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NECESSARY OPTIMALITY CONDITIONS IN GENERALIZED CONVEX MULTI-OBJECTIVE OPTIMIZATION INVOLVING NONCONVEX CONSTRAINTS

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Abstract. The aim of this paper is to derive necessary optimality conditions for Pareto efficient solutions of multi-objective optimization problems involving not necessarily convex constraints. The objective function is acting between a real linear topological pre-image space and a finite-dimensional image space and is assumed to be componentwise generalized convex (e.g., semi-strictly quasi-convex or quasi-convex). Günther and Tammer (2016, 2018) showed that the set of Pareto efficient solutions of a constrained multi-objective optimization problem can be computed completely by considering two corresponding unconstrained multi-objective optimization problems. By using these results and by applying methods of generalized differentiation, we show that it is possible to derive necessary optimality conditions for a problem with a nonconvex feasible set. These optimality conditions have a simple structure because the normal cone with respect to the constraints is not involved. Finally, we apply our results to multi-objective approximation problems with a not necessarily convex feasible set and derive necessary optimality conditions.

1. Introduction

In many real world problems, one aims to minimize several conflicting objective functions with respect to certain constraints. In order to have a representative mathematical model of a complex real world problem, it seems to make sense to deal in particular with nonconvex constraints. For instance, in the field of multi-objective location theory one can consider certain forbidden regions where it is not allowed to locate a new facility (see, e.g., Günther [13]), which leads to a nonconvex feasible set.

In this work, assuming that E is a real linear topological space, we start by considering a vector-valued objective function

$$f = (f_1, \cdots, f_m) : E \to \mathbb{R}^m \quad (m \in \mathbb{N}),$$

which should be minimized in the sense of multi-objective optimization over a nonempty feasible set $\Omega \subseteq E$. So, we are interested in the problem

$$(\mathcal{P}_\Omega) \qquad \begin{cases} f(x) = (f_1(x), \cdots, f_m(x)) \to \min \text{ w.r.t. } \mathbb{R}^m_+ \\ x \in \Omega, \end{cases}$$

where the image space $\mathbb{R}^m$ is partially ordered by the natural ordering cone

$$\mathbb{R}^m_+ := \{(y_1, \cdots, y_m) \in \mathbb{R}^m \mid \forall i \in \{1, \cdots, m\} : \; y_i \geq 0\}.$$

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In order to derive stronger results by exploiting the structure of the problem, we have to impose certain assumptions on the objective function f. In particular, standard componentwise notions of generalized convexity (e.g., componentwise semi-strict quasi-convexity or componentwise quasi-convexity) that are suitable for vector-valued functions acting between E and a finite-dimensional image space $\mathbb{R}^m$ will be used in our article. Generalized convexity assumptions are fulfilled for the objective function in many applications, for instance in the fields of production theory, utility theory and location theory (see, e.g., Cambini and Martein [5, Sec. 2.4]). In particular, the combination of the field of multi-objective optimization and the field of generalized convexity is interesting from the theoretical as well as practical point of view (see, e.g., Günther [13], Günther and Tammer [15], [16], Mäkelä *et al.* [21], Popovici [24], Puerto and Rodríguez-Chía [25]).

Using an idea of adding new objective functions to the formulations of multi-objective optimization problems (see, e.g., Klamroth and Tind [20], and Fliege [10]), Günther and Tammer [15] derived relationships between constrained and unconstrained multi-objective optimization, and developed a vectorial penalization approach for such problems, where the constraints are assumed to be convex. This approach uses an extended problem

$$(\mathcal{P}_E^{\oplus}) \qquad \begin{cases} f^{\oplus}(x) := (f_1(x), \cdots, f_m(x), \varphi(x)) \to \min \text{ w.r.t. } \mathbb{R}_+^{m+1} \\ x \in E \end{cases}$$

with an objective function

$$f^{\oplus} = (f, \varphi) : E \to \mathbb{R}^{m+1}$$

that involves a new scalar penalization function

$$\varphi : E \to \mathbb{R}$$

as an additional component function $f_{m+1} := \varphi$, where certain level sets of φ correspond to the set Ω. Instead of solving the initial problem $(\mathcal{P}_\Omega)$, one aims to minimize both functions f and $f^{\oplus}$ over E. Durea, Strugariu and Tammer [8] also exploited the idea of vectorial penalization for general vector optimization problems involving not necessarily convex constraints. The method by vectorial penalization was extended by Günther and Tammer [16] to a problem of type $(\mathcal{P}_\Omega)$ involving not necessarily convex constraints Ω by using new types of penalization functions. Beside the above mentioned approach, there is another approach that consists of adding penalization terms in the component functions of f (see, e.g., Apetrii *et al.* [2], and Ye [27]). However, in this work we will focus on the method by vectorial penalization. A more detailed view on this topic can also be found in the thesis by Günther [14]. It is known that complete solution sets of special nonconvex multi-objective location problems involving multiple forbidden regions can be constructed by applying the vectorial penalization method and using known results for corresponding unconstrained problems (see Günther [13]).

The aim of this paper is to derive necessary optimality conditions for Pareto efficient solutions of the multi-objective optimization problem $(\mathcal{P}_\Omega)$ involving closed but not necessarily convex constraints Ω. Applying results derived via the method by vectorial penalization by Günther and Tammer [16], we show that it is possible to obtain necessary optimality conditions for a problem $(\mathcal{P}_\Omega)$ with a nonconvex feasible set Ω by considering two unconstrained problems (to minimize f over E, and, respectively, $f^\oplus$ over E) and to use corresponding methods of generalized differentiation. These necessary optimality conditions have a simple structure because the normal cone with respect to the constraints is not involved. We note that also Durea, Strugariu and Tammer [8] studied methods to derive optimality conditions for vector optimization problems.

The paper is organized as follows:
In Section 2, we introduce notations and we recall facts from the fields of multi-objective optimization and generalized convexity.
The method by vectorial penalization is presented in Section 3. After formulating some important facts about the multi-objective optimization problem $(\mathcal{P}_E^\oplus)$, where an additional penalization function φ is added to the objective function f given in the initial problem $(\mathcal{P}_\Omega)$, we recall relationships between the problem $(\mathcal{P}_\Omega)$ involving a nonempty, closed (not necessarily convex) feasible set Ω, the problem that consists of minimizing f over E, and the extended multi-objective optimization problem $(\mathcal{P}_E^\oplus)$.
The main new results of this paper are included in Section 4, where we derive necessary optimality conditions for multi-objective optimization problems with a not necessarily convex feasible set that are completely given in terms of abstract subdifferentials and do not involve a normal cone with respect to the constraints.
Applications in approximation theory are studied in Section 5.
Section 6 contains some concluding remarks.

2. Preliminaries

Throughout this article, we will deal with certain standard notions of optimization that we now recall in this section. In what follows, we assume that E is a real topological linear space. For any two points $x^0, x^1 \in E$, the closed, open, half-open line segments are given by

$$[x^0, x^1] := \{(1-\lambda)x^0 + \lambda x^1 \mid \lambda \in [0,1]\}, \qquad]x^0, x^1[:= [x^0, x^1] \setminus \{x^0, x^1\},$$
$$[x^0, x^1[:= [x^0, x^1] \setminus \{x^1\}, \qquad]x^0, x^1] := [x^0, x^1] \setminus \{x^0\}.$$

Considering any nonempty set Ω in E, the interior, the closure and the boundary (in the topological sense) of Ω is denoted by $\operatorname{int}\Omega$, $\operatorname{cl}\Omega$ and $\operatorname{bd}\Omega$, respectively. In addition, the algebraic interior of Ω (or the core of Ω) is given as usual by

$$\operatorname{cor}\Omega := \{x \in \Omega \mid \forall\, v \in E\ \exists\, \delta > 0 :\ x + [0,\delta]\cdot v \subseteq \Omega\}.$$

It is easily seen that

$$\operatorname{int}\Omega \subseteq \operatorname{cor}\Omega \subseteq \Omega.$$

Assuming that Ω is a convex set in E, we actually have

$$\operatorname{int}\Omega = \operatorname{cor}\Omega$$

if one of the following conditions is satisfied (see, e.g., Barbu and Precupanu [4, Sec. 1.1.2], and Jahn [18, Lem. 1.3.2]):

(1) $\operatorname{int}\Omega \neq \emptyset$,
(2) E is a Banach space, and Ω is closed,
(3) E has finite dimension.

Let us concentrate on the multi-objective optimization problem $(\mathcal{P}_\Omega)$ which involves a vector-valued objective function $f = (f_1, \cdots, f_m) : E \to \mathbb{R}^m$ and a nonempty feasible set $\Omega \subseteq E$, as introduced in Section 1. In this article, the vector-valued minimization of f in $(\mathcal{P}_\Omega)$ is based on the well-known concept of Pareto efficiency with respect to the natural ordering cone $\mathbb{R}^m_+$. Therefore, it is convenient to introduce the set of Pareto efficient solutions of problem $(\mathcal{P}_\Omega)$ by

$$\mathrm{Eff}(\Omega \mid f) := \{x^0 \in \Omega \mid f[\Omega] \cap (f(x^0) - (\mathbb{R}^m_+ \setminus \{0\})) = \emptyset\},$$

while the set of weakly Pareto efficient solutions is given by

$$\mathrm{WEff}(\Omega \mid f) := \{x^0 \in \Omega \mid f[\Omega] \cap (f(x^0) - \operatorname{int}\mathbb{R}^m_+) = \emptyset\},$$

where

$$f[\Omega] := \{f(x) \in \mathbb{R}^m \mid x \in \Omega\}$$

denotes the image set of f over Ω. It is easy to observe that

$$\mathrm{Eff}(\Omega \mid f) \subseteq \mathrm{WEff}(\Omega \mid f).$$

Furthermore, we need the notions of level sets and level lines for a scalar function $h : E \to \mathbb{R}$. For any $s \in \mathbb{R}$, we define the following sets:

$$L_\leq(\Omega, h, s) := \{x \in \Omega \mid h(x) \leq s\} \quad \text{(lower-level set of } h \text{ to the level } s),$$
$$L_=(\Omega, h, s) := \{x \in \Omega \mid h(x) = s\} \quad \text{(level line of } h \text{ to the level } s),$$
$$L_<(\Omega, h, s) := \{x \in \Omega \mid h(x) < s\} \quad \text{(strict lower-level set of } h \text{ to the level } s).$$

Notice that we have

$$L_\sim(\Omega, h, s) = L_\sim(E, h, s) \cap \Omega \quad \text{for all } \sim \in \{\leq, =, <\}.$$

For notational convenience, throughout the article, for any natural number $m \in \mathbb{N}$, we define a special index set by

$$I_m := \{1, 2, \cdots, m\}.$$

Now, for any $x^0 \in \Omega$, we define the intersection of lower-level sets (respectively, level lines, strict lower-level sets) by

$$S_\leq(\Omega, f, x^0) := \bigcap_{i \in I_m} L_\leq(\Omega, f_i, f_i(x^0)),$$
$$S_=(\Omega, f, x^0) := \bigcap_{i \in I_m} L_=(\Omega, f_i, f_i(x^0)),$$
$$S_<(\Omega, f, x^0) := \bigcap_{i \in I_m} L_<(\Omega, f_i, f_i(x^0)).$$

In the next lemma, we recall useful characterizations of Pareto efficient solutions by using certain level sets and level lines of the component functions of $f = (f_1, \cdots, f_m) : E \to \mathbb{R}^m$ (cf. Ehrgott [9, Th. 2.30]).

Lemma 2.1 (cf. [9, Th. 2.30])). *For any $x^0 \in \Omega$, the following assertions hold:*

1°. $x^0 \in \mathrm{Eff}(\Omega \mid f)$ *if and only if* $S_\leq(E, f, x^0) \cap \Omega \subseteq S_=(E, f, x^0)$.

2°. $x^0 \in \mathrm{WEff}(\Omega \mid f)$ *if and only if* $S_<(E, f, x^0) \cap \Omega = \emptyset$.

Recall that a vector-valued function $f = (f_1, \cdots, f_m) : E \to \mathbb{R}^m$ is said to be

- componentwise convex if, for any $i \in I_m$, f_i is convex, i.e., for any $x^0, x^1 \in E$ and $\lambda \in [0, 1]$, we have
$$f_i((1-\lambda)x^0 + \lambda x^1) \leq (1-\lambda)f_i(x^0) + \lambda f_i(x^1).$$
- componentwise quasi-convex if, for any $i \in I_m$, f_i is quasi-convex, i.e., for any $x^0, x^1 \in E$ and $\lambda \in [0, 1]$, we have
$$f_i((1-\lambda)x^0 + \lambda x^1) \leq \max\left\{f_i(x^0), f_i(x^1)\right\}.$$
- componentwise semi-strictly quasi-convex if, for any $i \in I_m$, f_i is semi-strictly quasi-convex, i.e., for any $x^0, x^1 \in E$ such that $f_i(x^0) \neq f_i(x^1)$, and for any $\lambda \in]0, 1[$, we have
$$f_i((1-\lambda)x^0 + \lambda x^1) < \max\left\{f_i(x^0), f_i(x^1)\right\}.$$
- componentwise explicitly quasi-convex if it is both componentwise semi-strictly quasi-convex and componentwise quasi-convex.
- componentwise upper (lower) semi-continuous along line segments if, for any $i \in I_m$ and $x^0, x^1 \in E$, the function $f_i \circ l_{x^0,x^1} : [0, 1] \to \mathbb{R}$, where
$$l_{x^0,x^1}(\lambda) := (1-\lambda)x^0 + \lambda x^1 \quad \text{for all } \lambda \in [0, 1],$$
is upper (lower) semi-continuous.

It is a well-known fact that quasi-convexity and semi-strict quasi-convexity are generalizations of convexity. Given a convex function $h : \mathbb{R} \to \mathbb{R}$, we directly get that h is explicitly quasi-convex (i.e., both semi-strictly quasi-convex and quasi-convex). However, the reverse implication is not valid. To show this fact, consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := x^3$ for all $x \in \mathbb{R}$. Then, it is easily seen that h is explicitly quasi-convex but not convex. Under lower semi-continuity, each semi-strictly quasi-convex function is quasi-convex. For more details, we refer the reader to the book by Cambini and Martein [5].

Remark 2.2. For any $i \in I_m$, the following characterizations are valid:

1°. f_i is quasi-convex if and only if $L_\leq(E, f_i, s)$ is convex for all $s \in \mathbb{R}$.

2°. f_i is semi-strictly quasi-convex if and only if for any $s \in \mathbb{R}$ and $x^0, x^1 \in E$ such that $x^0 \in L_<(E, f_i, s)$ and $x^1 \in L_=(E, f_i, s)$ we have $]x^0, x^1[\in L_<(E, f_i, s)$.

Next, we recall some relationships between the problem $(\mathcal{P}_\Omega)$ and a corresponding problem with an objective function f and a feasible set given by the whole space E.

Proposition 2.3 ([16, Lem. 4.4])**.** *For any nonempty set* $\Omega \subseteq E$*, the following assertions hold:*

1°. *It holds that*
$$\Omega \cap \mathrm{Eff}(E \mid f) \subseteq \mathrm{Eff}(\Omega \mid f),$$
$$\Omega \cap \mathrm{WEff}(E \mid f) \subseteq \mathrm{WEff}(\Omega \mid f).$$

2°. *If $f : E \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex, then*

$$\begin{aligned}(\operatorname{cor}\Omega) \setminus \operatorname{Eff}(E \mid f) &\subseteq (\operatorname{cor}\Omega) \setminus \operatorname{Eff}(\Omega \mid f),\\ (\operatorname{cor}\Omega) \setminus \operatorname{WEff}(E \mid f) &\subseteq (\operatorname{cor}\Omega) \setminus \operatorname{WEff}(\Omega \mid f),\\ \operatorname{Eff}(\Omega \mid f) &\subseteq [\Omega \cap \operatorname{Eff}(E \mid f)] \cup \operatorname{bd}\Omega,\\ \operatorname{WEff}(\Omega \mid f) &\subseteq [\Omega \cap \operatorname{WEff}(E \mid f)] \cup \operatorname{bd}\Omega.\end{aligned}$$

According to Günther and Tammer [16, Lem. 4.4], the proof of Proposition 2.3 is based on the geometrical characterization of (weakly) Pareto efficient solutions given in Lemma 2.1 and the characterization of semi-strict quasi-convexity given in Remark 2.2.

3. The method by vectorial penalization

In this section, we recall the method by vectorial penalization for solving constrained multi-objective optimization problems. Throughout the section, we assume that the following assumptions hold:

$$\text{(3.1)} \qquad \begin{cases} \text{Let } E \text{ be a real topological linear space;} \\ \text{let } \Omega \subseteq E \text{ be a nonempty, closed set with } \Omega \neq E. \end{cases}$$

According to Günther and Tammer [15, 16], to the original objective function $f = (f_1, \cdots, f_m) : E \to \mathbb{R}^m$ of the problem $(\mathcal{P}_E)$ (defined as $(\mathcal{P}_\Omega)$ with E in the role of Ω) we now add a scalar penalization function $\varphi : E \to \mathbb{R}$ as a new component function $f_{m+1} := \varphi$. Then, we can consider the penalized multi-objective optimization problem $(\mathcal{P}_E^\oplus)$ with objective function $f^\oplus = (f, \varphi)$ and feasible set Ω, as considered in Section 1.

In what follows, we will need in certain results some of the following assumptions concerning the lower-level sets / level lines of the penalization function φ:

$$\text{(A1)} \qquad \forall x^0 \in \operatorname{bd}\Omega : \; L_\leq(E, \varphi, \varphi(x^0)) = \Omega,$$

$$\text{(A2)} \qquad \forall x^0 \in \operatorname{bd}\Omega : \; L_=(E, \varphi, \varphi(x^0)) = \operatorname{bd}\Omega,$$

$$\text{(A3)} \qquad \forall x^0 \in \Omega : \; L_=(E, \varphi, \varphi(x^0)) = L_\leq(E, \varphi, \varphi(x^0)) = \Omega,$$

$$\text{(A4)} \qquad \forall x^0 \in \operatorname{bd}\Omega \; \exists\, x^1 \in \operatorname{int}\Omega : \; [x^1, x^0) \subseteq L_<(E, \varphi, \varphi(x^0)).$$

3.1. Relationships between the sets of efficient solutions of the problems $(\mathcal{P}_\Omega)$, $(\mathcal{P}_E)$ and $(\mathcal{P}_E^\oplus)$.

We are going to recall important relationships between the sets of Pareto efficient solutions of the problems $(\mathcal{P}_\Omega)$, $(\mathcal{P}_E)$ and $(\mathcal{P}_E^\oplus)$.

The following result goes back to Günther and Tammer [16, Th. 5.1, Th. 5.6] and shows that the set of (weakly) Pareto efficient solutions of $(\mathcal{P}_\Omega)$ with a not necessarily convex feasible set $\Omega \subseteq E$ can be computed by solving at most two unconstrained problems $(\mathcal{P}_E)$ and $(\mathcal{P}_E^\oplus)$.

Proposition 3.1 ([16, Th. 5.1, Th. 5.6]). *Let* (3.1) *be satisfied and suppose that φ fulfills* (A1) *and* (A2). *Then, the following assertions hold:*

1°. *It holds that*

$$[\Omega \cap \operatorname{Eff}(E \mid f)] \cup \left[(\operatorname{bd}\Omega) \cap \operatorname{Eff}(E \mid f^\oplus)\right] \subseteq \operatorname{Eff}(\Omega \mid f).$$

2°. *If* $\operatorname{int}\Omega = \emptyset$ *or* $f : E \to \mathbb{R}^m$ *is componentwise semi-strictly quasi-convex, then*

$$[\Omega \cap \operatorname{Eff}(E \mid f)] \cup \left[(\operatorname{bd}\Omega) \cap \operatorname{Eff}(E \mid f^{\oplus})\right] \supseteq \operatorname{Eff}(\Omega \mid f).$$

3°. *Assume that* $\operatorname{int}\Omega \neq \emptyset$. *Let* $f : E \to \mathbb{R}^m$ *be componentwise upper semi-continuous along line segments. Furthermore, we suppose that* φ *fulfills* (A4). *Then,*

$$[(\operatorname{int}\Omega) \cap \operatorname{WEff}(E \mid f)] \cup \left[(\operatorname{bd}\Omega) \cap \operatorname{WEff}(E \mid f^{\oplus})\right] \subseteq \operatorname{WEff}(\Omega \mid f).$$

4°. *If* $\operatorname{int}\Omega = \emptyset$ *or* $f : E \to \mathbb{R}^m$ *is componentwise semi-strictly quasi-convex, then*

$$[(\operatorname{int}\Omega) \cap \operatorname{WEff}(E \mid f)] \cup \left[(\operatorname{bd}\Omega) \cap \operatorname{WEff}(E \mid f^{\oplus})\right] \supseteq \operatorname{WEff}(\Omega \mid f).$$

In Proposition 3.1, we considered a penalization function φ which fulfills (A1) and (A2). Now, we give some corresponding results for the case that φ satisfies (A3).

Proposition 3.2 ([16, Th. 5.3, Th. 5.4, Th. 5.7, Lem. 4.4]). *Let* (3.1) *be satisfied and suppose that* φ *fulfills* (A3). *Then, the following assertions are true:*

1°. *It holds that*

$$\Omega \cap \operatorname{Eff}(E \mid f^{\oplus}) = \operatorname{Eff}(\Omega \mid f).$$

2°.

$$[\Omega \cap \operatorname{Eff}(E \mid f)] \cup \left[(\operatorname{bd}\Omega) \cap \operatorname{Eff}(E \mid f^{\oplus})\right] \subseteq \operatorname{Eff}(\Omega \mid f).$$

3°. *If* $\operatorname{int}\Omega = \emptyset$ *or* $f : E \to \mathbb{R}^m$ *is componentwise semi-strictly quasi-convex, then*

$$[\Omega \cap \operatorname{Eff}(E \mid f)] \cup \left[(\operatorname{bd}\Omega) \cap \operatorname{Eff}(E \mid f^{\oplus})\right] \supseteq \operatorname{Eff}(\Omega \mid f).$$

4°. *It holds that*

$$\Omega = \Omega \cap \operatorname{WEff}(E \mid f^{\oplus}) \supseteq \operatorname{WEff}(\Omega \mid f).$$

5°. *If* $\operatorname{int}\Omega = \emptyset$ *or* $f : E \to \mathbb{R}^m$ *is componentwise semi-strictly quasi-convex, then*

$$[\Omega \cap \operatorname{WEff}(E \mid f)] \cup \left[(\operatorname{bd}\Omega) \cap \operatorname{WEff}(E \mid f^{\oplus})\right] \supseteq \operatorname{WEff}(\Omega \mid f),$$

where

$$(\operatorname{bd}\Omega) \cap \operatorname{WEff}(E \mid f^{\oplus}) = \operatorname{bd}\Omega.$$

In order to state some more results for weak Pareto efficiency, we need the concept of Pareto reducibility:

According to Popovici [24, Def. 1], $(\mathcal{P}_\Omega)$ is called Pareto reducible if the set of weakly Pareto efficient solutions of $(\mathcal{P}_\Omega)$ can be represented as the union of the sets of Pareto efficient solutions of its subproblems.

Considering the objective function

$$f_I = (f_{i_1}, \cdots, f_{i_k}) : E \to \mathbb{R}^k,$$

for a selection of indices $I = \{i_1, \dots, i_k\} \subseteq I_{m+1}$, $i_1 < \dots < i_k$, with cardinality $|I| = k \geq 1$, we define the problem

$$\begin{cases} f_I(x) = (f_{i_1}(x), \cdots, f_{i_k}(x)) \to \min \text{ w.r.t. } \mathbb{R}^k_+ \\ x \in \Omega. \end{cases} \tag{3.2}$$

In fact, (3.2) is a single-objective optimization problem when I is a singleton set, otherwise being a multi-objective one. In particular, we have $f_{I_m} = f$ and $f_{I_{m+1}} = f^{\oplus}$. Under the assumption $\emptyset \neq I \subseteq I_m$, problem (3.2) can be seen as a subproblem of the initial problem $(\mathcal{P}_\Omega)$.

Let us recall an important Pareto reducibility result derived by Popovici [24, Prop. 4].

Proposition 3.3 (cf. [24, Prop. 4]). *Let Ω be a nonempty, convex set in the linear topological space E. If $f : E \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments, then*

$$\mathrm{WEff}(\Omega \mid f) = \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(\Omega \mid f_I).$$

Due to Günther and Tammer [16, Th. 5.11], we have the following representation of the set $\mathrm{WEff}(\Omega \mid f)$ in terms of sets of Pareto efficient solutions of certain unconstrained problems.

Proposition 3.4 ([16, Th. 5.11]). *Let* (3.1) *be satisfied and suppose that φ fulfills* (A3). *Assume that Ω is convex. If $f : E \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments, then*

$$\mathrm{WEff}(\Omega \mid f) = \left[\bigcup_{\substack{\{m+1\} \subseteq I \subseteq I_{m+1}; \\ |I| \geq 2}} \mathrm{Eff}(E \mid f_I) \right] \cap \Omega.$$

Remark 3.5. Notice that

$$\Omega = \left[\bigcup_{\substack{\{m+1\} \subseteq I \subseteq I_{m+1}; \\ |I| \geq 2}} \mathrm{WEff}(E \mid f_I) \right] \cap \Omega$$

since

$$\Omega \cap \mathrm{WEff}(E \mid f_{I_{m+1}}) = \Omega \cap \mathrm{WEff}(E \mid f^{\oplus}) = \Omega$$

in view of Proposition 3.2 (4°).

3.2. Some examples for the penalization function φ.

Exact penalty principles in optimization often use a penalization function $\varphi : E \to \mathbb{R}$ (penalty term concerning Ω) which fulfills

$$x^0 \in \Omega \iff \varphi(x^0) = 0,$$
$$x^0 \in E \setminus \Omega \iff \varphi(x^0) > 0$$

(see, e.g., Apetrii *et al.* [2], Clarke [6], Durea *et al.* [8], Ye [27], and references therein). Notice that such a function φ satisfies condition (A3).

Within the method by vectorial penalization we can use further types of penalization functions that are not necessarily of the above type. Next, under the assumptions in (3.1), we present some examples (see Günther and Tammer [16, Sec. 4.3]).

Example 3.6. Assume, in addition, that Ω is convex and $x^1 \in \operatorname{int}\Omega$. Let a Minkowski gauge μ_B associated to the set $B := -x^1 + \Omega$ be given by

$$\mu_B(x) := \inf\{\lambda > 0 \,|\, x \in \lambda \cdot B\} \quad \text{for all } x \in E.$$

Then, $\varphi := \mu_B(\cdot - x^1)$ fulfills Assumptions (A1) and (A2).

Example 3.7. Let $(E, ||\cdot||)$ be a normed space with norm $||\cdot|| : E \to \mathbb{R}$. The distance to the set Ω is given by the function $d_\Omega : E \to \mathbb{R}$, where

$$d_\Omega(x) := \inf\{||x - x'|| \,|\, x' \in \Omega\} \quad \text{for all } x \in E.$$

The following properties of d_Ω are important:

- d_Ω is Lipschitz continuous with constant 1.
- d_Ω is convex if and only if Ω is convex.
- d_Ω fulfills

$$L_=(E, d_\Omega, 0) = L_\leq(E, d_\Omega, 0) = \Omega.$$

We conclude that d_Ω fulfills condition (A3).

Example 3.8. Consider again the normed space $(E, ||\cdot||)$. The Hiriart-Urruty function $\triangle_\Omega : E \to \mathbb{R}$ is given by

$$\triangle_\Omega(x) := d_\Omega(x) - d_{E\setminus\Omega}(x) = \begin{cases} d_\Omega(x) & \text{for } x \in E \setminus \Omega, \\ -d_{E\setminus\Omega}(x) & \text{for } x \in \Omega. \end{cases}$$

Let us recall some well-known properties of $\triangle_\Omega$ (see Hiriart-Urruty [17] and Zaffaroni [28]):

- $\triangle_\Omega$ is Lipschitz continuous with constant 1.
- $\triangle_\Omega$ is convex on E if and only if Ω is convex
- $\triangle_\Omega$ fulfills

$$L_\leq(E, \triangle_\Omega, 0) = \Omega,$$
$$L_=(E, \triangle_\Omega, 0) = \operatorname{bd}\Omega,$$
$$L_<(E, \triangle_\Omega, 0) = \operatorname{int}\Omega.$$

Therefore, $\varphi := \triangle_\Omega$ fulfills both conditions (A1) and (A2).

Example 3.9. Assume that $\Omega \neq \emptyset$ is given by a system of functional inequalities, i.e.,

$$\Omega = \{x \in E \mid g_1(x) \leq 0, \cdots, g_l(x) \leq 0\} = \bigcap_{i \in I_l} L_\leq(E, g_i, 0),$$

where $g_1, \cdots, g_l : E \to \mathbb{R}$, $l \in \mathbb{N}$, are continuous scalar functions. Define

$$\varphi := \max\{g_1, \cdots, g_l\}.$$

If φ is semi-strictly quasi-convex, and Slater's condition

$$L_<(E,\varphi,0) = \bigcap_{i\in I_l} L_<(E,g_i,0) \neq \emptyset$$

holds, then $\operatorname{int}\Omega \neq \emptyset$, and φ fulfills (A1), (A2) and (A4), which follows from Günther and Tammer [16, Cor. 6.15].

Example 3.10. Let Ω be a nonempty, closed set in E with $\Omega \neq E$. Consider $k \in E \setminus \{0\}$ such that the pair (Ω, k) fulfills $\Omega + [0,+\infty)\cdot k \subseteq \Omega$. Then, one can define the function $\phi_{\Omega,k} : E \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\phi_{\Omega,k}(x) := \inf\{s \in \mathbb{R} \mid x \in sk - \Omega\} \quad \text{for all } x \in E. \tag{3.3}$$

Important properties of the function $\phi_{\Omega,k}$ (see Gerth and Weidner [11], and Göpfert *et al.* [12, Th. 2.3.1]) are:

- $\phi_{\Omega,k}$ is lower semi-continuous on E and for every $s \in \mathbb{R}$ it holds
$$L_\leq(E,\phi_{\Omega,k},s) = sk - \Omega.$$
- $\phi_{\Omega,k}$ is convex if and only if Ω is convex.
- $\phi_{\Omega,k}(\lambda x) = \lambda\phi_{\Omega,k}(x)$ for all $x \in E$ and all $\lambda > 0$ if and only if Ω is a cone.
- $\phi_{\Omega,k}$ is subadditive if and only if $\Omega + \Omega \subseteq \Omega$.
- $\phi_{\Omega,k}$ is proper if and only if Ω does not contain lines parallel to k, i.e.,
$$\forall\, x \in E\, \exists\, t \in \mathbb{R} :\ x + tk \notin \Omega.$$
- $\phi_{\Omega,k}$ is finite-valued if and only if Ω does not contain lines parallel to k, and
$$\mathbb{R}\cdot k - \Omega = E.$$
- If $\Omega + (0,+\infty)\cdot k \subseteq \operatorname{int}\Omega$, then the function $\phi_{\Omega,k}$ is continuous, and for every $s \in \mathbb{R}$, we have
$$L_<(E,\phi_{\Omega,k},s) = sk - \operatorname{int}\Omega,$$
$$L_=(E,\phi_{\Omega,k},s) = sk - \operatorname{bd}\Omega.$$

Consider the function $\varphi : E \to \mathbb{R}$ defined by

$$\varphi(x) := \phi_{\Omega,k}(-x) \quad \text{for every } x \in E.$$

Suppose that $\phi_{\Omega,k}$ is finite-valued, hence φ is finite-valued as well. Then, we have

$$L_\leq(E,\varphi,0) = \Omega,$$

and if $\Omega + (0,+\infty)\cdot k \subseteq \operatorname{int}\Omega$, then

$$L_<(E,\varphi,0) = \operatorname{int}\Omega,$$
$$L_=(E,\varphi,0) = \operatorname{bd}\Omega,$$

which shows that φ satisfies both conditions (A1) and (A2).

In the following lemma, we collect important properties of the nonlinear functional (3.3) for the special case that Ω coincides with the natural ordering cone $\mathbb{R}^m_+$ (see Durea and Tammer [7, Lemma 2.1], Gerth and Weidner [11], and Göpfert *et al.* [12, Th. 2.3.1]). We will use these properties in Section 4 for deriving necessary optimality conditions.

Lemma 3.11. *Let $C := \mathbb{R}^m_+$. Consider, for any $k \in \operatorname{int} C$, the functional $\phi_{C,k} : \mathbb{R}^m \to \mathbb{R}$ that is given by*

$$\phi_{C,k}(y) := \inf\{s \in \mathbb{R} \mid y \in sk - C\}. \tag{3.4}$$

Then, the following assertions hold:

1°. *$\phi_{C,k}$ is convex and continuous.*

2°. *For any $u \in \mathbb{R}^m$, the Fenchel subdifferential $\partial\phi_{C,k}(u)$ is nonempty and can be represented as*

$$\partial\phi_{C,k}(u) = \{y^* \in \mathbb{R}^m_+ \mid \langle y^*, k\rangle = 1, \langle y^*, u\rangle = \phi_{C,k}(u)\},$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product in $\mathbb{R}^m$. In particular,

$$\partial\phi_{C,k}(0) = \{y^* \in \mathbb{R}^m_+ \mid \langle y^*, k\rangle = 1\}.$$

3°. *$\phi_{C,k}$ is Lipschitz continuous with constant $d_{\operatorname{bd} C}(k)^{-1}$, and for any $u \in \mathbb{R}^m$ and $y^* \in \partial\phi_{C,k}(u)$, one has*

$$\|k\|^{-1}_{\mathbb{R}^m} \le \|y^*\|_{\mathbb{R}^m} \le d_{\operatorname{bd} C}(k)^{-1}.$$

4°. *$\overline{x} \in \operatorname{WEff}(E \mid f)$ if and only if*

$$\phi_{C,k}(f(\overline{x}) - f(\overline{x})) = \min_{x\in E} \phi_{C,k}(f(x) - f(\overline{x})).$$

4. Necessary optimality conditions

We derive necessary optimality conditions for weakly Pareto efficient solutions of $(\mathcal{P}_\Omega)$ expressed in terms of generalized differentiation objects such as subdifferentials using the assertion given in Proposition 3.1. For any two real Banach spaces E and Y, let us denote by $\mathcal{F}(E,Y)$ a class of functions acting between E and Y having the property that by composition at left with a lower semi-continuous function from Y to $\mathbb{R}$ the resulting function is still lower semi-continuous. In this work, we do not consider a specific subdifferential, but we use a generic concept: An operator ∂ which associates with every $h : E \to \mathbb{R}$ and every $x \in E$ a subset $\partial h(x) \subseteq E^*$, where E^* is the topological dual space of E, such that the following axioms are satisfied:

(H1) If h is convex, then ∂h coincides with the Fenchel subdifferential, i.e.,

$$\partial h(x) = \{y^* \in E^* \mid \forall x' \in E : \langle y^*, x' - x\rangle + h(x) \le h(x')\}.$$

(H2) If h is locally Lipschitz continuous, and $\overline{x}$ is a local minimum point for h over E, then

$$0 \in \partial h(\overline{x}).$$

(H3) If $\eta : Y \to \mathbb{R}$ is convex and $\psi \in \mathcal{F}(E,Y)$, then for every $x \in E$,

$$\partial(\eta \circ \psi)(x) \subseteq \bigcup\nolimits_{y^* \in \partial\eta(\psi(x))} \partial(y^* \circ \psi)(x).$$

It is known that these axioms are fulfilled for several important subdifferentials on suitable classes of Banach spaces. We will not give technical details, but we mention the subdifferentials by Mordukhovich, Michel-Penot, Ioffe. Concerning more details, we refer the reader to Mordukhovich [22].

Using Proposition 3.1 and Lemma 3.11, we get the following necessary condition for weakly Pareto efficient solutions of problem $(\mathcal{P}_\Omega)$.

Theorem 4.1. *Let $(E, ||\cdot||_E)$ be a real Banach space, let $\Omega \subseteq E$ be a nonempty, closed set with $\Omega \neq E$ and $\operatorname{int}\Omega \neq \emptyset$. Assume that ∂ satisfies the axioms* (H1), (H2) *and* (H3). *Suppose that $\varphi : E \to \mathbb{R}$ fulfills both conditions* (A1) *and* (A2). *Let $f : E \to \mathbb{R}^m$ be componentwise semi-strictly quasi-convex, let $f \in \mathcal{F}(E, \mathbb{R}^m)$ and $f^{\oplus} \in \mathcal{F}(E, \mathbb{R}^{m+1})$. Take some $\overline{x} \in \operatorname{WEff}(\Omega \mid f)$. Assume that f and φ are locally Lipschitz continuous at $\overline{x}$. Then, the following necessary conditions hold:*

1°. *If $\overline{x} \in \operatorname{int}\Omega$, then for every $\varepsilon > 0$ there exist $y^* \in \mathbb{R}^m_+$ and $k \in \operatorname{int}\mathbb{R}^m_+$ with $\|y^*\|_{\mathbb{R}^m} < \varepsilon$ and $\langle y^*, k\rangle = 1$ such that*

$$0 \in \partial(y^* \circ f)(\overline{x}) = \partial\left(\sum_{i\in I_m} y_i^* f_i\right)(\overline{x}).$$

2°. *If $\overline{x} \in \operatorname{bd}\Omega$, then for every $\varepsilon > 0$ there exist $u^* := (y^*, s^*) \in \mathbb{R}^m_+ \times \mathbb{R}_+$ and $k \in \operatorname{int}\mathbb{R}^{m+1}_+$ with $\|u^*\|_{\mathbb{R}^{m+1}} < \varepsilon$ and $\langle u^*, k\rangle = 1$ such that*

$$0 \in \partial(u^* \circ f^{\oplus})(\overline{x}) = \partial\left(s^*\varphi + \sum_{i\in I_m} y_i^* f_i\right)(\overline{x}).$$

Proof. To prove assertion 1°, consider $\overline{x} \in (\operatorname{int}\Omega) \cap \operatorname{WEff}(\Omega \mid f)$. In view of Proposition 3.1 (4°), we get $\overline{x} \in (\operatorname{int}\Omega) \cap \operatorname{WEff}(E \mid f)$. Consider any $\varepsilon > 0$ and define $C := \mathbb{R}^m_+$. W.l.o.g. we take $k \in \operatorname{int} C$ such that $\varepsilon^{-1} < d_{\operatorname{bd} C}(k)$. Let us define a functional $z_{C,k} : \mathbb{R}^m \to \mathbb{R}$ by

$$z_{C,k}(y) := \phi_{C,k}(y - f(\overline{x})) \quad \text{for all } y \in \mathbb{R}^m.$$

Notice that $z_{C,k}$ Lipschitz continuous in view of Lemma 3.11 (3°), and f is assumed to be locally Lipschitz continuous at $\overline{x}$. Hence, $z_{C,k} \circ f$ is locally Lipschitz continuous at $\overline{x}$. Recalling that $\overline{x} \in \operatorname{WEff}(E \mid f)$, Lemma 3.11 (4°) yields that $\overline{x}$ is a minimum point for $z_{C,k} \circ f$ on E. Now, applying axiom (H2) for the abstract subdifferential ∂ yields

$$0 \in \partial(z_{C,k} \circ f)(\overline{x}). \tag{4.1}$$

The functional $z_{C,k}$ is convex taking into account Lemma 3.11 (1°), and $f \in \mathcal{F}(E, \mathbb{R}^m)$, hence axiom (H3) ensures the inclusion

$$\partial(z_{C,k} \circ f)(\overline{x}) \subseteq \bigcup\nolimits_{y^* \in \partial z_{C,k}(f(\overline{x}))} \partial(y^* \circ f)(\overline{x}). \tag{4.2}$$

By the convexity of $\phi_{C,k}$ in view of Lemma 3.11 (1°), and axiom (H1) for the abstract subdifferential ∂, we know that

$$\partial z_{C,k}(f(\overline{x})) = \partial\phi_{C,k}(0) \tag{4.3}$$

is a Fenchel subdifferential. Furthermore, due to Lemma 3.11 (2°, 3°) and $\varepsilon^{-1} < d_{\operatorname{bd} C}(k)$, for any $y^* \in \partial\phi_{C,k}(0)$, we have

$$\|y^*\|_{\mathbb{R}^m} \le d_{\operatorname{bd} C}(k)^{-1} < \varepsilon\ , \quad y^* \in \mathbb{R}^m_+ \ \text{ and } \ \langle y^*, k\rangle = 1. \tag{4.4}$$

Because of (4.1), (4.2), (4.3) and (4.4), assertion 1° of the theorem holds.

For proving assertion 2°, let $\overline{x} \in (\operatorname{bd}\Omega) \cap \operatorname{WEff}(\Omega \mid f)$. Due to Proposition 3.1 (4°), we get $\overline{x} \in (\operatorname{bd}\Omega) \cap \operatorname{WEff}(E \mid f^{\oplus})$. The rest of the proof uses similar ideas as given in the proof assertion 1°. □

Remark 4.2. By using the relationships between $(\mathcal{P}_\Omega)$ and the unconstrained problems $(\mathcal{P}_E)$ and $(\mathcal{P}_E^\oplus)$, we succeeded to give necessary conditions completely in terms of abstract subdifferentials. These conditions do not involve a normal cone with respect to the constraints Ω.

Remark 4.3. The reader could ask concerning similar results for the case that φ fulfills the condition (A3). In view of Proposition 3.2 (4°) (respectively, Proposition 3.2 (5°), Proposition 3.4, and Remark 3.5), one could derive, for any $\overline{x} \in \mathrm{WEff}(\Omega \mid f)$, a necessary condition

$$0 \in \partial(u^* \circ f^\oplus)(\overline{x}) \quad \text{for some } u^* \in \mathbb{R}_+^{m+1} \setminus \{0\}. \tag{4.5}$$

However, this condition (4.5) can be derived for any $\overline{x} \in \Omega$, as Proposition 3.2 (4°) shows. Consequently, a penalization function φ which fulfills conditions (A1) and (A2) (instead of (A3)) is more appropriate for deriving optimality conditions based on methods by generalized differentiation.

Remark 4.4. Using other penalization techniques, Durea, Strugariu and Tammer have shown necessary optimality conditions in [8, Th. 3.2, Th. 3.3] in terms of the normal (or limiting or Mordukhovich) coderivative and in [8, Th. 3.4] in terms of subdifferentials based on a generic concept. Therefore, the necessary optimality conditions in [8] have a different structure in comparison with the conditions in Theorem 4.1.

5. Application to Approximation Problems involving nonconvex constraints

As an application of our derived results, we present some necessary optimality conditions for multi-objective (not necessarily convex) approximation problems having a practical importance described by Khan, Tammer and Zălinescu [19].

In this section, we assume that the following assumptions hold:

$$\begin{cases} \text{Let } (Z, ||\cdot||_Z) \text{ be a real reflexive Banach space;} \\ \text{let } (E, ||\cdot||_E) \text{ be a real Banach space;} \\ \text{let } \Omega \subseteq E \text{ be a nonempty, closed set with } \Omega \neq E \text{ and } \operatorname{int}\Omega \neq \emptyset. \end{cases} \tag{5.1}$$

In order to formulate the vector-valued approximation problem, we introduce a vector-valued norm (see Jahn [18, Def. 1.35]) as an application $|||\cdot||| : Z \to \mathbb{R}_+^m$ which for all $z, z^1, z^2 \in Z$ and for all $\lambda \in \mathbb{R}$ satisfies:

(1) $|||z||| = 0 \iff z = 0$,

(2) $|||\lambda z||| = |\lambda| \; |||z|||$,

(3) $|||z^1 + z^2||| \in |||z^1||| + |||z^2||| - \mathbb{R}_+^m$.

It is easy to check that (2) and (3) guarantee that the vector-valued norm $|||\cdot|||$ is componentwise convex.

According to Jahn [18, Def. 2.21], for a vector-valued function $f : Z \to \mathbb{R}^m$ we can consider a subdifferential of f at a point $z^0 \in Z$ by

$$\partial^{\leq} f(z^0) := \left\{ T \in \mathrm{L}(Z, \mathbb{R}^m) \mid \forall z \in Z : f(z^0 + z) - f(z^0) - T(z) \in \mathbb{R}_+^m \right\}, \tag{5.2}$$

where $\mathrm{L}(Z,\mathbb{R}^m)$ denotes the space of linear continuous operators from Z into $\mathbb{R}^m$. For the particular case of above vector-valued norm $\|\!|\cdot|\!\|$, it has the following form (see Jahn [18, Ex. 2.22]):

$$(5.3)\quad \partial^{\leq}\|\!|z^0|\!\| = \left\{T \in \mathrm{L}(Z,\mathbb{R}^m) \mid T(z^0) = \|\!|z^0|\!\| \ \wedge\ (\forall z \in Z : \|\!|z|\!\| - T(z) \in \mathbb{R}^m_+)\right\}.$$

In what follows, we assume that the vector-valued norm $\|\!|\cdot|\!\| : Z \to \mathbb{R}^m_+$ is continuous. Since $\mathbb{R}^m_+$ has the Daniell property and because of the continuity of $\|\!|\cdot|\!\|$, the subdifferential of the vector-valued norm $\|\!|\cdot|\!\| : Z \to \mathbb{R}^m_+$ is nonempty at any $z^0 \in Z$, i.e., $\partial^{\leq}\|\!|z^0|\!\| \neq \emptyset$ (see Jahn [18, Th. 2.27]).

Since $\mathbb{R}^m_+$ has a weakly compact base, we adapt a special rule for the subdifferential of the vector-valued norm $\|\!|\cdot|\!\|$ by Valadier [26] which is useful in the sequel.

Lemma 5.1 (cf. [26], [18, Lem. 2.26])**.** *Let $(Z, ||\cdot||_Z)$ be a real reflexive Banach space. Suppose that the vector-valued norm $\|\!|\cdot|\!\| : Z \to \mathbb{R}^m_+$ is continuous. Then, for every $z \in Z$ and $y^* \in \mathbb{R}^m_+$, one has*

$$y^* \circ \partial^{\leq}\|\!|z|\!\| = \partial\langle y^*, \|\!|z|\!\|\rangle.$$

Notice that for every $y^* \in \mathbb{R}^m_+$, the mapping $\langle y^*, \|\!|\cdot|\!\|\rangle$ is convex, hence its subdifferential is the Fenchel subdifferential of convex analysis.

In the next lemma, we recall sufficient conditions for the fact that $f = (f_1, \cdots, f_m) : E \to \mathbb{R}^m$ is componentwise locally Lipschitz continuous.

Lemma 5.2 (cf. [23, Prop. 1.6])**.** *Assume that $(E, ||\cdot||_E)$ is a real Banach space. If $f = (f_1, \cdots, f_m) : E \to \mathbb{R}^m$ is componentwise convex and continuous, then f is componentwise locally Lipschitz continuous.*

Consider now a vector-valued function $g : E \to \mathbb{R}^m$ that is componentwise convex and continuous, linear operators $A_i \in \mathrm{L}(E, Z), i \in I_n$, $n \in \mathbb{N}$, and weights $\alpha_i \geq 0$, $i \in I_n$. The corresponding adjoint operator to A_i, $i \in I_n$, is denoted by A_i^*. For any nonempty, closed set $\Omega \subseteq E$, and a finite number of a priori given points $a^i \in Z, i \in I_n$, we formulate a vector-valued approximation problem by

$$(\mathcal{AP}_\Omega) \qquad \begin{cases} f^1(x) := g(x) + \sum_{i\in I_n} \alpha_i \|\!|A_i(x) - a^i|\!\| \to \min \ \text{ w.r.t. } \mathbb{R}^m_+ \\ x \in \Omega. \end{cases}$$

It is easily seen that $f^1 : E \to \mathbb{R}^m$ is a componentwise convex vector-valued function, hence f^1 is componentwise semi-strictly quasi-convex as well.

In the following, beside the axioms (H1), (H2), (H3) for the abstract subdifferential ∂, we need a fourth axiom (rule of sums):

(H4) If $v, w : E \to \mathbb{R}$ are locally Lipschitz continuous, then, for any $\overline{x} \in E$, we have

$$\partial(v + w)(\overline{x}) \subseteq \partial v(\overline{x}) + \partial w(\overline{x}).$$

Notice that also axiom (H4) is satisfied for several important subdifferentials on suitable classes of Banach spaces.

Theorem 5.3. *Let* (5.1) *be satisfied. Suppose that $\varphi : E \to \mathbb{R}$ is continuous and fulfills both conditions* (A1) *and* (A2). *Furthermore, assume that ∂ satisfies the axioms* (H1), (H2), (H3) *and* (H4). *Take some $\overline{x} \in \mathrm{WEff}(\Omega \mid f^1)$, i.e., $\overline{x}$ is a weakly* *efficient solution of problem* $(\mathcal{AP}_\Omega)$. *Assume that* φ *is locally Lipschitz continuous at* $\overline{x}$. *Then, the following necessary conditions hold:*

1°. *If* $\overline{x} \in \operatorname{int}\Omega$, *then for every* $\varepsilon > 0$ *there exist* $y^* \in \mathbb{R}^m_+$ *and* $k \in \operatorname{int}\mathbb{R}^m_+$ *with* $\|y^*\|_{\mathbb{R}^m} < \varepsilon$ *and* $\langle y^*, k\rangle = 1$ *such that*

$$0 \in \partial\langle y^*, g(\overline{x})\rangle + D(\overline{x}),$$

where

$$D(\overline{x}) := \Big\{ \sum_{i\in I_n} \alpha_i A_i^*(y^* \circ T_i) \mid \forall i \in I_n : \; T_i \in \mathrm{L}(Z, \mathbb{R}^m), \\ T_i(A_i(\overline{x}) - a^i) = \|A_i(\overline{x}) - a^i\|, \\ \forall z \in Z : \; \|z\| - T_i(z) \in \mathbb{R}^m_+ \Big\}.$$

2°. *If* $\overline{x} \in \operatorname{bd}\Omega$, *then for every* $\varepsilon > 0$ *there exist* $u^* := (y^*, s^*) \in \mathbb{R}^m_+ \times \mathbb{R}_+$ *and* $k \in \operatorname{int}\mathbb{R}^{m+1}_+$ *with* $\|u^*\|_{\mathbb{R}^{m+1}} < \varepsilon$ *and* $\langle u^*, k\rangle = 1$ *such that*

$$0 \in \partial\langle y^*, g(\overline{x})\rangle + D(\overline{x}) + \partial(s^*\varphi)(\overline{x}).$$

Proof. For the approximation problem $(\mathcal{AP}_\Omega)$, the conditions $f^1 \in \mathcal{F}(E, \mathbb{R}^m)$ and $(f^1)^\oplus \in \mathcal{F}(E, \mathbb{R}^{m+1})$ (where $(f^1)^\oplus := (f^1, \varphi) : E \to \mathbb{R}^{m+1}$) are fulfilled. Indeed, under the assumptions in Theorem 5.3, f^1 and $(f^1)^\oplus$ are continuous functions, hence the composition at left with a lower semi-continuous function from $\mathbb{R}^m$ to $\mathbb{R}$, respectively, from $\mathbb{R}^{m+1}$ to $\mathbb{R}$, is lower semi-continuous as well.

In order to show 1°, assume that $\overline{x} \in (\operatorname{int}\Omega) \cap \mathrm{WEff}(\Omega \mid f^1)$. By Theorem 4.1 (1°), for every $\varepsilon > 0$ there exist $y^* \in \mathbb{R}^m_+$ and $k \in \operatorname{int}\mathbb{R}^m_+$ with $\|y^*\|_{\mathbb{R}^m} < \varepsilon$ and $\langle y^*, k\rangle = 1$ such that

$$0 \in \partial(y^* \circ f_1)(\overline{x}). \tag{5.4}$$

The rule of sums (H4) yields the following inclusion

$$\partial(y^* \circ f^1)(\overline{x}) \subseteq \partial\langle y^*, g(\overline{x})\rangle + \sum_{i\in I_n} \partial\langle y^*, \alpha_i \|A_i(\overline{x}) - a^i\|\rangle \tag{5.5}$$

by the local Lipschitz continuity of the functions g and $\|A_i(\cdot) - a^i\|$, $i \in I_n$, at $\overline{x}$ taking into account Lemma 5.2. In addition, by (H1) and a constant factor rule for the Fenchel subdifferential of convex analysis (more precisely $\partial(\lambda h)(\overline{x}) = \lambda\partial h(\overline{x})$ for a convex function $h : E \to \mathbb{R}$ and $\lambda \geq 0$) it follows

$$\partial\langle y^*, g(\overline{x})\rangle + \sum_{i\in I_n} \partial\langle y^*, \alpha_i \|A_i(\overline{x}) - a^i\|\rangle \\ = \partial\langle y^*, g(\overline{x})\rangle + \sum_{i\in I_n} \alpha_i \partial\langle y^*, \|A_i(\overline{x}) - a^i\|\rangle. \tag{5.6}$$

Therefore, we get

$$
\begin{aligned}
0 &\in \partial(y^* \circ f^1)(\overline{x}) && \text{(apply (5.4))}\\
&\subseteq \partial\langle y^*, g(\overline{x})\rangle + \sum_{i\in I_n} \alpha_i \partial\langle y^*, \|A_i(\overline{x}) - a^i\|\rangle && \text{(apply (5.5) and (5.6))}\\
&= \partial\langle y^*, g(\overline{x})\rangle + \sum_{i\in I_n} \alpha_i (y^* \circ \left(\partial^{\leq}\|A_i(\overline{x}) - a^i\|\right)) && \text{(apply Lemma 5.1)}\\
&= \partial\langle y^*, g(\overline{x})\rangle + \sum_{i\in I_n} \alpha_i A_i^*(y^* \circ (\partial^{\leq}\|u_i\|)) \,|_{u_i = A_i(\overline{x})-a^i}\\
&= \partial\langle y^*, g(\overline{x})\rangle + D(\overline{x}) && \text{(apply (5.3)),}
\end{aligned}
$$

hence assertion 1° holds.

To prove assertion 2°, consider $\overline{x} \in (\mathrm{bd}\,\Omega) \cap \mathrm{WEff}(\Omega \mid f^1)$. By Theorem 4.1 (2°), for every $\varepsilon > 0$ there exist $u^* := (y^*, s^*) \in \mathbb{R}^m_+ \times \mathbb{R}_+$ and $k \in \mathrm{int}\,\mathbb{R}^{m+1}_+$ with $\|u^*\|_{\mathbb{R}^{m+1}} < \varepsilon$ and $\langle u^*, k\rangle = 1$ such that

$$0 \in \partial(u^* \circ (f^1)^{\oplus})(\overline{x}).$$

Here the rules (H1) and (H4) yield the inclusion

$$\partial(u^* \circ (f^1)^{\oplus})(\overline{x}) \subseteq \partial\langle y^*, g(\overline{x})\rangle + \sum_{i\in I_n} \alpha_i \partial\langle y^*, \|A_i(\overline{x}) - a^i\|\rangle + \partial(s^*\varphi)(\overline{x})$$

by the local Lipschitz continuity of the functions g, $\|A_i(\cdot) - a^i\|$, $i \in I_n$, and φ at $\overline{x}$ taking into account the result in Lemma 5.2. Now, the proof of the assertion 2° is analogous to the proof of 1°. □

Remark 5.4. The result in Theorem 5.3 differs from a necessary condition for solutions to set-valued approximation problems by Bao and Tammer in [3, Th. 5.1] where the limiting normal cone (known also as the basic or Mordukhovich normal cone) with respect to the feasible set Ω is involved. Furthermore, in comparison with a corresponding result by Durea and Tammer in [7, Th. 5.2] where also a normal cone with respect to the feasible set Ω is included, we have shown our necessary condition without using a normal cone.

Let us consider the following special case of problem $(\mathcal{AP}_\Omega)$ given by

$$\text{(5.7)} \qquad \begin{cases} f^2(x) := \|A(x) - a\| \to \min \ \text{ w.r.t. } \mathbb{R}^m_+ \\ x \in \Omega, \end{cases}$$

where $A \in \mathrm{L}(E, Z)$ and $a \in Z$. The corresponding adjoint operator to A is denoted by A^*.

Next, we derive necessary optimality condition for this class of problems.

Corollary 5.5. *Let* (5.1) *be satisfied. Suppose that* $\varphi : E \to \mathbb{R}$ *is continuous and fulfills both conditions* (A1) *and* (A2). *Furthermore, assume that* ∂ *satisfies the axioms* (H1), (H2), (H3) *and* (H4). *Take some* $\overline{x} \in \mathrm{WEff}(\Omega \mid f^2)$, *i.e.,* $\overline{x}$ *is a weakly efficient solution of problem (5.7). Assume that* φ *is locally Lipschitz continuous at* $\overline{x}$. *Then, the following necessary conditions hold:*

1°. *If* $\overline{x} \in \operatorname{int}\Omega$*, then for every* $\varepsilon > 0$ *there exist* $y^* \in \mathbb{R}^m_+$ *and* $k \in \operatorname{int}\mathbb{R}^m_+$ *with* $\|y^*\|_{\mathbb{R}^m} < \varepsilon$ *and* $\langle y^*, k\rangle = 1$ *such that*

$$0 \in \Big\{ A^*(y^* \circ T) \mid T \in \mathrm{L}(Z, \mathbb{R}^m),\, T(A(\overline{x}) - a) = \|A(\overline{x}) - a\|, \\ \forall z \in Z : \|z\| - T(z) \in \mathbb{R}^m_+ \Big\} =: D(\overline{x}).$$

2°. *If* $\overline{x} \in \operatorname{bd}\Omega$*, then for every* $\varepsilon > 0$ *there exist* $u^* := (y^*, s^*) \in \mathbb{R}^m_+ \times \mathbb{R}_+$ *and* $k \in \operatorname{int}\mathbb{R}^{m+1}_+$ *with* $\|u^*\|_{\mathbb{R}^{m+1}} < \varepsilon$ *and* $\langle u^*, k\rangle = 1$ *such that*

$$0 \in D(\overline{x}) + \partial(s^*\varphi)(\overline{x}).$$

Furthermore, we study the following multi-objective location problem as a special case of problem $(\mathcal{AP}_\Omega)$:

$$\begin{cases} f^3(x) := (\|x - a^1\|_E, \cdots, \|x - a^m\|_E) \to \min \text{ w.r.t. } \mathbb{R}^m_+ \\ x \in \Omega. \end{cases} \tag{5.8}$$

Notice that (5.8) can be seen as a special point-objective location problem (see, e.g, Alzorba et al. [1] and references therein).

In view of Jahn [18, Ex. 2.23], in a real Banach space $(E, ||\cdot||_E)$, the norm $||\cdot||_E$ is subdifferentiable with

$$\partial||x||_E = \begin{cases} \{x^* \in E^* \mid x^*(x) = ||x||_E \text{ and } ||x^*||_{E^*} = 1\} & \text{for } x \in E \setminus \{0\}, \\ \{x^* \in E^* \mid ||x^*||_{E^*} \le 1\} & \text{for } x = 0, \end{cases}$$

as a consequence of formula (5.3) (applied for $m = 1$).

Using the special structure of the subdifferential of the norm in E, we get a very simple necessary optimality condition for the multi-objective location problem (5.8) in Corollary 5.6.

Corollary 5.6. *Let* (5.1) *be satisfied. Suppose that* $\varphi : E \to \mathbb{R}$ *is continuous and fulfills both conditions* (A1) *and* (A2)*. Furthermore, assume that* ∂ *satisfies the axioms* (H1), (H2), (H3) *and* (H4)*. Take some* $\overline{x} \in \mathrm{WEff}(\Omega \mid f^3)$*, i.e.,* $\overline{x}$ *is a weakly efficient solution of problem (5.8). Assume that* φ *is locally Lipschitz continuous at* $\overline{x}$*. Then, the following necessary optimality conditions hold:*

1°. *If* $\overline{x} \in \operatorname{int}\Omega$*, then for every* $\varepsilon > 0$ *there exist* $y^* \in \mathbb{R}^m_+$ *and* $k \in \operatorname{int}\mathbb{R}^m_+$ *with* $\|y^*\|_{\mathbb{R}^m} < \varepsilon$ *and* $\langle y^*, k\rangle = 1$ *such that*

$$0 \in \Big\{ \sum_{i \in I_m} y_i^* T_i \mid \forall i \in I_m : T_i \in \mathrm{L}(E, \mathbb{R}), \\ T_i(\overline{x} - a^i) = \|\overline{x} - a^i\|_E, \|T_i\|_{E^*} \le 1 \Big\} =: D(\overline{x}).$$

2°. *If* $\overline{x} \in \operatorname{bd}\Omega$*, then for every* $\varepsilon > 0$ *there exist* $u^* := (y^*, s^*) \in \mathbb{R}^m_+ \times \mathbb{R}_+$ *and* $k \in \operatorname{int}\mathbb{R}^{m+1}_+$ *with* $\|u^*\|_{\mathbb{R}^{m+1}} < \varepsilon$ *and* $\langle u^*, k\rangle = 1$ *such that*

$$0 \in D(\overline{x}) + \partial(s^*\varphi)(\overline{x}).$$

Remark 5.7. The necessary optimality condition in Corollary 5.6 is very useful for deriving a simple algorithm for solving problem (5.8) (see [12, Sec. 4.3]). If the penalization function φ is given by a Minkowski gauge (see Example 3.6), then the subdifferential of $s^* \circ \varphi$ at $\overline{x}$ is easily to calculate.

6. Conclusion

We derived necessary optimality conditions for generalized convex multi-objective optimization problems involving not necessarily convex constraints. The necessary conditions are completely given in terms of abstract subdifferentials and do not involve a normal cone with respect to the constraints. This new feature allows to derive useful necessary optimality conditions for certain classes of multi-objective optimization problems, as illustrated for multi-objective approximations problems. In a forthcoming work, we aim to derive sufficient conditions for weakly Pareto efficient solutions by using appropriate generalized convexity concepts such as pseudo-convexity.

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