaces; see also [22]. See [5, 15] for the split common fixed point problem.

In this paper, motivated by this problem, methods and theorems, we consider the split common fixed point problem for families of generalized demimetric mappings in Banach spaces. Using the idea of Halpern iteration, we prove a strong convergence theorem for finding a solution of the split common fixed point problem for families of generalized demimetric mappings in Banach spaces. Furthermore, using the idea of Mann iteration, we obtain a weak convergence theorem for finding a solution of the problem in Banach spaces. Using these results, we obtain well-known and new strong and weak convergence theorems in Hilbert spaces and Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [18] that

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y \rangle; \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x-y, u-v \rangle = \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \le \|x-y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x-y \rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x \rangle \le 0$ holds for all $x \in H$ and $y \in C$; see [16]. The following result was proved by Takahashi and Toyoda [25].

Lemma 2.1 ([25]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \le \|x_n - u\|$ for all $n \in \mathbb{N}$ and $u \in C$, then $\{P_C x_n\}$ converges strongly to some $z \in C$, where P_C is the metric projection on H onto C.*

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$, respectively. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive.

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.5}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [16] and [17]. We know the following result.

Lemma 2.2 ([16])**.** *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy \rangle \ge 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy \rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.3 ([16])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x$;
(2) $\langle z - y, J(x-z) \rangle \ge 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of of E into 2^{E^*}. A multi-valued mapping A on E is said to be monotone if $\langle x-y, u^* - v^* \rangle \ge 0$ for all $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [3]; see also [17, Theorem 3.5.4].

Theorem 2.4 ([3]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [17].

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called generalized demimetric [8] if it satisfies (1.1), i.e.,

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$, where J is the duality mapping on E.

Let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [21] if, for any $x \in C$ and $q \in F(U)$,

$$\langle x - q, J(x - Ux)\rangle \geq \frac{1 - \eta}{2}\|x - Ux\|^2,$$

where $F(U)$ is the set of fixed points of U.

Examples 2.5. We know examples of generalized demimetric mappings.

(1) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let t be a real number with $0 \leq t < 1$. A mapping $U : C \to H$ is called a t-strict pseudo-contraction [4] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + t\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a t-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is $\frac{2}{1-t}$-generalized demimetric; see [8].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called generalized hybrid [9] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{2.6}$$

for all $x, y \in C$. Such a mapping U is called (α, β)-generalized hybrid. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 2-generalized demimetric. In fact, setting $x = u \in F(U)$ and $y = x \in C$ in (2.6), we have that

$$\alpha\|u - Ux\|^2 + (1 - \alpha)\|u - Ux\|^2 \leq \beta\|u - x\|^2 + (1 - \beta)\|u - x\|^2$$

and hence $\|Ux - u\|^2 \leq \|x - u\|^2$. From

$$\|Ux - u\|^2 = \|Ux - x\|^2 + \|x - u\|^2 + 2\langle Ux - x, x - u\rangle,$$

we have that

$$2\langle x-u, x-Ux\rangle \geq \|x-Ux\|^2$$

for all $x \in C$ and $u \in F(U)$. This means that U is 2-generalized demimetric. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [10, 11] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx-Ty\|^2 \leq \|Tx-y\|^2 + \|Ty-x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [19] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx-Ty\|^2 \leq \|x-y\|^2 + \|Tx-y\|^2 + \|Ty-x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [7].

(3) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is 1-generalized demimetric; see [8].

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is 1-generalized demimetric; see [8].

(5) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let T be a mapping from C into H. Suppose that T is Lipschitzian, that is, there exists $L > 0$ such that

$$\|Tx-Ty\| \leq L\|x-y\|$$

for all $x, y \in C$. Let $S = (L+1)I - T$. If $F(\frac{T}{L})$, then S is $(-2L)$-generalized demimetric; see [8, 24].

(6) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $\alpha > 0$. If B be an α-inverse strongly monotone mapping from C into H with $B^{-1}0 \neq \emptyset$, then $T = I + B$ is $\left(-\frac{1}{\alpha}\right)$-generalized demimetric; see [8, 24].

The following lemmas are important and crucial in the proofs of our main results.

Lemma 2.6 ([8]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. If a mapping $U : C \to E$ is θ-generalized demimetric and $\theta > 0$, then U is $\left(1 - \frac{2}{\theta}\right)$-demimetric.*

Lemma 2.7 ([8]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping of C into E. Then $F(T)$ is closed and convex.*

Lemma 2.8 ([8]). *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping from C into E and let $k \in \mathbb{R}$ with $k \neq 0$. Then $(1-k)I + kT$ is θk-generalized demimetric from C into E.*

We also know the following lemma from [26]:

Lemma 2.9 ([26]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 1)$ and let T be a k-demimetric mapping of C into H such that $F(T)$ is nonempty. Let λ be a real number with $0 < \lambda \leq 1-k$ and define $S = (1-\lambda)I + \lambda T$. Then S is a quasi-nonexpansive mapping of C into H.*

We also know the following lemmas:

Lemma 2.10 ([2], [29]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, \dots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.11 ([13]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ satisfies $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \cdots$ *and* $\tau(n) \to \infty$*;*
(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3. Strong convergence theorem

In this section, using the idea of Halpern iteration, we prove a strong convergence theorem of finding a solution of the split common fixed point problem for families of generalized demimetric mappings in Banach spaces. Let E be a Banach space, let C be a nonempty, closed and convex subset of E and let $\{U_n\}$ be a sequence of mappings of C into E such that $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$. The sequence $\{U_n\}$ is said to satisfy the condition (I) [1] if for any bounded sequence $\{z_n\}$ of C such that $\lim_{n\to\infty} \|z_n - U_n z_n\| = 0$, every weak cluster point of $\{z_n\}$ belongs to $\cap_{n=1}^{\infty} F(U_n)$.

Theorem 3.1. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F. Let $\{\theta_n\}$ and $\{\tau_n\}$ be sequences of real numbers with $\theta_n, \tau_n \neq 0$ and let $\{k_n\}$ and $\{h_n\}$ be sequences of real numbers with $\theta_n k_n > 0$ and $\tau_n h_n > 0$, respectively. Let $\{S_n\}$ be a sequence of θ_n-generalized demimetric mappings of H to H with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ satisfying the condition (I) and let $\{T_n\}$ be a sequence of τ_n-generalized demimetric mappings of F to F with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ satisfying the condition (I). Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$. Suppose that*

$$G := \cap_{n=1}^{\infty} F(S_n) \cap A^{-1} \cap_{n=1}^{\infty} F(T_n) \neq \emptyset.$$

Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 = x \in H$, let $\{x_n\} \subset H$ be a sequence generated by

$$\begin{cases} y_n = \big((1-\lambda_n)I + \lambda_n S_n\big)\big(x_n - r_n h_n A^* J_F (I - T_n) A x_n\big), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)\big(\beta_n x_n + (1-\beta_n) y_n\big), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c, d, e, f, g, \lambda_0 \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ *and* $\{\lambda_n\}, \{k_n\}$, $\{h_n\} \subset \mathbb{R}$ *satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1,\ 0 < c \leq |h_n| \leq d,\ \lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$0 < e \leq r_n \leq f < g \leq \frac{2}{\tau_n h_n \|A\|^2},\ 0 < \frac{\lambda_n}{k_n} \leq \frac{2}{\theta_n k_n} \ \ \textit{and}\ \ 0 < \lambda_0 \leq |\lambda_n|$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to a point* $z_0 \in G$, *where* $z_0 = P_G u$.

Proof. Since $S_n : H \to H$ is θ_n-generalized demimetric, $F(S_n)$ is closed and convex from Lemms 2.7. Then $\cap_{n=1}^{\infty} F(S_n)$ is closed and convex. Since $T_n : F \to F$ is τ_n-generalized demimetric, we also have from Lemma 2.7 that $F(T_n)$ is closed and convex. Since $A : H \to F$ is linear and continuous, $A^{-1} \cap_{n=1}^{\infty} F(T_n)$ is closed and convex. Then G is nonempty, closed and convex. Since G is nonempty, closed and convex, the metric projection P_G of H onto G is well-defined.

Since $T_n : F \to F$ is τ_n-generalized demimetric, we have from Lemma 2.8 that $(1-h_n)I + h_n T_n$ is $\tau_n h_n$-generalized demimetric. Since $S_n : H \to H$ is θ_n-generalized demimetric, we also have from Lemma 2.8 that $(1-k_n)I + k_n S_n$ is $\theta_n k_n$-generalized demimetric. Furthermore, from Lemma 2.6 and $\theta_n k_n > 0$, we have that $(1 - k_n)I + k_n S_n$ is $\left(1 - \frac{2}{\theta_n k_n}\right)$-demimetric in the sense of [21]. Since $0 < \frac{\lambda_n}{k_n} \leq \frac{2}{\theta_n k_n} = 1 - \left(1 - \frac{2}{\theta_n k_n}\right)$ and

$$(1-\lambda_n)I + \lambda_n S_n = \left(1 - \frac{\lambda_n}{k_n}\right) I + \frac{\lambda_n}{k_n}((1-k_n)I + k_n S_n),$$

we have from Lemma 2.9 that $(1-\lambda_n)I + \lambda_n S_n$ is quasi-nonexpansive.

Let $z \in G$. We have that $z = S_n z$ and $Az - T_n Az = 0$. Furthermore, putting $z_n = (I - r_n h_n A^* J_F (I - T_n)A)x_n$ and

$$y_n = ((1-\lambda_n)I + \lambda_n S_n)(I - r_n h_n A^* J_F (I - T_n)A)x_n$$

for all $n \in \mathbb{N}$, we have from $\tau_n h_n > 0$ that for all $z \in G$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\|y_n - z\|^2 &= \|((1-\lambda_n)I + \lambda_n S_n)z_n - ((1-\lambda_n)I + \lambda_n S_n)z\|^2 \\
&\leq \|x_n - r_n h_n A^* J_F (I - T_n)Ax_n - z\|^2 \\
&= \|x_n - z - r_n h_n A^* J_F (I - T_n)Ax_n\|^2 \\
&\leq \|x_n - z\|^2 - 2r_n h_n \langle Ax_n - Az, J_F(I - T_n)Ax_n\rangle \\
&\quad + r_n^2 h_n^2 \|A\|^2 \|(I - T_n)Ax_n\|^2 \\
&= \|x_n - z\|^2 - 2r_n \langle Ax_n - Az, J_F(I - ((1-h_n)I + h_n T_n)Ax_n\rangle \qquad (3.1) \\
&\quad + r_n^2 h_n^2 \|A\|^2 \|(I - T_n)Ax_n\|^2 \\
&\leq \|x_n - z\|^2 - 2r_n \frac{1}{\tau_n h_n} \|Ax_n - ((1-h_n)I + h_n T_n)Ax_n\|^2 \\
&\quad + r_n^2 h_n^2 \|A\|^2 \|(I - T_n)Ax_n\|^2 \\
&= \|x_n - z\|^2 - 2r_n h_n^2 \frac{1}{\tau_n h_n} \|Ax_n - T_n Ax_n\|^2 + r_n^2 h_n^2 \|A\|^2 \|(I - T_n)Ax_n\|^2
\end{aligned}$$

$$= \|x_n - z\|^2 + r_n h_n^2 (r_n \|A\|^2 - \frac{2}{\tau_n h_n}) \|(I - T_n)Ax_n\|^2 .$$

From $0 < e \le r_n \le f < g \le \frac{2}{\tau_n h_n \|A\|^2}$ we have that

$$\|y_n - z\| \le \|x_n - z\| \tag{3.2}$$

for all $n \in \mathbb{N}$.

Put $s_n = \beta_n x_n + (1-\beta_n)((1-\lambda_n)I + \lambda_n S_n)(x_n - r_n h_n A^* J_F (I - T_n)Ax_n)$. We have from (2.2) and (3.2) that

$$\begin{aligned} \|s_n - z\|^2 &= \|\beta_n(x_n - z) + (1-\beta_n)(y_n - z)\|^2 \\ &= \beta_n \|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\ &\le \beta_n \|x_n - z\|^2 + (1-\beta_n) \|x_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\ &= \|x_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\ &= \|x_n - z\|^2 . \end{aligned} \tag{3.3}$$

Using this, we get that

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(u_n - z) + (1-\alpha_n)(s_n - z)\| \\ &\le \alpha_n \|u_n - z\| + (1-\alpha_n) \|s_n - z\| \\ &\le \alpha_n \|u_n - z\| + (1-\alpha_n) \|x_n - z\| . \end{aligned}$$

Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}} \|u_n - z\| \le M$. Putting $K = \max\{\|x_1 - z\|, M\}$, we have that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \le K$. Suppose that $\|x_j - z\| \le K$ for some $j \in \mathbb{N}$. Then we have that

$$\begin{aligned} \|x_{j+1} - z\| &\le \alpha_j \|u_j - z\| + (1-\alpha_j)\|x_j - z\| \\ &\le \alpha_j K + (1-\alpha_j)K = K. \end{aligned}$$

By induction, we obtain that $\|x_n - z\| \le K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Furthermore, $\{Ax_n\}$, $\{y_n\}$ and $\{s_n\}$ are also bounded.

Take $z_0 = P_G u$. Since $y_n = ((1-\lambda_n)I + \lambda_n S_n)(I - r_n h_n A^* J_F (I - T_n)A)x_n$, we have from (3.3) that

$$\begin{aligned} \|x_{n+1} - z_0\|^2 &= \|\alpha_n(u_n - z_0) + (1-\alpha_n)(s_n - z_0)\|^2 \\ &\le \alpha_n \|u_n - z_0\|^2 + (1-\alpha_n) \|s_n - z_0\|^2 \\ &\le \alpha_n \|u_n - z_0\|^2 \\ &\quad + (1-\alpha_n)(\|x_n - z_0\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2) \\ &\le \alpha_n \|u_n - z_0\|^2 + \|x_n - z_0\| - \beta_n(1-\beta_n)\|x_n - y_n\|^2. \end{aligned}$$

Using this, we have that

$$\beta_n(1-\beta_n)\|x_n - y_n\|^2 \le \alpha_n \|u_n - z_0\|^2 + \|x_n - z_0\|^2 - \|x_{n+1} - z_0\|^2 . \tag{3.4}$$

Onthe other hand, we have that

$$\begin{aligned} x_{n+1} - x_n &= \alpha_n u_n + (1-\alpha_n)\{\beta_n x_n + (1-\beta_n)y_n\} - x_n \\ &= \alpha_n(u_n - x_n) + (1-\alpha_n)\{\beta_n x_n + (1-\beta_n)y_n - x_n\} \end{aligned}$$

$$
\begin{aligned}
&= \alpha_n(u_n - x_n) + (1-\alpha_n)\{(1-\beta_n)y_n - (1-\beta_n)x_n\} \\
&= \alpha_n(u_n - x_n) + (1-\alpha_n)(1-\beta_n)(y_n - x_n)
\end{aligned}
$$

and hence

$$
\|x_{n+1} - x_n\| \le \alpha_n\|u_n - x_n\| + (1-\alpha_n)(1-\beta_n)\|y_n - x_n\|. \tag{3.5}
$$

We will divide the proof into two cases.

Case 1: Set $\Gamma_n = \|x_n - z_0\|^2$ for all $n \in \mathbb{N}$. Suppose that there exists a natural number N such that $\Gamma_{n+1} \le \Gamma_n$ for all $n \ge N$. In this case, $\lim_{n\to\infty} \Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Using $\lim_{n\to\infty} \alpha_n = 0$ and $0 < a \le \beta_n \le b < 1$, we have from (3.4) that

$$
\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.6}
$$

From (3.5) we have that

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}
$$

We also have that

$$
\begin{aligned}
\|x_{n+1} - s_n\| &= \|\alpha_n u_n + (1-\alpha_n)s_n - s_n\| \\
&= \alpha_n\|u_n - s_n\| \to 0.
\end{aligned} \tag{3.8}
$$

Furthermore, using $\|s_n - y_n\| \le \beta_n\|y_n - x_n\| \to 0$, we have that

$$
\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{3.9}
$$

We show that $\limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle \le 0$, where $z_0 = P_G u$. Put

$$
l = \limsup_{n\to\infty}\langle u - z_0, y_n - z_0\rangle.
$$

Without loss of generality, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $l = \lim_{i\to\infty}\langle u - z_0, y_{n_i} - z_0\rangle$ and $\{y_{n_i}\}$ converges weakly to some point $w \in H$. From $\|x_n - y_n\| \to 0$, $\{x_{n_i}\}$ converges weakly to $w \in H$. On the other hand, from (3.1) we have that

$$
\begin{aligned}
r_n h_n^2\left(\frac{2}{\tau_n h_n} - r_n\|A\|^2\right)\|(I - T_n)Ax_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\
&= (\|x_n - z\| - \|y_n - z\|)(\|x_n - z\| + \|y_n - z\|) \\
&\le \|x_n - y_n\|\left(\|x_n - z\| + \|y_n - z\|\right).
\end{aligned} \tag{3.10}
$$

Then we get from $\|x_n - y_n\| \to 0$ that

$$
\lim_{n\to\infty} \|Ax_n - T_n Ax_n\| = 0. \tag{3.11}
$$

Since $\{x_{n_i}\}$ converges weakly to $w \in H$ and A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Using $\lim_{n\to\infty}\|Ax_n - T_nAx_n\| = 0$ and $\{T_n\}$ satisfies the condition (I), we have $Aw \in \cap_{n=1}^{\infty} F(T_n)$ and hence $w \in A^{-1}\cap_{n=1}^{\infty} F(T_n)$. We also prove $w \in \cap_{n=1}^{\infty} F(S_n)$. We have that

$$
\begin{aligned}
\|z_n - y_n\| &= \|z_n - ((1-\lambda_n)I + \lambda_n S_n)z_n\| \\
&= \|\lambda_n(z_n - S_n z_n)\| \\
&\ge \lambda_0\|z_n - S_n z_n\|.
\end{aligned} \tag{3.12}
$$

Furthemore, from $z_n = x_n - r_n h_n A^* J_F(Ax_n - T_n Ax_n)$, we have that

$$\begin{aligned}\|z_n - y_n\| &= \|x_n - r_n h_n A^* J_F(Ax_n - T_n Ax_n) - y_n\| \\ &\le \|x_n - y_n\| + |r_n h_n| \|A\| \|J_F(Ax_n - T_n Ax_n)\| \\ &= \|x_n - y_n\| + |r_n h_n| \|A\| \|Ax_n - T_n Ax_n\| \\ &\le \|x_n - y_n\| + f \cdot d \|A\| \|Ax_n - T_n Ax_n\|.\end{aligned}$$

Then we have from (3.6) and (3.11) that $\|z_n - y_n\| \to 0$. Using (3.12), we have that

$$\lim_{n\to\infty} \|z_n - S_n z_n\| = 0. \tag{3.13}$$

Since $\|z_n - x_n\| \to 0$ from $\|z_n - x_n\| = \|r_n h_n A^* J_F(Ax_n - T_n Ax_n)\| \to 0$, we also have that $\{z_{n_i}\}$ converges weakly to w. Since $\{S_n\}$ satisfies the condition (I), we have that $w \in \cap_{n=1}^{\infty} F(S_n)$. This implies that $w \in G$. Since $\{y_{n_i}\}$ converges weakly to $w \in G$, we have that

$$l = \lim_{i\to\infty} \langle u - z_0, y_{n_i} - z_0 \rangle = \langle u - z_0, w - z_0 \rangle \le 0.$$

On the other hand, we have from (2.1) that

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \|\alpha_n(u_n - z_0) + (1-\alpha_n)(s_n - z_0)\|^2 \\ &\le (1-\alpha_n)^2 \|s_n - z_0\|^2 + 2\alpha_n \langle u_n - z_0, x_{n+1} - z_0 \rangle \\ &\le (1-\alpha_n) \|x_n - z_0\|^2 + 2\alpha_n \langle u_n - z_0, x_{n+1} - z_0 \rangle \\ &= (1-\alpha_n) \|x_n - z_0\|^2 \\ &\quad + 2\alpha_n \Big(\langle u_n - u, x_{n+1} - z_0 \rangle + \langle u - z_0, x_{n+1} - z_0 \rangle \Big) \\ &= (1-\alpha_n) \|x_n - z_0\|^2 \\ &\quad + 2\alpha_n \Big(\langle u_n - u, x_{n+1} - z_0 \rangle + \langle u - z_0, x_{n+1} - y_n \rangle \\ &\quad + \langle u - z_0, y_n - z_0 \rangle \Big).\end{aligned}$$

Since $\sum_{n=1}^{\infty} \alpha_n = \infty$, by $u_n \to u$, (3.9) and Lemma 2.10 we obtain that $x_n \to z_0$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 2.11 that $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$. Thus we have from (3.4) that for all $n \in \mathbb{N}$,

$$\begin{aligned}&\beta_{\tau(n)}(1-\beta_{\tau(n)}) \|x_{\tau(n)} - y_{\tau(n)}\|^2 \\ &\qquad \le \alpha_{\tau(n)} \left\|u_{\tau(n)} - z_0\right\|^2 + \|x_{\tau(n)} - z_0\|^2 - \left\|x_{\tau(n)+1} - z_0\right\|^2.\end{aligned} \tag{3.14}$$

Using $\lim_{n\to\infty} \alpha_n = 0$ and $0 < a \le \beta_n \le b < 1$, we have from (3.14) that

$$\lim_{n\to\infty} \|y_{\tau(n)} - x_{\tau(n)}\| = 0. \tag{3.15}$$

As in the proof of Case 1 we have that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{3.16}$$

and

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - s_{\tau(n)}\| = 0. \tag{3.17}$$

Using $\|s_{\tau(n)} - y_{\tau(n)}\| \leq \beta_{\tau(n)}\|y_{\tau(n)} - x_{\tau(n)}\| \to 0$, we have that

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - y_{\tau(n)}\| = 0. \tag{3.18}$$

For $z_0 = P_G u$, let us show that $\limsup_{n\to\infty}\langle z_0 - u, y_{\tau(n)} - z_0\rangle \geq 0$. Put

$$l = \limsup_{n\to\infty}\langle z_0 - u, y_{\tau(n)} - z_0\rangle.$$

Without loss of generality, there exists a subsequence $\{y_{\tau(n_i)}\}$ of $\{y_{\tau(n)}\}$ such that $l = \lim_{i\to\infty}\langle z_0 - u, y_{\tau(n_i)} - z_0\rangle$ and $\{y_{\tau(n_i)}\}$ converges weakly to some point $w \in H$. From $\|y_{\tau(n)} - x_{\tau(n)}\| \to 0$, $\{x_{\tau(n_i)}\}$ converges weakly to $w \in H$. As in the proof of Case 1 we have that $w \in G$. Then we have

$$l = \lim_{i\to\infty}\langle z_0 - u, y_{\tau(n_i)} - z_0\rangle = \langle z_0 - u, w - z_0\rangle \geq 0. \tag{3.19}$$

As in the proof of Case 1, we also have that

$$\begin{aligned}
\left\|x_{\tau(n)+1} - z_0\right\|^2 &= \left\|\alpha_{\tau(n)}(u_{\tau(n)} - z_0) + (1 - \alpha_{\tau(n)})(s_{\tau(n)} - z_0)\right\|^2 \\
&\leq (1 - \alpha_{\tau(n)})^2 \left\|s_{\tau(n)} - z_0\right\|^2 \\
&\quad + 2\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, x_{\tau(n)+1} - z_0\rangle \\
&\leq (1 - \alpha_{\tau(n)}) \left\|x_{\tau(n)} - z_0\right\|^2 \\
&\quad + 2\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, x_{\tau(n)+1} - z_0\rangle.
\end{aligned}$$

From $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we have that

$$\alpha_{\tau(n)} \left\|x_{\tau(n)} - z_0\right\|^2 \leq 2\alpha_{\tau(n)}\langle u_{\tau(n)} - z_0, x_{\tau(n)+1} - z_0\rangle.$$

Since $\alpha_{\tau(n)} > 0$, we have that

$$\begin{aligned}
\left\|x_{\tau(n)} - z_0\right\|^2 &\leq 2\langle u_{\tau(n)} - z_0, x_{\tau(n)+1} - z_0\rangle \\
&= 2\Big(\langle u_{\tau(n)} - u, x_{\tau(n)+1} - z_0\rangle \\
&\quad + \langle u - z_0, x_{\tau(n)+1} - y_{\tau(n)}\rangle + \langle u - z_0, y_{\tau(n)} - z_0\rangle\Big).
\end{aligned}$$

Using $u_{\tau(n)} \to u$, (3.18) and (3.19), we have that

$$\limsup_{n\to\infty} \left\|x_{\tau(n)} - z_0\right\|^2 \leq 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$. From (3.16), we have also that $x_{\tau(n)} - x_{\tau(n)+1} \to 0$. Thus $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 2.11 again, we obtain that

$$\|x_n - z_0\| \leq \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

4. Weak convergence theorem

In this section, using the idea of Mann iteration, we prove a weak convergence theorem of finding a solution of the split common fixed point problem for families of generalized demimetric mappings in Banach spaces.

Theorem 4.1. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F. Let $\{\theta_n\}$ and $\{\tau_n\}$ be sequences of real numbers with $\theta_n, \tau_n \neq 0$ and let $\{k_n\}$ and $\{h_n\}$ be sequences of real numbers with $\theta_n k_n > 0$ and $\tau_n h_n > 0$, respectively. Let $\{S_n\}$ be a sequence of θ_n-generalized demimetric mappings of H to H with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ satisfying the condition (I) and let $\{T_n\}$ be a sequence of τ_n-generalized demimetric mappings of F to F with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ satisfying the condition (I). Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$. Suppose that*

$$G := \cap_{n=1}^{\infty} F(S_n) \cap A^{-1} \cap_{n=1}^{\infty} F(T_n) \neq \emptyset.$$

For any $x_1 = x \in H$, define

$$x_{n+1} = \beta_n x_n + (1-\beta_n)((1-\lambda_n)I + \lambda_n S_n)\big(x_n - r_n h_n A^* J_F(Ax_n - T_n Ax_n)\big)$$

for all $n \in \mathbb{N}$, where $a, b, c, d, e, f, g, \lambda_0 \in \mathbb{R}$, $\{\beta_n\} \subset [0,1]$, $\{r_n\} \subset (0,\infty)$ and $\{\lambda_n\}, \{k_n\}, \{h_n\} \subset \mathbb{R}$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1,\ 0 < c \leq |h_n| \leq d,$$

$$0 < e \leq r_n \leq f < g \leq \frac{2}{\tau_n h_n \|A\|^2},\ 0 < \frac{\lambda_n}{k_n} \leq \frac{2}{\theta_n k_n} \quad \text{and} \quad 0 < \lambda_0 \leq |\lambda_n|$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to $z_0 \in G$, where $z_0 = \lim_{n\to\infty} P_G x_n$.

Proof. As in the proof of Theorem 3.1, G is nonempty, closed and convex and hence the metric projection P_G of H onto G is well-defined.

Let $z \in G = \cap_{n=1}^{\infty} F(S_n) \cap A^{-1} \cap_{n=1}^{\infty} F(T_n)$. Then we have that $z = S_n z$ and $Az - T_n Az = 0$. Putting $z_n = x_n - r_n h_n A^* J_F(Ax_n - T_n Ax_n)$ and

$$y_n = ((1-\lambda_n)I + \lambda_n S_n)\big(x_n - r_n h_n A^* J_F(Ax_n - T_n Ax_n)\big)$$

for all $n \in \mathbb{N}$, as in the proof of Theorem 3.1, we have that

$$\begin{aligned} \|y_n - z\|^2 &= \|((1-\lambda_n)I + \lambda_n S_n)z_n - ((1-\lambda_n)I + \lambda_n S_n)z\|^2 \\ &\leq \|x_n - r_n h_n A^* J_F(Ax_n - T_n Ax_n) - z\|^2 \\ &= \|x_n - z - r_n h_n A^* J_F(Ax_n - T_n Ax_n)\|^2 \\ &= \|x_n - z\|^2 - 2\langle x_n - z, r_n h_n A^* J_F(Ax_n - T_n Ax_n)\rangle \\ &\quad + \|r_n h_n A^* J_F(Ax_n - T_n Ax_n)\|^2 \\ &\leq \|x_n - z\|^2 + r_n h_n^2 (r_n \|A\|^2 - \frac{2}{\tau_n h_n}) \|Ax_n - T_n Ax_n\|^2. \end{aligned} \tag{4.1}$$

From $0 < e \leq r_n \leq f < g \leq \frac{2}{\tau_n h_n \|A\|^2}$ we have that

$$\|y_n - z\| \leq \|x_n - z\|$$

for all $n \in \mathbb{N}$ and hence

$$\|x_{n+1} - z\| = \|\beta_n x_n + (1-\beta_n) y_n - z\|$$

$$\begin{aligned}&\leq \beta_n \|x_n - z\| + (1-\beta_n)\|y_n - z\|\\&\leq \beta_n \|x_n - z\| + (1-\beta_n)\|x_n - z\|\\&\leq \|x_n - z\|.\end{aligned}$$

Then $\lim_{n\to\infty}\|x_n - z\|$ exists. Thus $\{x_n\}$, $\{Ax_n\}$ and $\{y_n\}$ are bounded. Using the equality (2.2), we have that for $n \in \mathbb{N}$ and $z \in G$

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|\beta_n x_n + (1-\beta_n)y_n - z\|^2\\&= \beta_n \|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2\\&\leq \beta_n \|x_n - z\|^2 + (1-\beta_n)\|x_n - z\|^2\\&\quad + (1-\beta_n)r_n h_n^2(r_n\|A\|^2 - \frac{2}{\tau_n h_n})\|Ax_n - T_nAx_n\|^2\\&\quad - \beta_n(1-\beta_n)\|x_n - y_n\|^2\\&= \|x_n - z\|^2 + (1-\beta_n)r_n h_n^2(r_n\|A\|^2 - \frac{2}{\tau_n h_n})\|Ax_n - T_nAx_n\|^2\\&\quad - \beta_n(1-\beta_n)\|x_n - y_n\|^2.\end{aligned}$$

Therefore, we have that $\beta_n(1-\beta_n)\|x_n - y_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2$ and

$$(1-\beta_n)r_n h_n^2(\frac{2}{\tau_n h_n} - r_n\|A\|^2)\|Ax_n - T_nAx_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Using $0 < a \leq \beta_n \leq b < 1,\ 0 < c \leq |h_n| \leq d$ and

$$0 < e \leq r_n \leq f < g \leq \frac{2}{\tau_n h_n \|A\|^2},$$

we have that

$$\lim_{n\to\infty}\|x_n - y_n\|^2 = 0 \text{ and } \lim_{n\to\infty}\|Ax_n - T_nAx_n\|^2 = 0. \tag{4.2}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Since A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Using $\lim_{n\to\infty}\|Ax_n - T_nAx_n\| = 0$ and $\{T_n\}$ satisfies the condition (I), we have $Aw \in \cap_{n=1}^{\infty}F(T_n)$ and hence $w \in A^{-1}\cap_{n=1}^{\infty}F(T_n)$. We also prove $w \in \cap_{n=1}^{\infty}F(S_n)$. As in the proof of Theorem 3.1, we have that

$$\|z_n - y_n\| = \|z_n - ((1-\lambda_n)I + \lambda_n S_n)z_n\| \geq \lambda_0\|z_n - S_n z_n\|. \tag{4.3}$$

Furthemore, from $z_n = x_n - r_n h_n A^* J_F(Ax_n - T_nAx_n)$, we have that

$$\begin{aligned}\|z_n - y_n\| &= \|x_n - r_n h_n A^* J_F(Ax_n - T_nAx_n) - y_n\|\\&\leq \|x_n - y_n\| + f\cdot d\|A\|\|Ax_n - T_nAx_n\|.\end{aligned}$$

Then we have from (4.2) that $\|z_n - y_n\| \to 0$. Using (4.3), we have that

$$\lim_{n\to\infty}\|z_n - S_n z_n\| = 0. \tag{4.4}$$

Since $\|z_n - x_n\| \to 0$ from $\|z_n - x_n\| = \|r_n h_n A^* J_F(Ax_n - T_nAx_n)\| \to 0$, we also have that $\{z_{n_i}\}$ converges weakly to w. Since $\{S_n\}$ satisfies the condition (I), we have that $w \in \cap_{n=1}^{\infty}F(S_n)$. This implies that $w \in G$.

We next show that if $x_{n_i} \rightharpoonup x^*$ and $x_{n_j} \rightharpoonup y^*$, then $x^* = y^*$. We know $x^*, y^* \in G$ and hence $\lim_{n\to\infty} \|x_n - x^*\|$ and $\lim_{n\to\infty} \|x_n - y^*\|$ exist. Suppose $x^* \neq y^*$. Since H satisfies Opial's condition, we have that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - x^*\| &= \lim_{i\to\infty} \|x_{n_i} - x^*\| < \lim_{i\to\infty} \|x_{n_i} - y^*\| \\ &= \lim_{n\to\infty} \|x_n - y^*\| = \lim_{j\to\infty} \|x_{n_j} - y^*\| \\ &< \lim_{j\to\infty} \|x_{n_j} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\|.\end{aligned}$$

This is a contradiction. Then we have $x^* = y^*$. Therefore, $x_n \rightharpoonup x^* \in G$. Moreover, since for any $z \in G$

$$\|x_{n+1} - z\| \le \|x_n - z\|, \quad \forall n \in \mathbb{N},$$

we have from Lemma 2.1 that $P_G x_n \to z_0$ for some $z_0 \in G$. The property of metric projection implies that

$$\langle x^* - P_G x_n, x_n - P_G x_n\rangle \le 0.$$

Therefore, we have that $\|x^* - z_0\|^2 = \langle x^* - z_0, x^* - z_0\rangle \le 0$. This means that $x^* = z_0$. Therefore, $x_n \rightharpoonup z_0$, where $z_0 = \lim_{n\to\infty} P_G x_n$. □

5. Applications

In this section, using Theorems 3.1 and 4.1, we get well-known and new strong and weak convergence theorems which are connected with the split common fixed point problem in Hilbert spaces and Banach spaces. We know the following result obtained by Kocourek, Takahashi and Yao [9]; see also [28].

Lemma 5.1 ([9, 28]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Using Lemma 5.1, we have the following result.

Lemma 5.2. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to H$ be generalized hybrid mappings such that $F(S) \cap F(T) \neq \emptyset$ and let $\{\gamma_n\}$ be a sequence of real numbers. Assume that there exist $a, b \in \mathbb{R}$ such that $0 < a \le \gamma_n \le b < 1$ for all $n \in \mathbb{N}$. If $T_n = \gamma_n S + (1 - \gamma_n)T$ for all $n \in \mathbb{N}$, then $\cap_{n=1}^{\infty} F(T_n) = F(S) \cap F(T)$ and $\{T_n\}$ satisfies the condition (I).*

Proof. Since S and T are generalized hybrid mappings and $F(S) \cap F(T) \neq \emptyset$, S and T are quasi-nonexpansive mappings. Using this, we have from (2.2) that, for $z_0 \in F(S) \cap F(T)$, $z \in \cap_{n=1}^{\infty} F(T_n)$ and $n \in \mathbb{N}$,

$$\begin{aligned}\|z - z_0\|^2 &= \|T_n z - z_0\|^2 \\ &= \|(\gamma_n S + (1 - \gamma_n)T)z - z_0\|^2 \\ &= \|\gamma_n(Sz - z_0) + (1 - \gamma_n)(Tz - z_0)\|^2 \\ &= \gamma_n\|Sz - z_0\|^2 + (1 - \gamma_n)\|Tz - z_0\|^2 - \gamma_n(1 - \gamma_n)\|Sz - Tz\|^2 \\ &\le \gamma_n\|z - z_0\|^2 + (1 - \gamma_n)\|z - z_0\|^2 - \gamma_n(1 - \gamma_n)\|Sz - Tz\|^2 \\ &= \|z - z_0\|^2 - \gamma_n(1 - \gamma_n)\|Sz - Tz\|^2.\end{aligned}$$

This means that $\gamma_n(1-\gamma_n)\|Sz-Tz\|^2 \le 0$. Since $0 < a \le \gamma_n \le b < 1$ for all $n \in \mathbb{N}$, we have $Sz = Tz$. Since

$$\begin{aligned}
\|Sz - z\| &= \|\gamma_n Sz + (1-\gamma_n)Sz - z\| \\
&= \|\gamma_n Sz + (1-\gamma_n)Tz - z\| \\
&= \|(\gamma_n S + (1-\gamma_n)T)z - z\| \\
&= \|z - z\| \\
&= 0,
\end{aligned}$$

we have that $Sz = z$. Similarly, we have that $Tz = z$. This implies that $\cap_{n=1}^{\infty} F(T_n) \subset F(S) \cap F(T)$. It is obvious that $F(S) \cap F(T) \subset \cap_{n=1}^{\infty} F(T_n)$. Thus $\cap_{n=1}^{\infty} F(T_n) = F(S) \cap F(T)$.

Suppose that $\{z_n\}$ is a bounded sequence such that $z_n - T_n z_n \to 0$. Then we have from (2.1) and (2.2) that, for $z \in \cap_{n=1}^{\infty} F(T_n)$,

$$\begin{aligned}
\|z_n - z\|^2 &= \|z_n - T_n z_n + T_n z_n - z\|^2 \\
&\le \|T_n z_n - z\|^2 + 2\langle z_n - T_n z_n, z_n - z\rangle \\
&= \|\gamma_n S z_n + (1-\gamma_n)T z_n - z\|^2 + 2\langle z_n - T_n z_n, z_n - z\rangle \\
&= \gamma_n \|S z_n - z\|^2 + (1-\gamma_n)\|T z_n - z\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|S z_n - T z_n\|^2 + 2\langle z_n - T_n z_n, z_n - z\rangle \\
&\le \gamma_n \|z_n - z\|^2 + (1-\gamma_n)\|z_n - z\|^2 \\
&\quad - \gamma_n(1-\gamma_n)\|S z_n - T z_n\|^2 + 2\langle z_n - T_n z_n, z_n - z\rangle \\
&= \|z_n - z\|^2 - \gamma_n(1-\gamma_n)\|S z_n - T z_n\|^2 + 2\langle z_n - T_n z_n, z_n - z\rangle
\end{aligned}$$

and hence

$$\gamma_n(1-\gamma_n)\|S z_n - T z_n\|^2 \le 2\langle z_n - T_n z_n, z_n - z\rangle.$$

Since $z_n - T_n z_n \to 0$ and $\{z_n\}$ is bounded, we have that $Sz_n - Tz_n \to 0$. Using this, we have that

$$\begin{aligned}
\|z_n - S z_n\| &= \|z_n - T_n z_n + T_n z_n - S z_n\| \\
&\le \|z_n - T_n z_n\| + \|T_n z_n - S z_n\| \\
&= \|z_n - T_n z_n\| + (1-\gamma_n)\|T z_n - S z_n\| \\
&\to 0.
\end{aligned}$$

If a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ converges weakly to w, then we have from Lemma 5.1 and $z_n - Sz_n \to 0$ that $w \in F(S)$. Similarly, $w \in F(T)$. Thus every weak cluster point $\{z_n\}$ belongs to $F(S) \cap F(T) = \cap_{n=1}^{\infty} F(T_n)$. This completes the proof. □

Using Theorem 3.1, we get the following strong convergence theorems in Hilbert spaces and Banach spaces.

Theorem 5.3. *Let H_1 and H_2 be Hilbert spaces. Let $S, T : H_1 \to H_1$ be nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$ and let $U, V : H_2 \to H_2$ be nonspreading mappings with $F(U) \cap F(V) \neq \emptyset$. Let $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences of real numbers. Assume that there exists $s, t, u, v \in \mathbb{R}$ such that $0 < s \le \gamma_n \le t < 1$ and $0 < u \le \delta_n \le v < 1$ for all $n \in \mathbb{N}$. Let $A : H_1 \to H_2$ be a bounded linear operator* *such that $A \neq 0$. Suppose that $G := F(S) \cap F(T) \cap A^{-1}(F(U) \cap F(V)) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. For $x_1 = x \in H_1$, let $\{x_n\} \subset H_1$ be a sequence generated by*

$$\begin{cases} y_n = \big(\gamma_n S + (1-\gamma_n)T\big)\big(x_n - r_n A^*(Ax_n - (\delta_n U + (1-\delta_n)V)Ax_n)\big), \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)\big(\beta_n x_n + (1-\beta_n)y_n\big), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, e, f \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1, \ \lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$$

$$\text{and} \ \ 0 < e \leq r_n \leq f < \frac{1}{\|A\|^2}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in G$, where $z_0 = P_G u$.

Proof. Since S and T are nonexpansive mappings of H_1 into H_1, S and T are generalized hybrid. Since U and V are nonspreading mappings of H_2 into H_2, S and T are also generalized hybrid. From $F(S) \cap F(T) \neq \emptyset$ and $F(U) \cap F(V) \neq \emptyset$, $S_n = \gamma_n S + (1-\gamma_n)T$ and $T_n = \delta_n U + (1-\delta)V$ are quasi-nonexpansive mappings and hence they are 2-generalized demimetric mappings. Furthermore, $\{S_n\}$ and $\{T_n\}$ satisfy the condition (I) from Lemma 5.2. Putting $k_n = 1$, $h_n = 1$ and $\lambda_n = 1$ in Theorem 3.1, we obtain the desired result from Theorem 3.1. $\square$

Theorem 5.4. *Let H be a Hilbert space and let F be a uniformly convex and smooth Banach space. Let J_F be the duality mapping on F. Let G and B be maximal monotone operators of H into H and F into F^*, respectively. Let J_ν be the resolvent of G for $\nu > 0$ and let Q_μ be the metric resolvent of B for $\mu > 0$, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 = x \in H$, let $\{x_n\} \subset H$ be a sequence generated by*

$$x_{n+1} = \alpha_n u_n + (1-\alpha_n)\Big(\beta_n x_n + (1-\beta_n)J_{\nu_n}(x_n - r_n A^* J_F (I - Q_{\mu_n})Ax_n)\Big)$$

for all $n \in \mathbb{N}$, where $a, b, e, f, \lambda_0 \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\}, \{\nu_n\}, \{\mu_n\} \subset (0,\infty)$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1, \quad 0 < \lambda_0 \leq \nu_n, \mu_n, \quad \lim_{n\to\infty} \alpha_n = 0,$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < e \leq r_n \leq f < \frac{2}{\|A\|^2}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = P_{G^{-1}0 \cap A^{-1}(B^{-1}0)} u$.

Proof. Since Q_{μ_n} is the metric resolvent of B for $\mu_n > 0$, from (4) in Examples, Q_{μ_n} is 1-generalized demimetric. We also have that if $\{z_n\}$ is a bounded sequence in F such that $z_n - Q_{\mu_n} z_n \to 0$, then every weak cluster point of $\{z_n\}$ belongs to $B^{-1}0 = \cap_{n=1}^{\infty} F(Q_{\mu_n})$. In fact, suppose that $\{z_{n_i}\}$ is a subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup p$ and $z_n - Q_{\mu_n} z_n \to 0$. Since Q_{μ_n} is the metric resolvent of B, we have that

$$J_F(z_n - Q_{\mu_n} z_n)/\mu_n \in BQ_{\mu_n} z_n$$

for all $n \in \mathbb{N}$. From the monotonicity of B, we have

$$0 \leq \langle u - Q_{\mu_{n_i}} z_{n_i}, v^* - \frac{J_F(z_{n_i} - Q_{\mu_{n_i}} z_{n_i})}{\mu_{n_i}} \rangle$$

for all $(u, v^*) \in B$ and $i \in \mathbb{N}$. Taking the limit $i \to \infty$, we conclude that $\langle u - p, v^* \rangle \geq 0$ for all $(u, v^*) \in B$. Since B is a maximal monotone operator, we have $p \in B^{-1}0 = \cap_{n=1}^{\infty} F(Q_{\mu_n})$. This means that the family $\{Q_{\mu_n}\}$ satisfies the condition (I). Similarly, J_{ν_n} is 1-generalized demimetric and the family $\{J_{\nu_n}\}$ satisfies the condition (I). Putting $k_n = 1$, $h_n = 1$ and $\lambda_n = 1$ in Theorem 3.1, we have the desired result from Theorem 3.1. □

Theorem 5.5. *Let H be a Hilbert space and let F be a smooth, strictly convex and reflexive Banach space. Let J_F be the duality mapping on F. Let $\{\theta_n\}$ and $\{\tau_n\}$ be sequences of real numbers with $\theta_n, \tau_n \in (-\infty, 1)$. Let $\{S_n\}$ be a sequence of θ_n-demimetric mappings of H to H with $\cap_{n=1}^{\infty} F(S_n) \neq \emptyset$ satisfying the condition (I) and let $\{T_n\}$ be a sequence of τ_n-demimetric mappings of F to F with $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ satisfying the condition (I). Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$. Suppose that*

$$G := \cap_{n=1}^{\infty} F(S_n) \cap A^{-1} \cap_{n=1}^{\infty} F(T_n) \neq \emptyset.$$

Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 = x \in H$, let $\{x_n\} \subset H$ be a sequence generated by

$$\begin{cases} y_n = \big((1 - \lambda_n)I + \lambda_n S_n\big)\big(x_n - r_n A^* J_F (I - T_n) A x_n\big), \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)\big(\beta_n x_n + (1 - \beta_n) y_n\big), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, e, f, \lambda_0 \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$, $\{r_n\} \subset (0, \infty)$ and $\{\lambda_n\} \subset \mathbb{R}$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1, \ \lim_{n \to \infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$0 < e \leq r_n \leq f < g \leq \frac{1 - \tau_n}{\|A\|^2} \quad \text{and} \quad 0 < \lambda_0 \leq \lambda_n \leq 1 - \theta_n$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in G$, where $z_0 = P_G u$.

Proof. Since S_n is θ_n-demimetric, S_n is $\frac{2}{1-\theta_n}$-generalized demimetric and $\frac{2}{1-\theta_n} > 0$. Similarly, T_n is $\frac{2}{1-\tau_n}$-generalized demimetric and $\frac{2}{1-\tau_n} > 0$. Putting $k_n = 1$ and $h_n = 1$ in Theorem 3.1 and taking λ_n as $0 < \lambda_0 \leq \lambda_n \leq 1 - \theta_n$, we have the desired result from Theorem 3.1. □

Using Theorem 4.1, we get the following weak convergence theorems in Hilbert spaces and Banach spaces.

Theorem 5.6. *Let H_1 and H_2 be Hilbert spaces. Let $S, T : H_1 \to H_1$ be nonexpansive mappings with $F(S) \cap F(T) \neq \emptyset$ and let $U, V : H_2 \to H_2$ be nonspreading* *mappings with* $F(U) \cap F(V) \neq \emptyset$. *Let* $\{\gamma_n\}$ *and* $\{\delta_n\}$ *be sequences of real numbers. Assume that there exists* $s, t, u, v \in \mathbb{R}$ *such that* $0 < s \leq \gamma_n \leq t < 1$ *and* $0 < u \leq \delta_n \leq v < 1$ *for all* $n \in \mathbb{N}$. *Let* $A : H_1 \to H_2$ *be a bounded linear operator such that* $A \neq 0$. *Suppose that* $G := F(S) \cap F(T) \cap A^{-1}(F(U) \cap F(V)) \neq \emptyset$.

For any $x_1 = x \in H_1$, *define*

$$\begin{cases} z_n = x_n - r_n A^*(Ax_n - (\delta_n U + (1-\delta_n)V)Ax_n), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\gamma_n S + (1-\gamma_n)T)z_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, e, f \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1, \ \lim_{n\to\infty} \alpha_n = 0, \ \sum_{n=1}^{\infty} \alpha_n = \infty$$

$$\text{and} \quad 0 < e \leq r_n \leq f < \frac{1}{\|A\|^2}$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges weakly to a point* $z_0 \in F(S) \cap F(T) \cap A^{-1}(F(U) \cap F(V))$, *where* $z_0 = \lim_{n\to\infty} P_{F(T)\cap A^{-1}(F(U)\cap F(V))} x_n$.

Theorem 5.7. *Let* H *be a Hilbert space and let* F *be a uniformly convex and smooth Banach space. Let* J_F *be the duality mapping on* F. *Let* G *and* B *be maximal monotone operators of* H *into* H *and* F *into* F^*, *respectively. Let* J_ν *be the resolvent of* G *for* $\nu > 0$ *and let* Q_μ *be the metric resolvent of* B *for* $\mu > 0$, *respectively. Let* $A : H \to F$ *be a bounded linear operator such that* $A \neq 0$ *and let* A^* *be the adjoint operator of* A. *Suppose that* $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. *For any* $x_1 = x \in H$, *define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) J_{\nu_n}\big(x_n - r_n A^* J_F(Ax_n - Q_{\mu_n} Ax_n)\big)$$

for all $n \in \mathbb{N}$, *where* $a, b, e, f, \lambda_0 \in \mathbb{R}$, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *and* $\{r_n\}, \{\nu_n\}, \{\mu_n\} \subset (0,\infty)$ *satisfy the following:*

$$0 < a \leq \beta_n \leq b < 1, \quad 0 < \lambda_0 \leq \nu_n, \mu_n, \quad \lim_{n\to\infty} \alpha_n = 0,$$

$$\sum_{n=1}^{\infty} \alpha_n = \infty \quad \text{and} \quad 0 < e \leq r_n \leq f < \frac{2}{\|A\|^2}$$

for all $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges weakly to a point* $z_0 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, *where* $z_0 = \lim_{n\to\infty} P_{G^{-1}0\cap A^{-1}(B^{-1}0)} x_n$.

Theorem 5.8 ([27])**.** *Let* H *be a Hilbert space, let* F *be a smooth, strictly convex and reflexive Banach space and let* J_F *be the duality mapping on* F. *Let* $\{\theta_n\}$ *and* $\{\eta_n\}$ *be sequences of real numbers with* $\theta_n, \eta_n \in (-\infty, 1)$. *Let* $\{\lambda_n\}$ *be a sequence of real numbers such that for some* $\lambda_0 \in \mathbb{R}$, $0 < \lambda_0 \leq \lambda_n \leq 1 - \theta_n$ *for all* $n \in \mathbb{N}$. *Let* $\{T_n\}$ *be a sequence of* θ_n*-demimetric mappings of* H *to* H *with* $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$ *satisfying the condition (I) and let* $\{U_n\}$ *be a sequence of* η_n*-demimetric mappings of* F *to* F *with* $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$ *satisfying the condition (I). Let* $A : H \to F$ *be a bounded linear operator such that* $A \neq 0$. *Suppose that* $G := \cap_{n=1}^{\infty} F(T_n) \cap A^{-1} \cap_{n=1}^{\infty} F(U_n) \neq \emptyset$. *For any* $x_1 = x \in H$, *define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)((1-\lambda_n)I + \lambda_n T_n)\big(I - r_n A^* J_F(A - U_n A)\big)x_n$$

for all $n \in \mathbb{N}$, where $\{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1 \quad \text{and} \quad 0 < c \leq r_n\|AA^*\| \leq d < e \leq 1 - \eta_n$$

for some $a, b, c, d, e \in \mathbb{R}$. Then $\{x_n\}$ converges weakly to a point $z_0 \in G$, where $z_0 = \lim_{n\to\infty} P_G x_n$.

Problem. We do not know whether "Hilbert spaces" in Theorem 3.1 and Theorem 4.1 can be replaced by "Banach spaces" or not.

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