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MINIMAL SOLUTIONS OF SET OPTIMIZATION PROBLEMS

ELISABETH KÖBIS AND THANH TAM LE

Abstract. In this paper, a novel solution concept for set optimization problems is established. In particular, we introduce the notion of strong minimal solutions order to fill the gap between the existing concepts of strict and ideal minimal solutions. Moreover, we develop numerical procedures to obtain strong, strict and ideal minimal solutions of set optimization problems. Specifically, several extensions of the well-known Jahn-Graef-Younes method to set optimization are proposed under broad assumptions. Our methods can highly reduce the numerical effort that originates from the comparison of sets that is involved when solving set optimization problems.

1. Introduction

It is well known that set optimization has recently been investigated intensively because of its several important applications, see [7, 8, 9, 10, 11]. In this paper, we are concerned with the so-called set approach in order to define solutions of set optimization problems. The main idea of this concept is based on comparisons of sets in image space of the objective function, see [15, 16]. We investigate not only known but also novel solution concepts in order to fulfill the necessity of deriving types of solutions which will be useful from the mathematical point of view as well as from the practical perspective.

The main goal of this manuscript is deriving numerical methods for computing different types of solutions for set optimization problems. In the literature, there only exist few algorithms for solving set-valued optimization problems. A descent method given by Jahn [4] under convexity assumptions computes approximations of minimal elements. This approach is also further investigated in [8, 11]. In this paper, while using the so-called set approach, we extend the so-called Jahn-Graef-Younes method, which was introduced by Younes [17], Jahn and Rathje [6] for the case of vector optimization. The advantage of this procedure is that it is not necessary to utilize the transitive property of set relations as well as the convexity of the sets under consideration. In addition, our method can reduce the numerical effort while sorting out solutions which do not belong to the set of desired solutions.

This paper is organized as follows: In Section 2, we recall and introduce different solution concepts for set optimization problems. We also investigate properties and relationships among these solution notions. Section 3 involves existence results for set optimization, where the semicontinuity of the objective function plays a key role. In Section 4, we derive numerical methods for computing strong, strict and ideal minimal solutions of a given set optimization problem. For an illustration, we

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deal with a detailed example, where the objective function is not a convex-valued mapping. Section 5 concludes this work.

2. Preliminaries

Throughout this manuscript, let X, Y be linear spaces, and denote the *power set* of Y without the empty set by

$$\mathcal{P}(Y) := \{A \subset Y \mid A \text{ is nonempty}\}.$$

Let $S \subset X$ be a nonempty set, $F : S \rightrightarrows Y$ be a given set-valued map and $\preceq$ be a set relation, which is a binary relation among sets. If for some $A, B \in \mathcal{P}(Y)$, $A \preceq B$ and $B \preceq A$ holds, then we denote $A \sim B$. Furthermore, for $A \in \mathcal{P}(Y)$, we define $[A] := \{B \in \mathcal{P}(Y) \mid B \sim A\}$. We denote the family of sets $\bigcup_{x \in S} F(x)$ by $F(S)$. Let us consider the problem of minimizing the set-valued map F, denoted by

$$\min_{x \in S} F(x). \tag{2.1}$$

Several notions of minimality are collected in the definition below (see [5]).

Definition 2.1. Consider problem (2.1) in which the set relation $\preceq$ and $\bar{x} \in S$ are given. We say that:

(a) $\bar{x}$ is a minimal solution of problem (2.1) w.r.t. $\preceq$ if

$$x \in S, F(x) \preceq F(\bar{x}) \text{ implies } F(\bar{x}) \preceq F(x).$$

We denote by $\mathrm{Min}(F(S), \preceq)$ the set of all minimal solutions of problem (2.1) w.r.t. $\preceq$.

(b) $\bar{x}$ is a strong minimal solution of problem (2.1) w.r.t. $\preceq$ if

$$x \in S, F(x) \preceq F(\bar{x}) \text{ implies } F(\bar{x}) = F(x).$$

We denote the set of all strong minimal solutions of problem (2.1) w.r.t. $\preceq$ by $\mathrm{SoMin}(F(S), \preceq)$.

(c) $\bar{x}$ is a strict minimal solution of problem (2.1) w.r.t. $\preceq$ if

$$x \in S, F(x) \preceq F(\bar{x}) \text{ implies } \bar{x} = x.$$

We denote the set of all strict minimal solutions of problem (2.1) w.r.t. $\preceq$ by $\mathrm{SiMin}(F(S), \preceq)$.

(d) $\bar{x}$ is an ideal minimal solution of problem (2.1) w.r.t. $\preceq$ if

$$\forall\, x \in S \setminus \{\bar{x}\} : \; F(\bar{x}) \preceq F(x).$$

We denote the set of all ideal minimal solutions of problem (2.1) w.r.t. $\preceq$ by $\mathrm{IMin}(F(S), \preceq)$.

Remark 2.2. (1) The reason for deriving the definition of strong minimal solutions is that we would like to introduce a new useful solution concept concerning comparisons of elements in the image space, which includes the definition of strict minimal solutions where the comparison of elements in the pre-image space is involved. In addition, from the practical point of view, it is more appropriate if we consider this concept since we often take the set S containing different elements. Observe also that if $\preceq$ is antisymmetric then a minimal solution is also a strong minimal solution, see [5, Remark 5.1 (ii)]. In addition, we can see that if $\bar{x} \in \mathrm{SoMin}(F(S), \preceq)$, then for all $y \in S, F(y) \neq F(\bar{x})$ it holds that $F(y) \not\preceq F(\bar{x})$.

(2) Let $Y = \mathbb{R}^k$, $C = \mathbb{R}^k_+$, and $F : X \to Y$. Suppose that $\preceq$ is given by the partial ordering $\leq_C$, where for $y_1, y_2 \in Y$, we have $y_1 \leq_C y_2$ iff $y_1 \in y_2 - C$. Then $\bar{x} \in \mathrm{Min}(F(S), \preceq) = \mathrm{SoMin}(F(S), \preceq)$, which means that $\bar{x}$ is a Pareto optimal solution (see Ehrgott [1]). Moreover, $\bar{x} \in \mathrm{SiMin}(F(S), \preceq)$ means that $\bar{x}$ is a strictly Pareto optimal solution as defined by Ehrgott [1]. In that sense, the above definitions are meaningful extensions for minimality notions from the vector-valued to the set-valued case. In the same setting, if $\bar{x}$ is an ideal minimal solution, then $\bar{y} := f(\bar{x})$ is an ideal point as defined by Ehrgott [1].

The following result is straightforward.

Lemma 2.3. *The relation*

$$\mathrm{SiMin}(F(S), \preceq) \subseteq \mathrm{SoMin}(F(S), \preceq)$$

holds true. Moreover, if $\preceq$ is reflexive, then we have

$$\mathrm{SoMin}(F(S), \preceq) \subseteq \mathrm{Min}(F(S), \preceq) \text{ and } \mathrm{IMin}(F(S), \preceq) \subseteq \mathrm{Min}(F(S), \preceq).$$

The converse inclusions of the relations stated in Lemma 2.3 are generally not fulfilled, which the following example illustrates.

Example 2.4. Let $S = \{x_1, x_2\}$, $F : S \rightrightarrows \mathbb{R}^2$ be defined by

$$F(x) = \begin{cases} \{y \in \mathbb{R}^2 \mid y_1, y_2 \geq 0,\ y_2 \leq y_1 - 1\}, & x = x_1, \\ \{y \in \mathbb{R}^2 \mid y_1, y_2 \geq 0,\ y_2 = y_1 - 1\}, & x = x_2, \end{cases}$$

We use the set relation $\preceq^u_C$ defined by

$$A \preceq^u_C B \quad :\Longleftrightarrow \quad A \subseteq B - C$$

with $C := \mathbb{R}^2_+$. Then both $x_1, x_2 \in \mathrm{Min}(F(S), \preceq)$, but $x_1, x_2 \notin \mathrm{SoMin}(F(S), \preceq)$. Now assume that $F(x) := \{y \in \mathbb{R}^2 \mid y_2 \leq y_1 - 1\ y_1, y_2 \geq 0\}$ for $x \in S = \{x_1, x_2\}$. If $x_1 \neq x_2$, then $x_1, x_2 \in \mathrm{SoMin}(F(S), \preceq)$, but $x_1, x_2 \notin \mathrm{SiMin}(F(S), \preceq)$. If $x_1 = x_2$, then $x_1, x_2 \in \mathrm{SiMin}(F(S), \preceq)$.

Lemma 2.5. *Let the set relation $\preceq$ be reflexive and suppose that $\bar{x} \in \mathrm{IMin}(F(S), \preceq)$. Define $\overline{S} := \{x \in S \mid F(x) \sim F(\bar{x})\}$. Then*

$$\mathrm{IMin}(F(S), \preceq) \subseteq \overline{S}.$$

Conversely, if $\preceq$ is additionally transitive, then

$$\mathrm{IMin}(F(S), \preceq) \supseteq \overline{S}.$$

Proof. Let $x \in \mathrm{IMin}(F(S), \preceq)$. If $x = \bar{x}$, there is nothing to show. Therefore, we assume $x \neq \bar{x}$. From $x, \bar{x} \in \mathrm{IMin}(F(S), \preceq)$, we immediately obtain $F(x) \sim F(\bar{x})$. Conversely, let $x \in \overline{S}$. Again, we assume $x \neq \bar{x}$, since otherwise there is nothing to show. Then $F(x) \preceq F(\bar{x})$ and $F(\bar{x}) \preceq F(x)$. Because $\bar{x} \in \mathrm{IMin}(F(S), \preceq)$ and

$\preceq$ is transitive, we get $F(x) \preceq F(\bar{x}) \preceq F(\tilde{x})$ for all $\tilde{x} \in S \setminus \{\bar{x}\}$. This means that $x \in \text{IMin}(F(S), \preceq)$. □

We will use the result of the following Corollary in Algorithm 4.20.

Corollary 2.6. *If $\bar{x} \in \text{IMin}(F(S), \preceq)$ and $\preceq$ is reflexive and transitive, then*

$$\text{IMin}(F(S), \preceq) = \{x \in S \mid F(x) \sim F(\bar{x})\}.$$

The above result does not hold if $\preceq$ is not transitive, which the following example shows.

Example 2.7. Let $Y = \mathbb{R}^2$ and let the set relation be given by the possibly set order $\preceq =: \preceq_C^p$, (see also [7, Page 51]) for $A, B \in \mathcal{P}(Y)$ by

$$A \preceq_C^p B \quad :\Longleftrightarrow \quad \exists\, a \in A,\ \exists\, b \in B:\ a \in b - C$$

with $C := \mathbb{R}^2_+$. Let $S = \{x_1, x_2, x_3\}$ and $F(x_1) = \{(0,0), (2,2)\}$, $F(x_2) = \{(1,1)\}$ and $F(x_3) = \{(2, 0.5)\}$. Then $x_1 \in \text{IMin}(F(S), \preceq)$, since $F(x_1) \preceq_C^p F(x_2)$ and $F(x_1) \preceq_C^p F(x_3)$. It also holds that $F(x_1) \sim F(x_2)$, but $x_2 \notin \text{IMin}(F(S), \preceq)$, as $F(x_2) \not\preceq F(x_3)$.

Proposition 2.8. *Let $F : X \rightrightarrows Y$ be a set-valued map, $\preceq$ be a given set relation and $S \subseteq X$ be a nonempty subset of X. Then, we have the following inclusions:*

(i) *If $\preceq$ is reflexive, then* $\text{SiMin}(F(S), \preceq) \subseteq \big\{\bar{x} \in S \mid [F(\bar{x})] = \{F(\bar{x})\}\big\}$.

(ii) $\text{SoMin}(F(S), \preceq) \subseteq \big\{\bar{x} \in S \mid [F(\bar{x})] \subseteq \{F(x) : F(x) = F(\bar{x})\}\big\}$.

(iii) *If there are $x_1 \neq x_2, x_1, x_2 \in \text{IMin}(F(S), \preceq)$ then* $\text{SiMin}(F(S), \preceq) = \emptyset$.

(iv) *If there are $x_1 \neq x_2, x_1, x_2 \in \text{SiMin}(F(S), \preceq)$ then* $\text{IMin}(F(S), \preceq) = \emptyset$.

Proof. (i) Let $\bar{x} \in \text{SiMin}(F(S), \preceq)$ and suppose by contradiction that $[F(\bar{x})] \neq \{F(\bar{x})\}$. Then, we have $\{F(\bar{x})\} \not\subseteq [F(\bar{x})]$ or $[F(\bar{x})] \not\subseteq \{F(\bar{x})\}$. The first assertion immediately yields a contradiction to the reflexivity of $\preceq$. If $[F(\bar{x})] \not\subseteq \{F(\bar{x})\}$, then there exists some $x \in S$ such that $F(x) \preceq F(\bar{x})$. Taking into account $\bar{x} \in \text{SiMin}(F(S), \preceq)$, it holds that $x = \bar{x}$. That yields a contradiction to $[F(\bar{x})] \not\subseteq \{F(\bar{x})\}$.

(ii) Let $\bar{x} \in \text{SoMin}(F(S), \preceq)$ and suppose by contradiction that there is $x' \in S$ such that $F(x') \in [F(\bar{x})], F(x') \neq F(\bar{x})$. Since $F(x') \in [F(\bar{x})]$, we get that $F(x') \preceq F(\bar{x})$. Taking into account $\bar{x} \in \text{SoMin}(F(S), \preceq)$, it holds that $F(x') = F(\bar{x})$, a contradiction. Thus, $\text{SoMin}(F(S), \preceq) \subseteq \big\{\bar{x} \in S \mid [F(\bar{x})] \subseteq \{F(x) : F(x) = F(\bar{x})\}\big\}$.

(iii) Suppose by contradiction that there is $\bar{x} \in \text{SiMin}(F(S), \preceq)$. If $x_1 = \bar{x}$ ($x_2 = \bar{x}$, respectively), it is obvious that $F(x_2) \preceq F(\bar{x})$ ($F(x_1) \preceq F(\bar{x})$, respectively). This implies $x_2 = \bar{x} = x_1$, a contradiction to $x_1 \neq x_2$. If $x_1 \neq \bar{x}$ and $x_2 \neq \bar{x}$, We have that

$$x_1 \in \text{IMin}(F(S), \preceq) \Rightarrow F(x_1) \preceq F(\bar{x}) \Rightarrow \bar{x} = x_1$$

and

$$x_2 \in \text{IMin}(F(S), \preceq) \Rightarrow F(x_2) \preceq F(\bar{x}) \Rightarrow \bar{x} = x_2.$$

Therefore $x_1 = x_2$, a contradiction.

(iv) Suppose that there is $\bar{x} \in \text{IMin}(F(S), \preceq)$. If $x_1 = \bar{x}$ ($x_2 = \bar{x}$, respectively), it is obvious that $F(\bar{x}) \preceq F(x_2)$ ($F(\bar{x}) \preceq F(x_1)$, respectively). Because $x_2 \in \text{SiMin}(F(S), \preceq)$ ($x_1 \in \text{SiMin}(F(S), \preceq)$, respectively), this implies $x_2 = \bar{x} = x_1$, a contradiction. If $x_1 \neq \bar{x}$ and $x_2 \neq \bar{x}$, we have that

$$x_1 \in \text{SiMin}(F(S), \preceq) \Rightarrow F(\bar{x}) \preceq F(x_1) \Rightarrow x_1 = \bar{x}$$

and

$$x_2 \in \text{SiMin}(F(S), \preceq) \Rightarrow F(\bar{x}) \preceq F(x_2) \Rightarrow x_2 = \bar{x}.$$

Therefore $x_1 = x_2$, a contradiction.

□

Remark 2.9. Similar results as parts (iii) and (iv) in Proposition 2.8 for minimal elements of a family of sets are given in [14].

The following notion of *external stability* of a solution set will be used in some results in Section 4.

Definition 2.10 (External Stability)**.** We say that $\text{Min}(F(S), \preceq)$ ($\text{SoMin}(F(S), \preceq), \text{SiMin}(F(S), \preceq)$, $\text{IMin}(F(S), \preceq)$, respectively) is **externally stable** if for all $x \notin \text{Min}(F(S), \preceq)$ ($x \notin \text{SoMin}(F(S), \preceq)$, $x \notin \text{SiMin}(F(S), \preceq)$, $x \notin \text{IMin}(F(S), \preceq)$, respectively), there exists some $\bar{x} \in \text{Min}(F(S), \preceq)$ ($\bar{x} \in \text{SoMin}(F(S), \preceq)$, $\bar{x} \in \text{SiMin}(F(S), \preceq)$, $\bar{x} \in \text{IMin}(F(S), \preceq)$, respectively) such that $F(\bar{x}) \preceq F(x)$.

A sufficient condition for external stability of the set $\text{Min}(F(S), \preceq)$ is derived in [13]. We can easily obtain from Definition 2.1 that if the set $\text{IMin}(F(S), \preceq)$ is nonempty, then it is externally stable. In the following, we give sufficient conditions for the set of strong and strict minimal solutions of problem (2.1) to be externally stable.

Lemma 2.11. *Let $S \subset X$ be a set consisting of finitely many elements and let the set relation $\preceq$ be transitive and antisymmetric. Assume that the set* $\text{SoMin}(F(S), \preceq)$ *is nonempty, then* $\text{SoMin}(F(S), \preceq)$ *is externally stable.*

Proof. Let some $x \in S$ be given, and assume that x is not a strong minimal solution of $F(S)$ w.r.t. $\preceq$. Then there exists some $x_1 \in S$ such that $F(x_1) \preceq F(x)$ and $F(x) \neq F(x_1)$. If $x_1 \in \text{SoMin}(F(S), \preceq)$, then there is nothing to show. If $x_1 \notin \text{SoMin}(F(S), \preceq)$, then there exists some $x_2 \in S$ with $F(x_2) \preceq F(x_1)$ and $F(x_1) \neq F(x_2)$. This procedure continues until a strong minimal solution is found. Suppose that it does not. Because the set S is finite, we would eventually obtain an element which is equal to an element that has already been investigated. Let some $j \in \mathbb{N}$ be given such that $x_j \notin \text{SoMin}(F(S), \preceq)$. Then, there exists x_{j+1} such that $F(x_{j+1}) \preceq F(x_j)$ and $F(x_{j+1}) \neq F(x_j)$. Without loss of generality, we assume that $F(x_{j+1}) = F(x_i)$ for some $i \leq j$. Due to the transitivity of $\preceq$, we get $F(x_{j+1}) \preceq F(x_j) \preceq F(x_{j+1}) = F(x_i)$. By the antisymmetry of $\preceq$, we obtain $F(x_j) = F(x_{j+1})$, which is a contradiction. Therefore, the procedure stops with a strong minimal solution. □

Lemma 2.12. *Let $S \subset X$ be a set consisting of finitely many elements and let the set relation $\preceq$ be transitive and antisymmetric. Assume that the set* $\mathrm{SiMin}(F(S), \preceq)$ *is nonempty, then* $\mathrm{SiMin}(F(S), \preceq)$ *is externally stable.*

Proof. We follow the same lines given in the proof of Lemma 2.11. □

3. Existence results

Existence results for set optimization are investigated by many authors in the literature, see [7]. In this part, we briefly recall a recent result which is introduced in [5] to derive some corresponding conditions for the existence of a strong (strict) minimal solution set of the problem (2.1). We use the following definition of semicontinuity of a set-valued map w.r.t. a preorder $\preceq$ (see [5]).

Definition 3.1 (Semicontinuity)**.** Let $S \subseteq \mathbb{R}^n$. The set-valued mapping $F\colon S \rightrightarrows \mathbb{R}^m$ is called **semicontinuous** at $\bar{x} \in S$ w.r.t. the preorder $\preceq$ if $F(\bar{x}) \in \mathcal{V}$, where $\mathcal{V} := \{T \in \mathcal{A} \mid T \npreceq V\}$ for some $V \in \mathcal{P}(\mathbb{R}^m)$, implies that there exists a neighborhood U of $\bar{x}$ in $\mathbb{R}^n$ such that $F(x) \in \mathcal{V}$ for all $x \in U$. In other words, F is semicontinuous at $\bar{x}$ if

$$F(\bar{x}) \npreceq V \text{ for some } V \in \mathcal{P}(\mathbb{R}^m) \implies \exists\, U(\bar{x}) : \; F(x) \npreceq V \;\forall\, x \in U.$$

F is called semicontinuous on S w.r.t. $\preceq$ if F is semicontinuous w.r.t. $\preceq$ at every $\bar{x} \in S$.

An equivalent characterization of the semicontinuity w.r.t. the preorder $\preceq$ is presented as follows.

Proposition 3.2. [5, Proposition 4.3] *The following assertions are equivalent:*

(i) *F is semicontinuous at $\bar{x} \in S$ w.r.t. the preorder $\preceq$.*
(ii) *The level set w.r.t. the preorder of F at $F(\bar{x})$*

$$\mathcal{L}_F(F(\bar{x})) := \{x \in S |\; F(x) \preceq F(\bar{x})\}$$

is closed.

Now, we have the following existence result for the set $\mathrm{Min}(F(S), \preceq)$. The proof of this is based on the completeness property of the set $F(S)$, see [5, Theorem 4.1, Theorem 5.1] for more detail.

Corollary 3.3 ([5])**.** *Suppose that S is compact and that F is semicontinuous on S w.r.t. the preorder $\preceq$. Then* $\mathrm{Min}(F(S), \preceq) \neq \emptyset$.

Observe that if $\preceq$ is antisymmetric, then $\mathrm{Min}(F(S), \preceq) = \mathrm{SoMin}(F(S), \preceq)$, see also [5, Remark 5.1 (ii)]. Therefore, we obtain the following result concerning the existence of strong minimal solutions of the problem (2.1) by applying Corollary 3.3.

Lemma 3.4. *Suppose that S is compact and that F is semicontinuous on S w.r.t. the antisymmetric preorder $\preceq$. Then,* $\mathrm{SoMin}(F(S), \preceq) \neq \emptyset$.

The following lemma illustrates relationships between two sets of strict minimal solutions and strong minimal solutions of the problem (2.1).

Lemma 3.5. *If for all* $x_1, x_2 \in S$*,* $x_1 \neq x_2$*,* $F(x_1) \neq F(x_2)$*, then* $\mathrm{SiMin}(F(S), \preceq) = \mathrm{SoMin}(F(S), \preceq)$*.*

The following result is a consequence of Lemmas 3.4 and 3.5.

Lemma 3.6. *For all* $x_1, x_2 \in S$*,* $x_1 \neq x_2$*, let* $F(x_1) \neq F(x_2)$*. Suppose that* S *is compact and that* F *is semicontinuous on* S *w.r.t. the antisymmetric preorder* $\preceq$*. Then* $\mathrm{SiMin}(F(S), \preceq) \neq \emptyset$*.*

4. Numerical methods for determining minimal solutions

Efficient algorithms for finding minimal solutions of the problem (2.1) have already been proposed and thoroughly investigated in [13]. However, sometimes the set $\mathrm{Min}(F(S), \preceq)$ can be quite large, and one is interested in a smaller set. Therefore, in this section we are concerned with developing numerical methods for finding strong, strict and ideal minimal solutions of the problem (2.1).

In the literature, there already exist some algorithms for solving set-valued optimization problems. Jahn [4] proposes a descent method that generates approximations of minimal elements of set-valued optimization problems under convexity assumptions on the considered sets. Specifically, in [4], the set less relation is used and characterized by means of linear functionals. More recently, in [8, 11], the authors propose a similar descent method for obtaining approximations of minimal solutions of set-valued optimization problems.

In this section, we are concerned with finding strong / strict / ideal minimal solutions of the problem (2.1). Note that a finite family of sets $F(S)$ can also be computed by an appropriate discretization of the outcome sets of the considered (continuous) set optimization problem.

Remark 4.1. Note that it is also possible to use scalarizing methods to compare elements in the family of sets $F(S) = \{F(x) | \ x \in S\}$. These methods are also investigated in many publications for several kinds of set relations, see, for instance [8, 11, 12, 13, 14].

4.1. Strong Minimal Solutions.

It is our goal to extend the well-known Jahn-Graef-Younes method, which was introduced in the dissertation by Younes [17], Jahn and Rathje [6] (compare also Jahn [3, Section 12.4]) for determining minimal elements in the vector-valued case, where $Y = \mathbb{R}^n$. The Jahn-Graef-Younes method selects minimal elements of a set of finitely many elements. Its advantage is that this method reduces the numerical effort by excluding elements which cannot be minimal for a given set. Eichfelder [2] formulated corresponding algorithms for vector-valued problems with a variable ordering structure. A first extension of this method to set optimization is given in [13], where algorithms that deal with minimal solutions of the problem (2.1) are proposed.

In this section, we extend this method to the set-valued case in order to obtain strong minimal elements of a family of finitely many sets. We propose several extensions of the Jahn-Graef-Younes method under different, very broad assumptions. We extend the idea of such a method to set optimization problems, where we assume that a family of finitely many sets $F(S)$ is given and minimal, strong minimal and strict minimal elements, respectively, of $F(S)$ are to be identified. When the family of sets $F(S)$ is given by a large number of elements, it may take a long time to compare the sets pairwise according to Definition 2.1. Our approach is especially useful if each comparison of sets is rather expensive, as our proposed method can reduce the number of comparisons of sets.

The following algorithm filters out solutions of the problem (2.1) which are not strong minimal.

Algorithm 4.2 (Jahn-Graef-Younes method for sorting out points that are not strong minimal solutions of the problem (2.1))**.**

Input: $S := \{x_1, \dots, x_m\} \subset X$, mapping $F : X \rightrightarrows Y$, set relation $\preceq$
% initialization
$\mathcal{T} := \{x_1\}$
% iteration loop
for $j = 2 : 1 : m$ **do**
  if $\left(F(x) \preceq F(x_j),\ x \in \mathcal{T} \implies F(x_j) = F(x) \right)$ **then**
    $\mathcal{T} := \mathcal{T} \cup \{x_j\}$
  end if
end for
Output: $\mathcal{T}$

Algorithm 4.2 is a reduction method which sorts out sets that cannot be minimal. In the if-statement of Algorithm 4.2, each element is compared only with elements that have been considered so far (which belong to the set $\mathcal{T}$), so it is not necessary to compare all elements with each other pairwise, which can reduce the computation time of determining minimal elements significantly. The theorem below shows that all strong minimal solutions of the problem (2.1) are contained in the output set $\mathcal{T}$ generated by Algorithm 4.2.

Theorem 4.3. (1) *Algorithm 4.2 is well-defined.*
(2) *Algorithm 4.2 generates a nonempty set $\mathcal{T} \subseteq S$.*
(3) *Every strong minimal solution of problem* (2.1) *also belongs to the set $\mathcal{T}$ generated by Algorithm 4.2.*

Proof. As (1) and (2) are obvious, we only prove part (3). Let x_j be a strong minimal solution of problem (2.1), but assume that $x_j \notin \mathcal{T}$. Clearly $j \neq 1$. As x_j is a strong minimal solution of problem (2.1), we have

$$x \in S,\ F(x) \preceq F(x_j) \implies F(x_j) = F(x).$$

Since $\mathcal{T} \subseteq S$, we have

$$F(x) \preceq F(x_j),\ x \in \mathcal{T} \implies F(x_j) = F(x).$$

But then the condition in the if-statement is fulfilled and x_j is added to $\mathcal{T}$, which is a contradiction to our assumption. □

Remark 4.4. Notice that the set relation $\preceq$ does not need to be transitive in Algorithm 4.2, in contrast to descent methods (see Jahn [4]), which rely on the transitivity of the considered set relation. Moreover, notice that the if-condition in Algorithm 4.2 can be replaced by

$$\forall\, x \in \mathcal{T},\ F(x) \not\preceq F(x_j) \text{ or } F(x) = F(x_j).$$

When the for-loop derived in Algorithm 4.2 is performed backwards on the outcome set $\mathcal{T}$, then it is possible to obtain all strong minimal solutions of the problem (2.1). This procedure is described in the next algorithm. Here, the external stability assumption on the set $\mathrm{SoMin}(F(S), \preceq)$ is essential in order to compute all strong minimal solutions of the problem (2.1).

Algorithm 4.5 (Jahn-Graef-Younes method with backward iteration for finding strong minimal solutions of the problem (2.1), where $\mathrm{SoMin}(F(S), \preceq)$ is externally stable)**.**

Input: $S := \{x_1, \ldots, x_m\} \subset X$, mapping $F : X \rightrightarrows Y$, set relation $\preceq$
% initialization
$\mathcal{T} := \{x_1\}$
% forward iteration loop
for $j = 2 : 1 : m$ **do**
  if $\Big(F(x) \preceq F(x_j),\ x \in \mathcal{T} \implies F(x_j) = F(x)\Big)$ **then**
    $\mathcal{T} := \mathcal{T} \cup \{x_j\}$
  end if
end for
$\{x_1, \ldots, x_p\} := \mathcal{T}$
$\mathcal{U} := \{x_p\}$
% backward iteration loop
for $j = p - 1 : -1 : 1$ **do**
  if $\Big(F(x) \preceq F(x_j),\ x \in \mathcal{U} \implies F(x_j) = F(x)\Big)$ **then**
    $\mathcal{U} := \mathcal{U} \cup \{x_j\}$
  end if
end for
Output: $\mathcal{U}$

Remark 4.6. In the worst-case, the computational complexity of Algorithm 4.5 is $\mathcal{O}(m^2)$, depending essentially on the cardinality of the set $\mathcal{T}$, generated after the forward iteration.

Theorem 4.7. *Let the set of strong minimal solutions* $\mathrm{SoMin}(F(S), \preceq)$ *be nonempty and externally stable. Then, the output* $\mathcal{U}$ *of Algorithm 4.5 consists of exactly all strong minimal solutions of problem* (2.1)*.*

Proof. Let $\mathcal{U} := \{x_1, \ldots, x_q\}$. By (3) of Theorem 4.3, we know that all strong minimal solutions of problem (2.1) are contained in $\mathcal{T}$ as well as in $\mathcal{U}$. Now, we prove that every element of $\mathcal{U}$ is also a strong minimal solution of problem (2.1). Let $x_j \in \mathcal{U}$ be arbitrarily chosen. By the forward iteration of Algorithm 4.5, we obtain

$$\forall\ i < j\ (i \geq 1):\ F(x_i) \preceq F(x_j) \Longrightarrow F(x_j) = F(x_i).$$

The backward iteration of Algorithm 4.5 yields

$$\forall\ i > j\ (i \leq q):\ F(x_i) \preceq F(x_j) \Longrightarrow F(x_j) = F(x_i).$$

This means that

$$\forall\ i \neq j\ (1 \leq i \leq q):\ F(x_i) \preceq F(x_j) \Longrightarrow F(x_j) = F(x_i). \tag{4.1}$$

(4.1) implies that

$$\forall\ x_i \in \mathcal{U} \setminus \{x_j\}:\ F(x_i) \preceq F(x_j) \quad \Longrightarrow \quad F(x_j) = F(x_i).$$

Therefore, we can conclude that $x_j \in \text{SoMin}(F(\mathcal{U}), \preceq)$. Now suppose that $x_j \notin \text{SoMin}(F(S), \preceq)$. Then, as $\text{SoMin}(F(S), \preceq)$ was assumed to be externally stable, there exists a strong minimal solution x in $\text{SoMin}(F(S), \preceq)$ (especially, $F(x) \neq F(x_j)$) with the property $F(x) \preceq F(x_j)$. Since x is a strong minimal solution of problem (2.1), Theorem 4.3, 3. implies that $x \in \mathcal{U}$. Therefore, by (4.1), $F(x_j) = F(x)$, a contradiction. □

Remark 4.8. Notice that in Theorem 4.7 we do not pose any assumptions on the set relation $\preceq$. In a corresponding result ([13, Theorem 4.12]), which is concerned with finding *minimal solutions* of a family of finitely many sets, antisymmetry of the relation $\preceq$ is required.

Finally, we propose the following algorithm that does not rely on antisymmetry or external stability of the set relation $\preceq$. The idea stems from Eichfelder [2, Algorithm 1], who gave a similar numerical procedure for finding minimal elements in vector optimization with a variable ordering structure. In the following algorithm, a third for-loop is added which compares the elements that were obtained in the set $\mathcal{U}$ by Algorithm 4.5 with all remaining elements in $S \setminus \mathcal{U}$.

Algorithm 4.9 (Jahn-Graef-Younes method with backward iteration for finding strong minimal solutions of problem (2.1))**.**

Input: $S := \{x_1, \ldots, x_m\} \subset X$, mapping $F : X \rightrightarrows Y$, set relation $\preceq$
% initialization
$\mathcal{T} := \{x_1\}$
% forward iteration loop
for $j = 2 : 1 : m$ **do**
  if $\left(F(x) \preceq F(x_j),\ x \in \mathcal{T} \implies F(x_j) = F(x)\right)$ **then**
    $\mathcal{T} := \mathcal{T} \cup \{x_j\}$
  end if
end for
$\{x_1, \ldots, x_p\} := \mathcal{T}$
$\mathcal{U} := \{x_p\}$
% backward iteration loop

for $j = p-1 : -1 : 1$ **do**
  if $\left(F(x) \preceq F(x_j),\ x \in \mathcal{U} \implies F(x_j) = F(x)\right)$ **then**
    $\mathcal{U} := \mathcal{U} \cup \{x_j\}$
  end if
end for
$\{x_1, \dots, x_q\} := \mathcal{U}$
$\mathcal{V} := \emptyset$
% final comparison
for $j = 1 : 1 : q$ **do**
  if $\left(F(x) \preceq F(x_j),\ x \in S \setminus \mathcal{U} \implies F(x_j) = F(x)\right)$ **then**
    $\mathcal{V} := \mathcal{V} \cup \{x_j\}$
  end if
end for
Output: $\mathcal{V}$

Theorem 4.10. *Algorithm 4.9 consists of exactly all strong minimal solutions of problem* (2.1).

Proof. Let x_j be an arbitrary element in $\mathcal{V}$. We show that $x_j \in \mathrm{SoMin}(F(S), \preceq)$. It holds that $x_j \in \mathcal{U}$, as $\mathcal{V} \subseteq \mathcal{U}$, and

$$F(x) \preceq F(x_j),\ x \in S \setminus \mathcal{U} \implies F(x_j) = F(x).$$

Suppose that $x_j \notin \mathrm{SoMin}(F(S), \preceq)$. Then, there exists some $x \in S$ such that $F(x) \preceq F(x_j)$ and $F(x_j) \neq F(x)$. If $x \notin \mathcal{U}$, then this is a contradiction. If $x \in \mathcal{U}$, then $x \in \mathrm{SoMin}(F(\mathcal{U}), \preceq)$ (compare the proof of Theorem 4.7). Since $x_j \in \mathcal{U}$, and x_j is also strong minimal in $\mathcal{U}$, we obtain from $F(x) \preceq F(x_j)$ that $F(x_j) = F(x)$, a contradiction.

Conversely, let $x_j \in \mathrm{SoMin}(F(S), \preceq)$. Then, we get

$$F(x) \preceq F(x_j),\ x \in S \implies F(x_j) = F(x).$$

Now suppose that $x_j \notin \mathcal{V}$. Then there exists some $x \in S \setminus \mathcal{U}$ with $F(x) \preceq F(x_j)$ and $F(x_j) \neq F(x)$. As $x_j \in \mathrm{SoMin}(F(S), \preceq)$, we get $F(x_j) = F(x)$, a contradiction. □

Example 4.11. We will illustrate the method for deriving the set of strong solutions for the following problem:

$$\text{Min } F(S) \text{ w.r.t. } \preceq^l_{\mathbb{R}^2_+}, \tag{4.2}$$

where $a, b > 0, r > 0,\ S = \{(x_1, y_1), \dots, (x_k, y_k)\} \subseteq \mathbb{R}^2,\ j \in \{1, \dots, k\}$ and $F : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ determined as

$$F(x_j, y_j) = \begin{cases} \{(u,v) \in \mathbb{R}^2 \mid (u - x_j)^2 + (v - y_j)^2 = r^2\} \text{ if mod } (j,3) = 0 \\ \{(u,v) \in \mathbb{R}^2 \mid |u - x_j|, |v - y_j| \leq \frac{a}{2}\} \text{ if mod } (j,3) = 1 \\ \{(u,v) \in \mathbb{R}^2 \mid u \geq x_j - \frac{b}{2},\ v \geq y_j - \frac{b}{2},\ v + u \leq y_i + x_i\} \\ \qquad \cup \{u + v \geq (x_i + y_i),\ v - u \geq y_i - x_i,\ |v - y_i| \leq \frac{b}{2}\} \text{ else,} \end{cases}$$

and $A \preceq^l_{\mathbb{R}^2_+} B$ is the lower set order relation defined by $A + \mathbb{R}^2_+ \supseteq B$ for $A, B \in \mathcal{P}(\mathbb{R}^2)$.

x_j	y_j
0.49569	0.008598
0.161157	0.008707
0.003015	0.947034
0.069465	0.039968
0.003893	0.220735
0.008142	0.037274
0.001533	0.849506
0.995213	0.003356
0.366188	0.015826
0.306639	0.005631
0.881342	0.004196
0.143987	0.01434
0.006202	0.161529
0.153872	0.027098

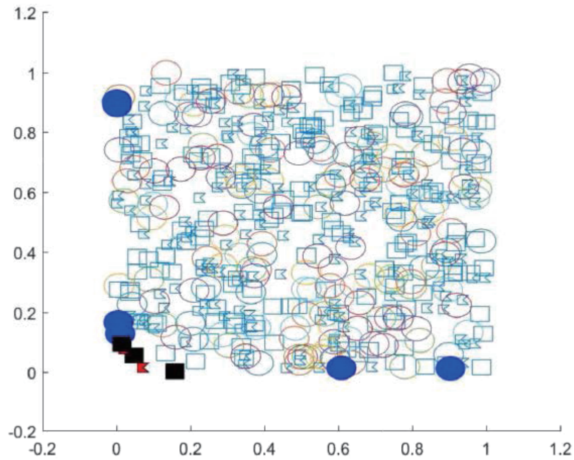

Figure 1. Strong solutions of the problem (4.2) and their images

Observe that the set-valued function F is not convex-valued. We use the Algorithm 4.5 to derive the set of strong solutions of problem (4.2). The Figure 1 depicts the images of strong solutions for the problem (4.2), where $k = 500; r = 0.04; a = 0.05; b = 0.03$. We also show the result concerning solutions of this problem when the input data is taken arbitrarily. In this case the set of strong solutions and strict solutions of (4.2) are identical (Lemma 3.5). In addition, taking into account Proposition 2.8, it holds that $\mathrm{IMin}(F(S), \preceq^l_{\mathbb{R}^2_+}) = \emptyset$.

4.2. **Strict Minimal Solutions.** In this section, we are concerned with computing strict minimal solutions of the problem (2.1). The following method is a filtering algorithm that sorts out solutions of the problem (2.1) which are not strict minimal.

Algorithm 4.12 (Jahn-Graef-Younes method for sorting out points that are not strict minimal solutions of the problem (2.1))**.**

Input: $S := \{x_1, \ldots, x_m\} \subset X$, mapping $F : X \rightrightarrows Y$, set relation $\preceq$
% initialization
$\mathcal{T} := \{x_1\}$
% iteration loop
for $j = 2 : 1 : m$ **do**
  if $\left(F(x) \preceq F(x_j),\ x \in \mathcal{T} \implies x_j = x\right)$ **then**
    $\mathcal{T} := \mathcal{T} \cup \{x_j\}$
  end if
end for
Output: $\mathcal{T}$

We refrain from giving the proof of the following theorem, as it is similar to that of Theorem 4.3.

Theorem 4.13. (1) *Algorithm 4.12 is well-defined.*
(2) *Algorithm 4.12 generates a nonempty set $\mathcal{T} \subseteq S$.*
(3) *Every strict minimal solution of problem* (2.1) *also belongs to the set $\mathcal{T}$ generated by Algorithm 4.12.*

Remark 4.14. Moreover, notice that the if-condition in Algorithm 4.12 can be replaced by

$$\forall\, x \in \mathcal{T},\ F(x) \not\preceq F(x_j) \text{ or } x = x_j.$$

Algorithm 4.15 (Jahn-Graef-Younes method with backward iteration for finding strict minimal solutions of the problem (2.1), where $\mathrm{SoMin}(F(S), \preceq)$ is externally stable)**.**

Input: $S := \{x_1, \ldots, x_m\} \subset X$, mapping $F : X \rightrightarrows Y$, set relation $\preceq$
% initialization
$\mathcal{T} := \{x_1\}$
% forward iteration loop
for $j = 2 : 1 : m$ **do**
  if $\left(F(x) \preceq F(x_j),\ x \in \mathcal{T} \implies x_j = x\right)$ **then**
    $\mathcal{T} := \mathcal{T} \cup \{x_j\}$
  end if
end for
$\{x_1, \ldots, x_p\} := \mathcal{T}$
$\mathcal{U} := \{x_p\}$
% backward iteration loop
for $j = p - 1 : -1 : 1$ **do**
  if $\left(F(x) \preceq F(x_j),\ x \in \mathcal{U} \implies x_j = x\right)$ **then**
    $\mathcal{U} := \mathcal{U} \cup \{x_j\}$
  end if
end for
Output: $\mathcal{U}$

Remark 4.16. The computational complexity of Algorithm 4.15 depends essentially on the cardinality of the set $\mathcal{T}$, generated after the forward iteration. In the worst-case, the computational complexity of this algorithm is $\mathcal{O}(m^2)$.

Theorem 4.17. *Let the set of strong minimal elements* $\operatorname{SoMin}(F(S), \preceq)$ *be nonempty and externally stable. Then, the output* $\mathcal{U}$ *of Algorithm 4.15 consists of exactly all strong minimal solutions of problem* (2.1)*.*

Algorithm 4.18 (Jahn-Graef-Younes method with backward iteration for finding strict minimal solutions of problem (2.1))**.**

Input: $S := \{x_1, \dots, x_m\} \subset X$, mapping $F : X \rightrightarrows Y$, set relation $\preceq$
% initialization
$\mathcal{T} := \{x_1\}$
% forward iteration loop
for $j = 2 : 1 : m$ **do**
  if $\left(F(x) \preceq F(x_j),\ x \in \mathcal{T} \implies x_j = x\right)$ **then**
    $\mathcal{T} := \mathcal{T} \cup \{x_j\}$
  end if
end for
$\{x_1, \dots, x_p\} := \mathcal{T}$
$\mathcal{U} := \{x_p\}$
% backward iteration loop
for $j = p - 1 : -1 : 1$ **do**
  if $\left(F(x) \preceq F(x_j),\ x \in \mathcal{U} \implies x_j = x\right)$ **then**
    $\mathcal{U} := \mathcal{U} \cup \{x_j\}$
  end if
end for
$\{x_1, \dots, x_q\} := \mathcal{U}$
$\mathcal{V} := \emptyset$
% final comparison
for $j = 1 : 1 : q$ **do**
  if $\left(F(x) \preceq F(x_j),\ x \in S \setminus \mathcal{U} \implies x_j = x\right)$ **then**
    $\mathcal{V} := \mathcal{V} \cup \{x_j\}$
  end if
end for
Output: $\mathcal{V}$

Theorem 4.19. *Algorithm 4.18 consists of exactly all strict minimal solutions of problem* (2.1)*.*

4.3. **Ideal Minimal Solutions.** In this section, we propose an algorithmic procedure that finds all ideal minimal solutions of the problem (2.1), provided that this solution set is nonempty. In the following algorithm we make use of the statement given in Corollary 2.6, which says that if an ideal minimal solution $\bar{x} \in S$ is found, then the set of ideal minimal solutions consists of all point $x \in S$ which satisfy $F(x) \sim F(\bar{x})$ provided that $\preceq$ is reflexive and transitive, and vice versa. If

$x_j \in \text{IMin}(F(S), \preceq)$, then

$$x \notin \text{IMin}(F(S), \preceq) \quad \Longrightarrow \quad x \notin \{\widetilde{x} \in S \mid F(\widetilde{x}) \sim F(x_j)\}.$$

Therefore, in the if-loop of the following algorithm, we proceed as follows: In order to check whether a point x_j is an ideal minimal solution, we have to check whether $F(x_j) \preceq F(x)$ holds for $x \in \text{IMin}(F(S), \preceq)$.

Algorithm 4.20 (Method for obtaining ideal minimal solutions of the problem (2.1))**.**

Input: $S := \{x_1, \ldots, x_m\} \subset X$, mapping $F : X \rightrightarrows Y$, set relation $\preceq$
% initialization
$\mathcal{T} := \emptyset$
$\text{num}(\mathcal{T}) := 0$
% iteration loop
for $j = 1 : 1 : m$ **do**
  if $\text{num}(\mathcal{T}) = 0$ **then**
    if $\forall\, x \in S \setminus \{x_j\} :\ F(x_j) \preceq F(x)$ **then**
     $\mathcal{T} := \mathcal{T} \cup \{x_j\}$
     $\text{num}(\mathcal{T})\text{:=}\text{num}(\mathcal{T}) + 1$
    end if
  else if $\exists\, x \in \mathcal{T} \setminus \{x_j\} :\ F(x_j) \preceq F(x)$ **then**
   $\mathcal{T} := \mathcal{T} \cup \{x_j\}$
   $\text{num}(\mathcal{T})\text{:=}\text{num}(\mathcal{T}) + 1$
  end if
end for

Output: $\mathcal{T}$

Remark 4.21. By using directly Definition 2.1, the computational complexity of Algorithm 4.20 is $\mathcal{O}(m^2)$. In this algorithm, instead of comparing all pairs in the image space $F(S)$, we are concerned with elements in a smaller set $F(\mathcal{T})$ when $\mathcal{T}$ is nonempty. By doing this, the computing time will be reduced. However, it is more convenient if we step by step eliminate the points that are not ideal minimal solutions of the problem (2.1) as follows. Recall also that we have from Proposition 2.8 (iv) the following implication

$$x_1 \neq x_2, x_1, x_2 \in \text{SiMin}(F(S), \preceq) \quad \Longrightarrow \quad \text{IMin}(F(S), \preceq) = \emptyset.$$

Therefore, the following algorithm will be used in case the number of *distinct* strict minimal solutions, computed by Algorithm 4.15, is smaller than two.

Algorithm 4.22 (Jahn-Graef-Younes method with backward iteration for obtaining ideal minimal solutions of the set-valued optimization problem (2.1))**.**

Input: $S := \{x_1, \ldots, x_m\} \subset X$, mapping $F : X \rightrightarrows Y$, set relation $\preceq$

% initialization
$\mathcal{T} := \{x_1\}$
% iteration loop
for $j = 2:1:m$ **do**
    if $\forall\, x \in \mathcal{T}:\ F(x_j) \preceq F(x)$ **then**
        $\mathcal{T} := \mathcal{T} \cup \{x_j\}$; **break**
    end if
$\mathcal{T} := \{x_1, ..., x_k\}$
$\mathcal{U} := \{x_k\}$
end for
for $l = k-1 : -1 : 1$ **do**
    if $\forall x \in \mathcal{U}:\ F(x_l) \preceq F(x)$ **then**
        $\mathcal{U} := \mathcal{U} \cup \{x_l\}$
Output: $\mathcal{U}$

Remark 4.23. The computational complexity of Algorithm 4.22 depends essentially on the cardinality of the set $\mathcal{T}$, generated after the forward iteration. In the worst-case, the computational complexity of this algorithm is $\mathcal{O}(m^2)$.

Theorem 4.24. *The following assertions hold true for the Algorithm 4.22.*

(a) $\text{IMin}(\text{F(S)}, \preceq) \subseteq \mathcal{T}$.
(b) $\mathcal{U} = \text{IMin}(F(S), \preceq)$, *provided that* $\text{IMin}(\text{F(S)}, \preceq)$ *is externally stable and* $\preceq$ *is transitive.*

Proof. (a) Suppose that there exists $x_j \in \text{IMin}(F(S), \preceq)$ but $x_j \notin \mathcal{T}$. By Definition 2.1, it holds that

$$\forall x \in S \setminus \{x_j\} : F(x_j) \ \preceq F(x).$$

Obviously, $\mathcal{T} \subseteq S \setminus \{x_j\}$. Therefore,

$$\forall x \in \mathcal{T} : F(x_j) \ \preceq F(x).$$

However, by this property, x_j will be added to the set $\mathcal{T}$, a contradiction.

(b) First, we will prove that $\text{IMin}(F(S), \preceq) \subseteq \mathcal{U}$. Indeed, let $x_j \in \text{IMin}(F(S), \preceq), j \in \{1, \ldots, k\}$ and suppose that $x_j \notin \mathcal{U}$. Of course, $j \neq k$ and since $x_j \in \text{IMin}(F(S), \preceq)$, we have that $F(x_j) \preceq F(x)$, $\forall x \in \mathcal{U}$. By this condition, x_j is added to $\mathcal{U}$, a contradiction.
To this end, we will prove that $\mathcal{U} \subseteq \text{IMin}(F(S), \preceq)$. Take $x_j \in \mathcal{U}$. We get from the forward iteration that

$$\forall i < j, x_i \neq x_j :\ F(x_j) \preceq F(x_i)$$

Moreover, the backward iteration yields

$$\forall i > j, x_i \neq x_j :\ F(x_j) \preceq F(x_i).$$

Therefore, for all $x \in \mathcal{U} \setminus \{x_j\}$ we have that $F(x_j) \preceq F(x)$, i.e., $x_j \in \text{IMin}(F(\mathcal{U}), \preceq)$. Suppose by contradiction that $x_j \notin \text{IMin}(F(S), \preceq)$. Since $\text{IMin}(F(S), \preceq)$ is externally stable, there is $\bar{x} \in \text{IMin}(F(S), \preceq) \subseteq \mathcal{U}$ such that $F(\bar{x}) \preceq F(x_j)$. In addition, since $\text{IMin}(F(S), \preceq) \subseteq \mathcal{U}$ such that $F(\bar{x}) \preceq F(x_j)$, $\bar{x} \in \mathcal{U}$ and thus $F(x_j) \preceq F(\bar{x})$. This implies that $F(x_j) \sim F(\bar{x})$. Taking into account the transitivity and reflexivity of $\preceq$, it follows from Corollary 2.6 that $x_j \in \text{IMin}(F(S), \preceq)$, a contradiction.

□

5. Conclusions

In this paper, we investigate the problem of minimizing a set-valued map. In particular, we are concerned with strong, strict and ideal minimal solutions of the problem (2.1). We propose numerical algorithms that reduce the numerical effort while sorting out solutions that are not strong, strict or ideal minimal solutions and extended this method to select the sets which are strong, strict or ideal minimal solutions. Our approach can be regarded as an extension of the well-known Jahn-Graef-Younes method. More research shall be done on the implementations of our proposed algorithms to specific applications of set optimization problems.

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E. Köbis
Institute of Mathematics, Martin–Luther–University Halle–Wittenberg, Theodor-Lieser-Str. 5, 06120 Halle (Saale), Germany
E-mail address: elisabeth.koebis@mathematik.uni-halle.de

T. T. Le
Institute of Mathematics, Martin–Luther–University Halle–Wittenberg, Theodor-Lieser-Str. 5, 06120 Halle (Saale), Germany;
University of Transport and Communications, Viet Nam
E-mail address: le.thanh-tam@mathematik.uni-halle.de