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RIEMANN SUBMERSION AND MASLOV QUANTIZATION CONDITION

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Abstract. We discuss the behaviour of Lagrangian submanifolds satisfying a condition, so called Maslov quantization condition, under submersions and give mutual existence results of eigenvalues of Laplacians on the total space and the base space of a Riemannian submersion as a consequence of the Eigenvalue Theorem by A. Weinstein. Also we discuss the most simple case of a sub-Riemannian structure on the similar existence theorem of eigenvalues for the natural sub-Laplacian. In the Appendix we explain the Maslov class from the point of the Maslov index defined for arbitrary paths, which makes rather simple to deal with the functorial property of the Maslov class, and indicate a relation of the α-construction by Hörmander.

1. Introduction

In general, if $\dim N > \dim M$, smooth maps between Riemannian manifolds $\varphi : M \to N$ do not commute with the respective Laplacians Δ^M on M and Δ^N on N, that is $\varphi^* \circ \Delta^N \neq \Delta^M \circ \varphi^*$. If there is such a map that $\varphi^* \circ \Delta^N = \Delta^M \circ \varphi^*$ (on $C^\infty(N)$), then φ must be a *Riemannian submersion* and moreover it is required that the fibers of the submersion are minimal (cf. [21], see also [11, 15, 16]).

Here we mean that a map $\varphi : M \to N$ is a Riemannian submersion, if it satisfies the condition that for any $x \in M$,

$$T_x(M)/\mathrm{Ker}\, d\varphi_x \cong (\mathrm{Ker}\, d\varphi_x)^\perp \text{ is isometric to } T_{\varphi(x)}(N) \tag{1.1}$$

through the map $d\varphi$.

In case that the commutativity holds, then if $f \in C^\infty(N)$ is an eigenfunction of the Laplacian Δ^N, then $\varphi^*(f)$ is an eigenfunction of Δ^M with the same eigenvalue.

Our concern in this paper is to study the existence of eigenvalues or a correspondence of eigenvalues of Laplacians on the base space and the total space of a Riemannian submersion which need not satisfy the above minimality condition.

If we consider the base space of a Riemannian submersion as the configuration space of a physics system and the total space as describing the more precise structure of a point (= a state) in the base space, then it will be natural to expect there must be some correspondence of eigenvalues of Laplacians on the base space and the total space, since we may think that through the Riemannian submersion the physics system is described together (total space is understood as *multidimensional universe* in physics, for example see [3] in relation to physics aspect).

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Our arguments are based on the existence of Lagrangian submanifolds satisfying a condition, so called "Maslov quantization condition" and the "Eigenvalue Theorem" by A. Weinstein (cf. [18, 19, 22]). We explain an aspect of the behaviour of Lagrangian submanifolds under submersions and the main results are direct consequences of the Eigenvalue Theorem.

In §2 we recall local parametrizations of Lagrangian submanifolds by phase functions and explain the construction of the conic Lagrangian submanifold from a compact Lagrangian submanifold. Here we treat a part of the proof of the Eigenvalue Theorem 5.1 in a simple way (cf. [5, 13, 22, 23]).

In §3 we discuss the behaviour of Lagrangian submanifolds under submersions and in §4 their behaviour with respect to the Maslov quantization condition.

In §5 we explain our main results after recalling the Eigenvalue Theorem from [22] with a short description of its proof, although we do not enter into the details on the theory of Fourier integral operators and the theory of Lagrangian distributions (cf. [10, 13]), since our main subject is to describe the behaviour of Lagrangian submanifolds under Riemannian submersions.

Finally in §6 we discuss a similar existence theorem of eigenvalues of the sub-Laplacian for the most simple case of a sub-Riemannian structure.

In the Appendix, we give a definition of the Maslov class of two Lagrangian subbundles in a symplectic vector bundle based on the Maslov index for arbitrary paths (cf. [4, 5, 7, 9, 13]) and give proofs of several basic properties of the Maslov class.

2. Lagrangian submanifold and its parametrization by phase functions

Let X be a smooth manifold and $\pi^X : T^*(X) \to X$ the natural projection map from the cotangent bundle $T^*(X)$ to the base manifold X.

The intrinsic (or canonical) one-form on the cotangent bundle $T^*(X)$ is denoted by θ^X. It is also called the Liouville one-form, and we denote its differential $d\theta^X$ by $\omega^X := d\,\theta^X$. The two-form ω^X defines the intrinsic symplectic structure on the space of the cotangent bundle.

Let $x_1, \ldots, x_n$ be local coordinates defined on an open subset $U \subset X$ (put $\dim X = n$), then we always consider the local coordinates on the open subset $(\pi^X)^{-1}(U) = T^*(U) \cong U \times \mathbb{R}^n$ given by $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$, which corresponds to the cotangent vector $\sum_{i=1}^{n} \xi_i dx_i \in T^*_x(X)$. So often we denote the element in $T^*(X)$ by $(x, \xi) \longleftrightarrow \sum \xi_i dx_i$.

With these coordinates on $(\pi^X)^{-1}(U)$ the canonical one-form θ^X is expressed as $\sum \xi_i dx_i$.

There are two meanings of the expression $\sum \xi_i dx_i$, however they will be distinguished suitably without confusion.

A closed $n-$dimensional submanifold L $(n = \dim X)$ in the cotangent bundle $T^*(X)$ is said to be a *Lagrangian submanifold*, if the two-form ω^X vanishes on it, that is at each point $(x, \xi) \in L$ the tangent space $T_{(x,\xi)}(L)$ is a Lagrangian subspace in the symplectic vector space $T_{(x,\xi)}(T^*(X))$. In this case the canonical one-form is a closed form on L and it defines a de Rham cohomology class $[\theta^X{}_{|L}] \in H^1_{dR}(L)$.

Let L be a Lagrangian submanifold. Then for any point $p \in L$, there exists a coordinate neighborhood U of $\pi^X(p)$, an open subset D in $\mathbb{R}^k$ and a smooth function $\phi \in C^\infty(U \times D)$ satisfying the following properties:

Put

$$C_\phi = \left\{ (x,\eta) \in U \times D \;\middle|\; \frac{\partial \phi(x,\eta)}{\partial \eta_j} = \phi_{\eta_j} = 0,\ 1 \le j \le k = \dim D \right\}$$

Then,

$$\text{the one-forms } \left\{ d\phi_{\eta_j} \right\}_{j=1}^k \text{ are linearly independent on } C_\phi, \tag{2.1}$$

$$\text{the map } \rho_\phi : C_\phi \ni (x,\eta) \longmapsto (x;\phi_x) \longleftrightarrow \sum_{j=1}^n \frac{\partial \phi(x,\eta)}{\partial x_j} dx_j \tag{2.2}$$

$$= \sum_{j=1}^n \phi_{x_j} dx_j \in L \subset T^*(X) \tag{2.3}$$

and the map ρ_ϕ is a *diffeomorphism* between C_ϕ and $\rho_\phi(C_\phi) =: L_\phi \subset L$, when we take U (and D) small enough, since the differential $d\rho_\phi$ is injective.

Relations of the map ρ_ϕ and other maps can be seen from the commutative diagram:

$$\begin{tikzcd}
T^*(U \times D) \arrow[d, shift right, "\pi^{U\times D}"'] & {\pi_U}^*(T^*(U)) \arrow[l, hook', "\chi_{\pi_U}"'] \arrow[rr, "\mathfrak{p}_{\pi_U}"] & & T^*(U) \arrow[r, hook] & T^*(X) \arrow[dd, "\pi^X"] \\
& C_\phi \arrow[ul, hook', "d\phi_{|C_\phi}"'] \arrow[dl, hook'] \arrow[r, "\rho_\phi"] & L_\phi \arrow[ur, hook] \arrow[r, hook] \arrow[dr, "\pi^X{}_{|L_\phi}"'] & L \arrow[ur, hook] \arrow[dr, "\pi^X{}_{|L}"] & \\
U \times D \arrow[uu, shift right, "d\phi"'] \arrow[rrr, "\pi_U"'] & & & U \arrow[r, hook] & X
\end{tikzcd} \tag{2.4}$$

where $\pi_U : U \times D \to U$ is the projection map. The map $\mathfrak{p}_{\pi_U}$ is the natural projection map from the induced bundle ${\pi_U}^*(T^*(U))$ on $U \times D$ to the original bundle $T^*(U)$ and it is a submersion. The map $\chi_{\pi_U} := (d\pi_U)^*$ is the dual map of the differential $d\pi_U : T(U \times D) \to {\pi_U}^*(T(U))$ and it is an embedding. Also note that

$$\pi^X \circ \mathfrak{p}_{\pi_U} = \pi_U \circ \pi^{U\times D} \circ \chi_{\pi_U}.$$

The subset C_ϕ can be characterized as

$$C_\phi = \left\{ (x,\eta) \in U \times D \mid d\phi\,(x,\eta) \in \chi_{\pi_U}({\pi_U}^*(T^*(U)) \right\}.$$

We call C_ϕ or $\rho_\phi(C_\phi) = L_\phi$ a local parametrization of L by a phase-function-triple $(U \times D, \phi, \rho_\phi)$, where we always assume that the open subset U is taken small enough for the map ρ_ϕ to be a diffeomorphism.

By the properties of the phase function, we see from the diagram above that on $d\phi(C_\phi)$ the map $\mathfrak{p}_{\pi_U}$ is injective and there is a unique smooth map $s_\phi : L_\phi \to$

${\pi_U}^*(T^*(U))$ such that $\chi_{\pi_U} \circ s_\phi \circ \rho_\phi = d\phi$ on C_ϕ. Using this fact and a general formula that $(df)^*(\theta^X) = df$, $f \in C^\infty(X), df : X \to T^*(X)$, we have

Proposition 2.1. *Let L be a Lagrangian submanifold in $T^*(X)$ and ϕ_i $(i = 1, 2)$ two phase functions around a point $p \in L$ defined on $U_i \times D_i$, where $\pi^X(p) \in U_1 \bigcap U_2$ and D_i open subset in $\mathbb{R}^{k_i}$ $(i = 1, 2)$. They parametrize locally around a point $p \in L$ together with the maps $\rho_{\phi_i} : C_{\phi_i} \to L_{\phi_i}$.*

Put the functions $\psi_i \in C^\infty(L_{\phi_i})$ by $\psi_i := \phi_i \circ \rho_{\phi_i}^{-1}$. By the definition of the map ρ_{ϕ_i}, we have $d\psi_i = {\theta^X}_{|L}$ on L_{ϕ_i}, and

$$0 = d\psi_1 - d\psi_2 = d(\psi_1 - \psi_2).$$

Hence the difference $\psi_1 - \psi_2$ is locally constant on $L_{\phi_1} \bigcap L_{\phi_2}$.

Now let L be a Lagrangian submanifold and $\{L_{\phi_i}\}$ a covering by local parametrizations defined by the phase-function-triples

$$\begin{aligned}\mathrm{Pft}(L) &:= \{(U_i \times D_i, \phi_i, \rho_{\phi_i})\}_{i \in S}\,,\\ L &= \bigcup_{i \in S} L_{\phi_i}.\end{aligned}$$

Then,

Proposition 2.2. *The set of locally constant functions*

$$\left\{\ c_{ij} = \phi_j \circ \rho_{\phi_j}^{-1} - \phi_i \circ \rho_{\phi_i}^{-1} : L_{\phi_j} \cap L_{\phi_i} \to \mathbb{R}\ \right\}_{i\,j \in S}$$

defines an 1-Čech cocycle with the values in $\mathbb{R}$, that is its cohomology class corresponds to the de Rham cohomology class of ${\theta^X}_{|L}$ according to the fine resolution of $\mathbb{R}$-constant sheaf $\mathbb{R}_L$ on L by the sheaves of differential forms on L.

Remark 2.3. *From the diagram* (2.4) *we can see the basic relation* (cf. [13])*:*

$$k - rank\left(\frac{\partial^2 \phi_i}{\partial \eta_j \eta_\ell}\right) = n - rank\,(d{\pi^X}_{|L_\phi}),$$

since the tangent vectors $\sum_{\ell=1}^n a_\ell \frac{\partial}{\partial x_\ell} + \sum_{j=1}^k b_j \frac{\partial}{\partial \eta_j}$ of C_{ϕ_i} mapped to zero by the map ${\pi^X}_{|L_{\phi_i}} \circ \rho_{\phi_i} = \pi_{U|C_{\phi_i}}$ is of the form $\sum_{j=1}^k b_j \frac{\partial}{\partial \eta_j}$ and satisfies $\sum_{j=1}^k b_j \frac{\partial^2 \phi_i(x,\eta)}{\partial \eta_j \partial \eta_\ell} = 0$ for $1 \leq \ell \leq k$.

The positive real numbers $\lambda \in \mathbb{R}_+$ act on $T^*(X)\backslash\{0\}$,

$$(x;\xi) \longleftrightarrow \sum \xi_i dx_i \longmapsto \lambda \sum \xi_i dx_i = \sum \lambda \xi_i dx_i \longleftrightarrow (x; \lambda\xi),$$

where $T^*(X)\backslash\{0\}$ means the zero section removed cotangent space and we call it the punctured cotangent bundle and sometimes we denote it by $T_0^*(X)$. By this action we call the space $T_0^*(X)$ a cone bundle over the quotient space $T_0^*(X)/\mathbb{R}_+$ which becomes naturally a contact manifold and its contact form is defined in the natural way from the canonical one-form θ^X. We call it the cotangent sphere bundle denoting by $S^*(X)$.

If a Lagrangian submanifold $L \subset T_0^*(X)$ (closed in $T_0^*(X)$) is invariant under the action of $\mathbb{R}_+$, we call it a *conic Lagrangian submanifold*. On such a Lagrangian submanifold the canonical one-form θ^X vanishes. Conversely, if the canonical one-form vanishes on a closed n-dimensional submanifold in $T_0^*(X)$, then it is a conic Lagrangian submanifold and in this case the phase functions ϕ can be taken as defined on an open cone, that is, the phase function $\phi \in C^\infty(U \times D)$, where D is an open set in $\mathbb{R}^k \backslash \{0\}$ and is invariant under the scalar action of $\mathbb{R}_+$ on $\mathbb{R}^k \backslash \{0\}$. Moreover the phase function $\phi = \phi(x, \eta) \in C^\infty(U \times D)$ is homogeneous of degree 1 with respect to the variable $\eta \in D \subset \mathbb{R}^k \backslash \{0\}$.

We call a Lagrangian submanifold $L \subset T_0^*(X)$ "*quasi-integral*" (or "*integral*)", if there exists a positive constant c_0 (integral in case $c_0 = 1$) such that the de Rham cohomology class $c_0[\theta^X{}_{|L}]$ is in $\check{H}^1(L, \mathbb{Z}_L) \subset \check{H}^1(L, \mathbb{R}_L) \cong H^1_{dR}(L)$, where the inclusion is the induced map from the natural inclusion map $\mathbb{Z}_L \subset \mathbb{R}_L$ of constant sheaves and the natural isomorphism between the de Rham cohomology group and the $\check{C}$ech cohomology group (cf. [1]). This is equivalent to assume that for any smooth closed curve $\{\gamma(t)\}$ in L, the integral

$$c_0 \int_\gamma \theta^X \ \in \ \mathbb{Z}. \tag{2.5}$$

Hence, in this case the Lagrangian submanifold

$$L_0 := c_0 \cdot L = \{(x, c_0 \xi) \mid (x, \xi) \in L\}$$

is integral and the cohomology class $[\theta^X{}_{|L_0}] \in \check{H}^1(L_0, \mathbb{Z}_{L_0})$.

Remark 2.4. *By the induced map $\check{H}^1(c_0 \cdot L, \mathbb{Z}_{c_0 \cdot L}) \to \check{H}^1(L, \mathbb{Z}_L)$ from the diffeomorphism $L \overset{\approx}{\to} c_0 \cdot L$, the class $[\theta^X{}_{|c_0 \cdot L}]$ is mapped to the class $c_0 \cdot [\theta^X{}_{|L}]$.*

By (2.5), if $L \subset T_0^*(X)$ is an integral Lagrangian submanifold then for any positive integer $k \in \mathbb{N}$, $k \cdot L$ is also integral.

Proposition 2.5. *Let L be an integral Lagrangian submanifold in $T_0^*(X)$. Then,*

(1) *There exists a function $\vartheta : L \to U(1)$ (mod $C^\infty(L)$) such that ϑ is mapped to the cohomology class $[\theta^X{}_{|L}]$, that is ϑ expresses the cohomology class $[\theta^X{}_{|L}]$ through the connecting homomorphism $\delta : C^\infty(L, U(1)) \to \check{H}^1(L, \mathbb{Z}_{\mathbb{L}})$ associated with the exact sequence of sheaves on L:*

$$\{0\} \longrightarrow \mathbb{Z}_L \longrightarrow \mathcal{F}(L, \mathbb{R}) \overset{f \mapsto e^{2\pi\sqrt{-1}f}}{\longrightarrow} \mathcal{F}(L, U(1)) \longrightarrow \{0\},$$

where $\mathcal{F}(L, \mathbb{R})$ is the sheaf of germs of real valued smooth functions on L and $\mathcal{F}(L, U(1))$ is a sheaf of germs of smooth functions taking values in $U(1)$.

In fact, once we fix a set of a covering of L by local parametrizations $\{L_{\phi_i}\}$, by the phase-function-triples $\mathrm{Pft}(L) := \{(U_i \times D_i, \phi_i, \rho_{\phi_i})\}$, *then a function ϑ is given by $\vartheta = e^{2\pi\sqrt{-1}\,\phi_i \circ \rho_{\phi_i}^{-1}}$ on L_{ϕ_i}, since*

$$e^{2\pi\sqrt{-1}\left(\phi_j \circ \rho_{\phi_j}^{-1} - \phi_i \circ \rho_{\phi_i}^{-1}\right)} \equiv 1$$

on $L_{\phi_i} \bigcap L_{\phi_j}$.

(2) *Let*

$$\hat{L} = \left\{ (x; \tau \cdot \xi\,,\, \overline{\vartheta(x;\xi)}\,; \tau) \;\middle|\; (x;\xi) \in L, \tau > 0 \right\}.$$

Then $\hat{L}$ is a conic Lagrangian submanifold in $T_0^*(X) \times T_0^*(U(1))$.

In fact, it is covered by local parametrizations defined by the phase-function-triples

$$\mathrm{Pft}(\hat{L}) := \{ (U_i \times \hat{D}_i, \hat{\phi}_i, \rho_{\hat{\phi}_i}) \},$$

where we define a conic open subset $\hat{D}_i \subset \mathbb{R}^{k_i+1} \backslash \{0\}$ by

$$\hat{D}_i = \Big\{ (v, \tau) \in \mathbb{R}^{k_i} \times \mathbb{R}_+ \ \Big| \ 1/\tau \cdot v \in D_i \Big\}$$

and a phase function $\hat{\phi}_i$ by

$$\begin{aligned} C^\infty(U_i \times \mathbb{R} \times \hat{D}_i) \ &\ni \ \hat{\phi}_i(x, t, v, \tau) \\ &:= \tau \phi_i(x, 1/\tau \cdot v) + \tau\, t, \ \ (x, 1/\tau \cdot v) \in U_i \times D_i, \ \ \tau > 0, \ \ t \in \mathbb{R}. \end{aligned}$$

Proof. The assertion with respect to the (local) parametrization is essentially proved in the papers [13] and also [23].

Let's consider the equations:

$$\frac{\partial \hat{\phi}_i(x, t, v, \tau)}{\partial v_j} = \frac{\partial \phi_i}{\partial \eta_j}(x, 1/\tau \cdot v) = 0$$

and

$$\frac{\partial \hat{\phi}_i(x, t, v, \tau)}{\partial \tau} = \phi_i(x, 1/\tau \cdot v) - \sum_{j=1}^{k} \frac{v_j}{\tau} \frac{\partial \phi_i}{\partial \eta_j} + t \ = \ \phi_i(x, 1/\tau \cdot v) \ + \ t = 0.$$

Then we can characterize the set

$$C_{\hat{\phi}_i} = \{ \ (x, t, v, \tau) \mid (x, 1/\tau \cdot v) \in C_{\phi_i}, t + \phi_i(x, 1/\tau \cdot v) = 0, \ \tau \in \mathbb{R}_+ \ \}.$$

Note that we may assume that the range of the phase functions ϕ_i on $U_i \times D_i$ are included in a sufficiently small interval so that the values $e^{-2\pi\sqrt{-1}\phi_i}$ are included in a small arc and the maps $\rho_{\hat{\phi}_i}$ are given as

$$\begin{aligned} \rho_{\hat{\phi}_i} : C_{\hat{\phi}_i} \ni (x, t, v, \tau) \mapsto &\Big(\tau \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j}(x, 1/\tau \cdot v) dx_j \ , \ \tau dt \Big) \\ &\qquad \in \hat{L}_{\hat{\phi}_i} \subset T_0^*(U_i) \times T_0^*(U(1)), \end{aligned}$$

where $\tau dt \in T^*_{e^{-2\pi\sqrt{-1}\phi_i(x, 1/\tau \cdot v)}}(U(1))$.

$\square$

Corollary 2.6. *Let $L \subset T_0^*(X)$ be an integral Lagrangian submanifold, then the corresponding conic Lagrangian submanifold $\widehat{k \cdot L}$ is given as*

$$k \cdot L = \Big\{ (x\,; \tau k \xi \,, \overline{\vartheta^k(x\,; \xi)}, \tau) \ \Big| \ (x; \xi) \in L, \vartheta : L \to U(1), \ \tau > 0 \ \Big\},$$

where $\vartheta : L \to U(1)$ is the map constructed in (1) *in the preceding Proposition* (2.5).

Let $L \subset T_0^*(X)$ be a Lagrangian submanifold and we assume that $k \cdot L$ is integral with a positive integer k. Let $\vartheta : k \cdot L \to U(1)$ be the map constructed in the above Proposition 2.5 and consider a manifold

$$\overline{L} = \left\{ (x; \xi \,,\, e^{2\pi\sqrt{-1}\, s}) \in L \times U(1) \mid \vartheta(x\,; k \cdot \xi) = e^{2\pi\sqrt{-1}\, k\, s} \right\}. \tag{2.6}$$

Since the map $\vartheta(x\,; k \cdot \xi)$ is given locally by $\vartheta(x\,; k \cdot \xi) = e^{2\pi\sqrt{-1}\, k\ \phi \circ \rho_\phi{}^{-1}(x,\xi)}$ with a non-degenerate phase function ϕ of L, $d\phi \neq 0$ (because $L \subset T_0^*(X)$), the subset $\overline{L}$ is a smooth submanifold in $L \times U(1)$ and is a k-hold covering of L.

Now consider the map

$$\begin{aligned} \tilde{\rho} : \overline{L} \times \mathbb{R}_+ &\longrightarrow T_0^*(X) \times T_0^*(U(1)), \\ \left(x; \xi \,,\, e^{2\pi\sqrt{-1}\, s}, \tau \right) &\longmapsto \left(x;\, \tau\xi \,,\, e^{-2\pi\sqrt{-1}\, s}; \tau \right) \\ &= \left(\tau \sum_{i=1}^{n} \xi_i dx_i \,,\, \tau dt \right) \in T_0^*(X) \times T_0^*(U(1)), \end{aligned} \tag{2.7}$$

where $dt \in T^*_{e^{-2\pi\sqrt{-1}\, s}}(U(1))$. Then

Proposition 2.7. *The map $\tilde{\rho}$ is an embedding and the image is a closed conic Lagrangian submanifold in $T_0^*(X) \times T_0^*(U(1))$.*

If $k = 1$, that is if L is integral, the submanifold $\overline{L}$ is the graph of the map ϑ constructed in Proposition 2.5 and the image by this map coincides with the conic Lagrangian submanifold $\hat{L}$, so we also denote $\tilde{\rho}(\overline{L} \times \mathbb{R}_+)$ by $\hat{L}$.

3. Submersion and Lagrangian submanifold

Let $\varphi : M \to N$ be a surjective submersion ($\dim M = m \geq \dim N = n$, $d := m - n$). Then we have a commutative diagram:

$$\begin{tikzcd} T_0^*(M) \arrow[drr, "\pi^M"'] & & \varphi^*(T_0^*(N)) \arrow[ll, hook', "\chi_\varphi"'] \arrow[r, "\mathfrak{p}_\varphi"] \arrow[d, "\pi^M \circ \chi_\varphi"] & T_0^*(N) \arrow[d, "\pi^N"] \\ & & M \arrow[r, "\varphi"'] & N, \end{tikzcd}$$

where the map $\mathfrak{p}_\varphi$ is the natural projection from the induced bundle to the original (cotangent)bundle. It is also a submersion. Since the map $d\varphi$ is surjective by the assumption, the space $\varphi^*(T_0^*(N))$ is regarded as a submanifold in $T_0^*(M)$ through the dual map $\chi_\varphi = (d\varphi)^*$ of the differential $d\varphi : T(M) \to \varphi^*(T(N))$ (see (2.4)).

In the followings we assume the manifolds are compact and orientable.

We can find a local coordinates system $(x, y) = (x_1, \ldots, x_n; y_1, \ldots, y_{m-n})) \in U \times V \cong W \subset M$, $U \times V \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$, around a point $p \in W \subset M$ and $\varphi(p) \in \overline{U} \cong U$, $\overline{U} \subset N$, such that the map φ is realized as the projection $(x, y) \overset{\pi_U}{\longmapsto} x$.

Then the canonical two-forms ω^M and ω^N are expressed locally as

$$\begin{aligned} \omega^M &= \sum d\xi_i \wedge dx_i + \sum d\beta_j \wedge dy_j, \text{ and} \\ \omega^N &= \sum d\xi_i \wedge dx_i \end{aligned}$$

with respect to the induced coordinates on $(\pi^M)^{-1}(W)$ and $(\pi^N)^{-1}(\overline{U})$, respectively. By these expressions it is apparent that

$$\mathfrak{p}_\varphi{}^*(\omega^N) = \chi_\varphi{}^*(\omega^M). \tag{3.1}$$

Let L be a Lagrangian submanifold in $T_0^*(N)$. Since the map $\mathfrak{p}_\varphi$ is a submersion, $\mathfrak{p}_\varphi{}^{-1}(L)$ is a submanifold and $\dim \mathfrak{p}_\varphi{}^{-1}(L) = \dim N + (\text{fiber dim of } \varphi) = \dim M$. More generally,

Proposition 3.1. *If L is an isotropic submanifold (that is, it is closed and the canonical one-form vanishes on it), then $\mathfrak{p}_\varphi{}^{-1}(L)$ is also isotropic.*

Also if L is conic, then $\mathfrak{p}_\varphi{}^{-1}(L)$ is conic, and if L is compact, then $\mathfrak{p}_\varphi{}^{-1}(L)$ is also compact, since we assumed M is compact.

On the other hand, let $\tilde{L}$ be a Lagrangian submanifold included in

$$\tilde{L} \subset \varphi^*(T_0^*(N)) \overset{\chi_\varphi}{\hookrightarrow} T_0^*(M).$$

Then,

Proposition 3.2. *We assume $\mathfrak{p}_\varphi{}^{-1}(\mathfrak{p}_\varphi(\tilde{L})) = \tilde{L}$ (as a set), then $\mathfrak{p}_\varphi(\tilde{L})$ is a Lagrangian submanifold in $T_0^*(N)$.*

In particular, if the fibers of the submersion φ are connected, then the condition $\mathfrak{p}_\varphi{}^{-1}(\mathfrak{p}_\varphi(\tilde{L})) = \tilde{L}$ is automatically satisfied.

Proof. Let $\pi_U : U \times V \ni (x, y) \longmapsto x \in U$ be local coordinates as before realizing the map φ, where $U \times V \cong W \subset M$. Then points within this coordinates are identified with the points $(x, y\,; \xi, \beta) \longleftrightarrow \sum \xi_i dx_i + \sum \beta_j dy_j \in T^*_{(x,y)}(W)$ and mapped as

$$\mathfrak{p}_\varphi : \varphi^*(T_0^*(N))_{(x,y)} \ni (x, y; \xi, 0) \longmapsto (x; \xi) \longleftrightarrow \sum \xi_i dx_i \in T_x^*(N).$$

By the assumption, we have $\tilde{L} \bigcap W = \mathfrak{p}_\varphi{}^{-1}\big(\mathfrak{p}_\varphi(\tilde{L})\big) \bigcap W \cong \mathfrak{p}_\varphi(\tilde{L}) \times V$. Hence $\mathfrak{p}_\varphi(\tilde{L})$ must be a submanifold in N and so there are functions $\{f_i(x)\}_{i=1}^n$ depending on the variables x such that these are the local defining functions of $\mathfrak{p}_\varphi(\tilde{L})$. Now the Lagrangian property of $\mathfrak{p}_\varphi(L)$ follows as before by making use of (3.1). $\square$

So in the both directions, a submanifold being Lagrangian is rather a mild condition in relation to submersions.

Let $\varphi : M \to N$ be a submersion as before (both are compact and orientable).

Let L be an integral Lagrangian submanifold in $T_0^*(N)$ and we take a small coordinates neighborhood W of an arbitrary point $p \in \mathfrak{p}_\varphi{}^{-1}(L)$, $M \supset W \cong U \times V \subset \mathbb{R}^n \times \mathbb{R}^{m-n}$ such that the projection map $\pi_U : U \times V \to U \cong \overline{U} \subset N$ realizes the submersion φ and also there is an open subset D in $\mathbb{R}^k$ and a phase function $\phi \in C^\infty(U \times D)$ parametrizing the Lagrangian submanifold L around a point $\mathfrak{p}_\varphi(p)$.

Then the pull-back of ϕ to $U \times V \times D$ is a phase function of $\mathfrak{p}_\varphi{}^{-1}(L)$ parametrizing a neighborhood around the point p, which we denote by $\tilde{\phi} := \mathfrak{p}_\varphi{}^*(\phi)$, that is $\tilde{\phi}(x, y, \eta) = \phi(x, \eta)$. Then the map $\rho_{\tilde{\phi}}$ and $C_{\tilde{\phi}} \cong C_\phi \times D$ is

$$\rho_{\tilde{\phi}} : C_{\tilde{\phi}} = \left\{ (x, y, \eta, 0) \;\Big|\; \frac{\partial \phi}{\partial \eta_j}(x, \eta) = 0, 1 \le j \le k = \dim D \right\} \ni (x, y, \eta, 0)$$

$$\longmapsto \left(x, y\,; \frac{\partial \phi(x,\eta)}{\partial x_1}, \dots, \frac{\partial \phi(x,\eta)}{\partial x_n}, 0 \right) = (x, y\,; {\phi'}_x, 0) \in {\mathfrak{p}_\varphi}^{-1}(L)_{\tilde{\phi}}.$$

By the form of the phase functions for ${\mathfrak{p}_\varphi}^{-1}(L)$,

Proposition 3.3. *If $L \subset T_0^*(N)$ is integral, then the Lagrangian submanifold ${\mathfrak{p}_\varphi}^{-1}(L)$ is also integral. If $\tilde{L}$ is an integral Lagrangian submanifold in $\varphi^*(T_0^*(N))$ satisfying the condition ${\mathfrak{p}_\varphi}^{-1}(\mathfrak{p}_\varphi(\tilde{L})) = \tilde{L}$, then $\mathfrak{p}_\varphi(\tilde{L})$ is also an integral Lagrangian submanifold.*

Let $\varphi : M \to N$ be a submersion as above. Then the map $\varphi \times Id : M \times U(1) \to N \times U(1)$ is also a submersion. We denote $\varphi \times Id$ by $\hat{\varphi}$. Then from all the arguments above we have

Proposition 3.4. *Although there is no uniqueness of the construction of the conic Lagrangian submanifold $\hat{L}$ in $T_0^*(N) \times T_0^*(U(1))$ from a given integral Lagrangian submanifold $L \subset T_0^*(N)$, if once we fix a covering $\{L_{\phi_i}\}$ of L by local parametrizations defined by phase-function-triples $\mathrm{Pft}(L) = \{(U_i \times D_i, \phi_i, \rho_\phi)\}$, then the resulting conic Lagrangian submanifolds $\widehat{{\mathfrak{p}_\varphi}^{-1}(L)}$ and ${\mathfrak{p}_{\hat\varphi}}^{-1}(\hat{L})$ in $T_0^*(M) \times T_0^*(U(1))$ coincide, where the map $\mathfrak{p}_{\hat\varphi}$ is the natural projection map from the induced bundle:*

$$\begin{tikzcd}
T_0^*(M \times U(1)) \arrow[dr, "\pi^{M\times U(1)}"'] & \hat{\varphi}^*(T_0^*(N \times U(1))) \arrow[l, hook', "\chi_{\hat\varphi} = (d\hat\varphi)^*"'] \arrow[r, "\mathfrak{p}_{\hat\varphi}"] \arrow[d, "\pi^{M\times U(1)} \circ \chi_{\hat\varphi}"] & T_0^*(N \times U(1)) \arrow[d, "\pi^{N\times U(1)}"] \\
 & M \times U(1) \arrow[r, "\hat\varphi = \varphi \times Id"'] & N \times U(1).
\end{tikzcd}$$

4. Maslov quantization condition

Let X be a compact oriented Riemannian manifold with a Riemannian metric g^X. We denote its dual inner product on the cotangent bundle $T^*(X)$ by $Q_x^X(\xi,\, \eta)$ $\big(\, \xi := \sum \xi_i dx_i,\ \eta := \sum \eta_i dx_i \,\in\, T_x^*(X) \,\big)$.

In this section we recall the Maslov quantization condition. This has a long history and there are many articles. Among them here we only cite [5, 10, 22].

The function $T^*(X) \ni \sum \xi_i\, dx_i \mapsto Q_x^X(\xi, \xi)$ coincides with the principal symbol of the Laplacian Δ^X, $\sigma(\Delta^X)(x; \xi) = Q_x^X(\xi, \xi)$.

Let L be a compact Lagrangian submanifold in $T_0^*(X)$, then the Maslov quantization condition for L says that

$$(4.1)\qquad \begin{cases} \text{Mas[1]: } \sigma_{\Delta^X}\big|_L \equiv E_L = \text{constant } (>0) \text{ on } L, \\ \text{Mas[2]: for any (smooth) closed curve } \{\gamma\} \text{ in } L, \\ \qquad \dfrac{1}{2\pi} \displaystyle\int_\gamma \theta^X \;-\; \dfrac{1}{4} < \mathfrak{m}_L, \gamma > \;\in\; \mathbb{Z}, \\ \qquad \text{where } \mathfrak{m}_L \text{ is a cohomology class } \in H^1(L, \mathbb{Z}) \text{, called} \\ \qquad \text{Maslov class of } L \text{, which is explained in the Appendix,} \\ \text{Mas[3]: there exists a positive "\textit{invariant}" measure } d\mu_L \text{ on } L, \\ \qquad \text{that is the measure } d\mu_L \text{ is a nowhere vanishing highest} \\ \qquad \text{degree differential form which is "\textit{invariant}" under the} \\ \qquad \text{geodesic flow action.} \end{cases}$$

Note that by the condition Mas[1], L itself is invariant under the geodesic flow action.

The condition Mas[2] is usually called the Maslov quantization condition (also it is sometimes called corrected Bohr-Sommerfeld quantization condition, see [10]).

Proposition 4.1. *Let* $\mathrm{Pft}(L) := \{ (U_i \times D_i, \phi_i, \rho_{\phi_i}) \}$ *be a set of phase-function-triples defining a covering of the Lagrangian submanifold* L. *Then by the condition* Mas[2] *the functions* $\{e^{\sqrt{-1}\rho_{\phi_i}^{-1}\circ\phi_i}\}$, *each of which is defined on* $L_{\phi_i} = \rho_{\phi_i}(C_{\phi_i})$, *define a global section of the Maslov line bundle* (*see A.3*), *which we denote by* $\mathbf{s}_{\mathrm{Pft}(L)}$, *although it depends on the chosen covering of* L *by phase-function-triples* $\mathrm{Pft}(L) = \{ (U_i \times D_i, \phi_i, \rho_{\phi_i}) \}$.

Proposition 4.2. *Let* $L \subset T_0^*(X)$ *be a compact integral Lagrangian submanifold. We denote the composition of the map* $\tilde{\rho}^{-1} : \hat{L} \to \overline{L} \times \mathbb{R}_+ = L \times \mathbb{R}_+$ *and the projection* $L \times \mathbb{R}_+ \to L$ *by* $p_L : \hat{L} \to L$. *Then*

$$p_L^*(\mathfrak{m}_L) = \mathfrak{m}_{\hat{L}}, \tag{4.2}$$

$$p_L^*(\mathbf{s}_{\mathrm{Pft}(L)}) = \mathbf{s}_{\mathrm{Pft}(\hat{L})}, \tag{4.3}$$

when we fix a covering of L *by phase-function-triples* $\mathrm{Pft}(L) = \{(U_i \times D_i, \phi_i, \rho_{\phi_i})\}$ *and the covering of* $\hat{L}$ *by associated phase-function-triples,* $\mathrm{Pft}(\hat{L}) = \{(U_i \times \hat{D}_i, \hat{\phi}_i, \tilde{\rho}_{\hat{\phi}_i})\}$. *The map* p_L^* *on the space of sections is defined in the obvious way.*

The proof of the above Proposition 4.2 is given based on the following data in Proposition 4.3 and Proposition 4.4 and we explain it in the Appendix.

We denote the vertical subbundle of the projection map $\pi^X : T^*(X) \to X$ by $\mathcal{V}^X := \mathrm{Ker}\ d\pi^X$. Let $L \subset T_0^*(X)$ be a compact integral Lagrangian submanifold. we express bases of the tangent spaces of $T(L)$ and $T(\hat{L})$ and intersections with each of vertical subbundles $\mathcal{V}^X$ and $\mathcal{V}^{X\times U(1)}(:= \mathrm{Ker}\ d\pi^{X\times U(1)})$ in terms of local coordinates and a phase function. So let $(U \times D, \phi, \rho_\phi)$ be a phase-function-triple and consider the map

$$\begin{aligned} &W_\phi : U' \times D'' \to U \times D \overset{\rho_\phi}{\to} U \times \mathbb{R}^n \cong T^*(U), \\ &W_\phi : U' \times D'' \ni (x', \eta'') \mapsto (x', x''(x', \eta''), \eta'(x', \eta''), \eta'') \longmapsto \\ &\big(x', x''(x', \eta''); \phi_{x_1}(x', x'', \eta', \eta''), \dots, \phi_{x_n}(x', x'', \eta', \eta'')\big) \in U \times \mathbb{R}^n, \end{aligned} \tag{4.4}$$

where we may assume that variables $x'' = (x_{p+1}, \dots, x_n)$ and $\eta' = (\eta_1, \dots, \eta_r)$ are solved by the variables $x' = (x_1, \dots, x_p)$ and $\eta'' = (\eta_{r+1}, \dots, \eta_N)$ in the equations $\frac{\partial\phi}{\partial\eta_\ell} = 0,\ \ell = 1, \dots, N$. Then we have

$$\frac{\partial}{\partial x_i} \longmapsto \frac{\partial}{\partial x_i} + \sum_{j=p+1}^{n} \frac{\partial x_j}{\partial x_i}\frac{\partial}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial \phi_{x_k}}{\partial x_i}\frac{\partial}{\partial \xi_k},\ 1 \le i \le p, \tag{4.5}$$

$$\frac{\partial}{\partial \eta_\beta} \longmapsto \sum_{j=p+1}^{n} \frac{\partial x_j}{\partial \xi_\beta}\frac{\partial}{\partial x_j} + \sum_{k=1}^{n}\sum_{j=p+1}^{n} \frac{\partial^2\phi}{\partial x_j \partial x_k}\frac{\partial x_j}{\partial \eta_\beta}\frac{\partial}{\partial \xi_k}, \quad \beta = r+1, \dots, N. \tag{4.6}$$

Hence

Proposition 4.3. *The intersection* $T_{(x;\xi)}(L)\cap\mathcal{V}^X_{(x;\xi)}$ *is*

$$T_{(x;\xi)}(L)\cap\mathcal{V}_{(x;\xi)}=\left\{\sum_{k=1}^{n}\left(\sum_{\beta=r+1}^{N}\sum_{j=p+1}^{n}c_\beta\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial x_j}{\partial\eta_\beta}\right)\frac{\partial}{\partial\xi_k}\right\},$$

where the coefficients $\{c_\beta\}_{\beta=r+1}^N$ *must satisfy the condition*

$$\sum_{\beta=r+1}^{N}c_\beta\frac{\partial x_j(x',\eta'')}{\partial\eta_\beta}=0,\ \ j=p+1,\dots,n. \tag{4.7}$$

Since the embedding $L\times\mathbb{R}_+\to\hat{L}\subset T^*(X)\times T^*(U(1))=T^*(X\times U(1))$ is given locally by

$$(x',\eta'',\tau)\mapsto(x;\xi,\tau)\mapsto(x;\,\tau\cdot\xi\,,\,-2\pi\phi(x,\xi);\tau),$$

we have

$$\frac{\partial}{\partial x_i}\longmapsto\frac{\partial}{\partial x_i}+\sum_{j=p+1}^{n}\frac{\partial x_j}{\partial x_i}\frac{\partial}{\partial x_j}+\tau\sum_{k=1}^{n}\frac{\partial\phi_{x_k}}{\partial x_i}\frac{\partial}{\partial\xi_k}-2\pi\frac{\partial\phi}{\partial x_i}\frac{\partial}{\partial t}, \tag{4.8}$$

$$\begin{aligned}\frac{\partial}{\partial\eta_\beta}\longmapsto\sum_{j=p+1}^{n}\frac{\partial x_j}{\partial\xi_\beta}\frac{\partial}{\partial x_j}+\tau\sum_{k=1}^{n}\sum_{j=p+1}^{n}\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial x_j}{\partial\eta_\beta}\frac{\partial}{\partial\xi_k}&\\-2\pi\frac{\partial\phi}{\partial\eta_\beta}\frac{\partial}{\partial t},\ \ \beta=r+1,\dots,N,&\end{aligned} \tag{4.9}$$

$$\frac{\partial}{\partial\tau}\longmapsto\sum_{k=1}^{n}\xi_k\frac{\partial}{\partial\xi_k}+\frac{\partial}{\partial\tau}. \tag{4.10}$$

Hence again we have

Proposition 4.4.

$$T_{(x;\xi,\,t;\tau)}(\hat{L})\cap T_{(x,t;\,\xi,\tau)}(T^*(X\times U(1))) \tag{4.11}$$

$$=\left\{\sum_{k=1}^{n}\left(\sum_{\beta=r+1}^{N}\sum_{j=p+1}^{n}c_\beta\frac{\partial^2\phi}{\partial x_j\partial x_k}\frac{\partial x_j}{\partial\eta_\beta}\right)\frac{\partial}{\partial\xi_k}\right\}, \tag{4.12}$$

where $t=-\phi(x,\eta)$, $(x,\eta)\in C_\phi$ *and the coefficients* $\{c_\beta\}_{\beta=r+1}^N$ *satisfy the same condition as* (4.7).

Proof. It will be enough to prove

$$\sum_{\beta=s+1}^{N}c_\beta\frac{\partial\phi}{\partial\eta_\beta}=0.$$

Then on C_ϕ

$$\sum_{\beta=r+1}^{N}c_\beta\frac{\partial\phi(x',x''(x',\eta''),\eta'(x',\eta''),\eta'')}{\partial\eta_\beta}$$

$$= \sum_{\beta=r+1}^{N} c_\beta \left\{ \sum_{j=p+1}^{n} \frac{\partial\phi}{\partial x_j}\frac{\partial x_j}{\partial\eta_\beta} + \sum_{\alpha=1}^{r} \frac{\partial\phi}{\partial\eta_\alpha}\frac{\partial\eta_\alpha}{\partial\eta_\beta} + \frac{\partial\phi}{\partial\eta_\beta} \right\} = 0,$$

by the condition (4.7). □

In the proof of Theorem 5.1 both of the conditions Mas[1] and Mas[3] are required for the construction of an operator.

Proposition 4.5. *Let L be a compact Lagrangian submanifold in $T_0^*(X)$. Let $\{\gamma(s)\}$ be a closed curve in L, then for any positive real number $A > 0$*

$$< \mathfrak{m}_L,\, \gamma > = < \mathfrak{m}_{A\cdot L},\, A\cdot\gamma >,$$

where $A\cdot\gamma$ is the dilation by the positive constant A.

A proof is given in the Appendix.

Remark 4.6. *Let L be a Lagrangian submanifold in $T_0^*(X)$ satisfying the condition* Mas[2] *above. Then, according to the Maslov class being*

(1) $4\,\mathfrak{m}_L \equiv 0$ $(mod\ 4)$, *or*

(2) $2\,\mathfrak{m}_L \equiv 0$ $(mod\ 4)$, *or*

(3) $\mathfrak{m}_L \equiv 0$ $(mod\ 4)$,

the Lagrangian submanifold $\frac{2}{\pi}\cdot L$ is integral (case (1)), *$\frac{1}{\pi}\cdot L$ is integral* (case (2)), *or $\frac{1}{2\pi}\cdot L$* (case (3)) *is integral.*

Let k be any positive integer. Then, in the case (3) *the Lagrangian submanifolds $k\cdot L$ satisfy the condition* Mas[2]. *In the case* (2), *the Lagrangian submanifolds $(2k+1)\cdot L$ satisfy the condition* Mas[2]. *In the case* (1), *the Lagrangian submanifolds $(4k+1)\cdot L$ satisfy the condition* Mas[2]. *Also if L satisfies conditions* Mas[1] *and* Mas[3], *then $k\cdot L$ always satisfies the conditions* Mas[1] *and* Mas[3].

Let $\varphi : M \to N$ be a Riemannian submersion between compact orientable manifolds with the dual inner products Q^M and Q^N on $T^*(M)$ and $T^*(N)$ respectively.

Let $(x,y) \in U\times V \cong W \subset M$ be a local coordinates neighborhood such that the submersion $\varphi : M \to N$ is expressed as the projection map $\pi_U : (x,y) \mapsto x \in U \cong \overline{U} \subset N$ as in the beginning of §4. Let

$$\begin{pmatrix} g^M(\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}) & g^M(\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_\alpha}) \\ g^M(\frac{\partial}{\partial y_\alpha},\frac{\partial}{\partial x_i}) & g^M(\frac{\partial}{\partial y_\alpha},\frac{\partial}{\partial y_\beta}) \end{pmatrix} =: \begin{pmatrix} g_{ij}(x,y) & h_{i\alpha}(x,y) \\ h_{\alpha i}(x,y) & v_{\alpha\beta}(x,y) \end{pmatrix} =: \begin{pmatrix} G^M & H \\ {}^tH & V \end{pmatrix} \tag{4.13}$$

be the metric tensor of g^M with respect to the coordinates (x,y).

Let

$$H_i = \frac{\partial}{\partial x_i} + \sum b_{i\alpha}(x,y)\frac{\partial}{\partial y_\alpha},\ i = 1,\ldots,n = \dim N$$

be vector fields defined on $U \times V$ and assume that these are orthogonal to the fibers of the map φ. Then the condition that the map φ is a Riemannian submersion is expressed as

$$(1) \quad g^M\left(H_i, \frac{\partial}{\partial y_\beta}\right) = 0, \ i = 1, \dots, n \text{ and } \beta = 1, \dots, d = \dim M - \dim N,$$

and for $1 \le i, j \le n$

$$(2) \quad g^M(H_i, H_j) = g^N\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) =: g_{ij}(x).$$

These are rewritten in terms of component matrices:

$$G^M + B\,{}^t\!H = G^N, \text{ and } H + BV = 0, \tag{4.14}$$

where we put $G^N = G^N(x) = \Big(g_{ij}(x)\Big)$ and $B = B(x,y) = \Big(b_{i\alpha}(x,y)\Big)$. Hence using these relations we have

$$\begin{pmatrix} G^M & H \\ {}^t\!H & V \end{pmatrix}^{-1} = \begin{pmatrix} (G^N)^{-1} & (G^N)^{-1}B \\ {}^t\!B\,(G^N)^{-1} & V^{-1} + {}^t\!B\,(G^N)^{-1}B \end{pmatrix}.$$

Now the dual norm function Q^M on $T^*(M)$ is given by

$$\begin{aligned} E^M := Q^M_{x,y}(\xi, \eta\,; \xi, \eta) = &\sum_{i,j} \left(G^N(x)^{-1}\right)_{i\,j} \xi_i\,\xi_j + 2\sum_{i,\alpha} \left((G^N)^{-1}B\right)_{i\,\alpha} \xi_i\,\eta_\alpha \\ &+ \sum_{\alpha,\beta} \left(V^{-1} + {}^tB(G^N)^{-1}B\right)_{\alpha\,\beta} \eta_\alpha\,\eta_\beta, \end{aligned}$$

where $\left(G^N(x)^{-1}\right)_{i\,j}$ means the $(i\ j)$ component of the matrix $G^N(x)^{-1}$ and so on. So on the subspace $\varphi^*(T^*_0(N))$, the solution curves of the Hamilton vector field with the Hamiltonian E^M satisfy the equations

$$\begin{cases} \dfrac{d}{dt}x_i(t) = \dfrac{\partial E^M}{\partial \xi_i} = \displaystyle\sum_j \,(G^N(x)^{-1})_{i\,j}\xi_j, \\ \dfrac{d}{dt}y_\alpha(t) = \dfrac{\partial E^M}{\partial \eta_\alpha} = 2\displaystyle\sum_i \left(G^N(x)^{-1}B(x,y)\right)_{i\,\alpha}\xi_i, \\ \dfrac{d}{dt}\xi_i(t) = -\dfrac{\partial E^M}{\partial x_i} = -\displaystyle\sum_{i'\,j} \dfrac{\partial (G^N(x)^{-1})_{i'\,j}}{\partial x_i}\xi_{i'}\,\xi_j, \\ \dfrac{d}{dt}\eta_\alpha(t) = -\dfrac{\partial E^M}{\partial y_\alpha} \equiv 0. \end{cases} \tag{4.15}$$

Also the solution curves of the Hamilton vector field with the Hamiltonian $E^N = Q_x^N(\xi,\xi)$ satisfy the equations

$$(4.16)\qquad \begin{cases} \dfrac{d}{dt}x_i(t) = \dfrac{\partial E^N}{\partial \xi_i} = \displaystyle\sum_j (G^N(x)^{-1})_{i\,j}\xi_j, \\[2ex] \dfrac{d}{dt}\xi_i(t) = -\dfrac{\partial E^N}{\partial x_i} = -\displaystyle\sum_{i'\,j} \dfrac{\partial (G^N(x)^{-1})_{i'\,j}}{\partial x_i}\xi_{i'}\,\xi_j. \end{cases}$$

Then, since $E^M = Q^M_{x,y}(\xi,0;\xi,0) = Q^N_x(\xi,\xi) = E^N$ by comparing (4.15) and (4.16) we have the commutative diagram:

$$(4.17)\qquad \begin{array}{ccccc} T_0^*(M) & \xleftarrow{\chi_\varphi} & \varphi^*(T_0^*(N)) & \xrightarrow{\mathfrak{p}_\varphi} & T_0^*(N) \\ \downarrow{\scriptstyle \Phi_t^M} & & \downarrow{\scriptstyle \Phi_t^M} & & {\scriptstyle \Phi_t^N}\downarrow \\ T_0^*(M) & \xleftarrow{\chi_\varphi} & \varphi^*(T_0^*(N)) & \xrightarrow{\mathfrak{p}_\varphi} & T_0^*(N), \end{array}$$

where we denote by $\{\Phi_t^M\}$ and $\{\Phi_t^N\}$ the geodesic flows (= bicharacteristic flows of the Laplacians) on M and N respectively.

Proposition 4.7. *Let $\varphi : M \to N$ be a Riemannian submersion as above and let L be a compact Lagrangian submanifold in $T_0^*(N)$ satisfying the conditions* Mas[1] *and* Mas[2]*. Then the Lagrangian submanifold $\tilde{L} := {\mathfrak{p}_\varphi}^{-1}(L) \subset \varphi^*(T_0^*(N))$ also satisfies the conditions* Mas[1] *and* Mas[2]*.*

Proof. From the above arguments it will be seen that the condition Mas[1] is satisfied by the Lagrangian submanifold ${\mathfrak{p}_\varphi}^{-1}(L)$ with the same value E_L.

Since ${\mathfrak{p}_\varphi}^*(\theta^N) = {\chi_\varphi}^*(\theta^M)$ we have the equality of the integral

$$\int_{\tilde{\gamma}} \theta^M = \int_{\mathfrak{p}_\varphi(\tilde{\gamma})} \theta^N,$$

for arbitrary loops $\tilde{\gamma}$ in ${\mathfrak{p}_\varphi}^{-1}(L)$. This equality together with the fact ${\mathfrak{p}_\varphi}^*(\mathfrak{m}_L) = \mathfrak{m}_{\tilde{L}}$ implies the condition Mas[2] for ${\mathfrak{p}_\varphi}^{-1}(L)$. □

Proposition 4.8. *If a compact Lagrangian submanifold $\tilde{L}$ in $\varphi^*(T_0^*(N))$ satisfying the condition that ${\mathfrak{p}_\varphi}^{-1}(\mathfrak{p}_\varphi(\tilde{L})) = \tilde{L}$ and the conditions* Mas[1] *and* Mas[2]*, then by noting that all the loops in $L = \mathfrak{p}_\varphi(\tilde{L})$ come from loops in $\tilde{L}$ we know that L also satisfies the conditions* Mas[1] *and* Mas[2]*.*

Let $d\mu$ be a smooth measure on a manifold X. We treat it as a "*distribution*"

$$C_0^\infty(X) \ni f \longmapsto \int_X f\,d\mu =: < d\mu,\, f > .$$

Then through a general differentiable map $h : X \to Y$ (with compact fibers or the support $d\mu$ is compact), we can define a *distribution $h_*(d\mu)$ on Y* by the formula

$$< h_*(d\mu),\, g > := \int_X h^*(g) d\mu = < d\mu,\, h^*(g) >,\ g \in C_0^\infty(N),$$

which is called the *push-forward* of the measure $d\mu$. If h is a submersion, then the push-forward $h_*(d\mu)$ is again a smooth measure.

Now let $\varphi : M \to N$ be a Riemannian submersion as before with compact fibers.

Proposition 4.9. *Let $\tilde{L}$ be a Lagrangian submanifold in $\varphi^*(T_0^*(N))$ satisfying the same conditions in Proposition* (4.8) *and we assume there exists a $\{\Phi_t^M\}$ action invariant smooth measure $d\tilde{\mu}_{\tilde{L}}$ on $\tilde{L}$. Then by the commutativity given in the diagram* (4.17) *the push-forwarded smooth measure $(\mathfrak{p}_\varphi)_*(d\tilde{\mu}_{\tilde{L}})$ is a geodesic flow $\{\Phi_t^N\}$-action invariant measure on $\mathfrak{p}_\varphi(\tilde{L})$.*

On the other hand, let $L \subset T_0^*(N)$ be a Lagrangian submanifold carrying a smooth measure $d\mu_L$ which is invariant under the geodesic flow $\{\Phi_t^N\}$ action. In this case it is not apparent of the existence of the geodesic flow $\{\Phi_t^M\}$ action invariant measure on ${\mathfrak{p}_\varphi}^{-1}(L)$. Here we explain a possible candidate for such a measure.

Since the map $\mathfrak{p}_\varphi : {\mathfrak{p}_\varphi}^{-1}(L) = \varphi^*(L) \to L$ is a submersion ($\varphi^*(L)$ denotes the fiber product of M and L by the submersion φ) we have an exact sequence of vector bundles

$$\{0\} \longrightarrow \mathcal{V} \longrightarrow T({\mathfrak{p}_\varphi}^{-1}(L)) \xrightarrow{d\mathfrak{p}_\varphi} {\mathfrak{p}_\varphi}^*(T(L)) \longrightarrow \{0\}$$

on ${\mathfrak{p}_\varphi}^{-1}(L) = \varphi^*(L)$, where $\mathcal{V} = \operatorname{Ker} d\mathfrak{p}_\varphi$ and it can be identified with the induced bundle $(\pi^M)^*(\operatorname{Ker} d\varphi)$ of $\operatorname{Ker} d\varphi$ on M by the restriction of the differential $d\pi^M$ to $\varphi^*(L)$. This can be seen by the following way. Let $\{\tilde{\gamma}\}$ be a smooth curve in $\varphi^*(L)$ mapped to a point by $\mathfrak{p}_\varphi$, then the curve $\{\pi^M(\tilde{\gamma})\}$ is also mapped to a point by the map φ. Hence the differential $d\pi^M$ maps $\mathcal{V}$ to $\operatorname{Ker} d\varphi$. We can see this is injective, since if a curve $\{\tilde{\gamma}\}$ is mapped to a point by the map $\mathfrak{p}_\varphi$ and also mapped to a point by the map π^M, then it is a constant curve in $\varphi^*(L)$ by the definition of the fiber product, which implies that the natural map $d\pi^M{}_{|\varphi^*(L)} : \mathcal{V} \longrightarrow (\pi^M)^*(\operatorname{Ker} d\varphi)$ is injective. By comparing the dimensions of the bundles $\mathcal{V}$ and $(\pi^M)^*\big(\operatorname{Ker} d\varphi\big)$ we know that the two bundles $\mathcal{V}$ and $(\pi^M)^*\big(\operatorname{Ker} d\varphi\big)$ on $\varphi^*(L)$ are isomorphic.

Now by the assumption that φ is a Riemannian submersion, the Riemannian volume forms dv^M on M and dv^N on N have a relation through the isomorphism

$$\bigwedge^{max} T^*(M) \cong \bigwedge^{max} \operatorname{Ker} d\varphi \otimes \bigwedge^{max} (\varphi^*(T^*(N)))$$

that there exists the non-where vanishing unique section $dF : M \to \bigwedge^{max} \operatorname{Ker} d\varphi$ such that $dv^M = dF \otimes \varphi^*(dv^N)$. By the relation (4.14), we have

$$\det \begin{pmatrix} G^M & H \\ {}^t\!H & V \end{pmatrix} = \det V \cdot \det G^N. \tag{4.18}$$

Using this relation, the volume form dv^M is expressed in local coordinates (x, y) as

$$\begin{aligned} dv^M &= \sqrt{\det \begin{pmatrix} G^M & H \\ {}^t\!H & V \end{pmatrix}} dy_1 \wedge \cdots \wedge dy_d \wedge dx_1 \wedge \cdots \wedge dx_n \\ &= \sqrt{\det V} dy_1 \wedge \cdots \wedge dy_d \wedge \sqrt{\det G^N}\, dx_1 \wedge \cdots \wedge dx_n \\ &= \sqrt{\det V}\, dy_1 \wedge \cdots \wedge dy_d \wedge \varphi^*(dv^N). \end{aligned}$$

Hence $dF = \sqrt{\det V} dy_1 \wedge \cdots \wedge dy_d$.

The possible candidate of an invariant measure on $\varphi^*(L)$ under the geodesic flow $\{\Phi_t^M\}$ action is

$$f \cdot (\pi^M)^*(dF) \otimes \mathfrak{p}_\varphi{}^*(d\mu_L), \tag{4.19}$$

with a suitable positive valued function $f \in C^\infty(\mathfrak{p}_\varphi{}^{-1}(L))$.

Example 4.10. One typical example we have such an invariant measure is the case that M is a C_ℓ-manifold, i.e., the geodesic flow is periodic with the common period $\ell > 0$, like the case of $S^{2n+1} \to P\mathbb{C}^n$ and $S^{4n+3} \to P\mathbb{H}^n$ (Hopf fibrations).

Remark 4.11. *Let $\tilde{L}$ be a compact Lagrangian submanifold in $T_0^*(M)$. There will be various cases that two submanifolds $\tilde{L}$ and $\varphi^*(T_0^*(N))$ intersect cleanly and the image $\mathfrak{p}_\varphi(S)$ of the intersection $S = \tilde{L} \cap \varphi^*(T_0^*(N))$ can be a Lagrangian submanifold in $T^*(N)$. The case above is a special case. As one of another special case we may consider the following situation, that is we assume that $\tilde{L}$ and $\varphi^*(T_0^*(N))$ intersect transversally (in this case* $\dim S = \dim N$*), and that the intersection $\tilde{L} \cap \varphi^*(T_0^*(N))$ intersect cleanly with the vertical foliation of the submersion $\mathfrak{p}_\varphi$ with the zero dimensional intersections with each leaf (= fibers of the submersion $\mathfrak{p}_\varphi$), then $\mathfrak{p}_\varphi : S \to T_0^*(N)$ is a Lagrangian immersion. In such a case if $\tilde{L}$ satisfies the condition* Mas[2]*, then $\mathfrak{p}_\varphi(S)$ also satisfies the condition* Mas[2].

Remark 4.12. *Let $g : X \to T^*(X)$ be a closed one-form ($\pi^X \circ g = Id$), then the image $g(X) \subset T^*(X)$ is a Lagrangian submanifold. There are many deep studies on "exact" Lagrangian submanifolds, for example see* [8] *and also* [1] *for basic properties. If a class $\in H^1(X, \mathbb{Z})$ has a representative by a nowhere vanishing closed one-form g, then $g(X) \subset T_0^*(X)$ is an integral Lagrangian submanifold whose Maslov class is zero. Hence $\frac{1}{2\pi} g(X)$ satisfies the condition* Mas[2] *(without correction term $\mathfrak{m}_{g(X)}/4$). Also we can find a Riemanian metric with respect to which the Lagrangian submanifold $g(X)$ satisfies the conditions* Mas[1]*, however it is not clear among such metrics whether we can find a suitable metric satisfying the condition* Mas[3]*. So it is not trivial to find "non-trivial" examples of compact Lagrangian submanifolds satisfying all the conditions* Mas[1]$\sim$Mas[3]*. There are examples of such Lagrangian submanifolds satisfying all the conditions* Mas[1]$\sim$Mas[3] *on the manifolds with the completely integrable geodesic flow, where such a Lagrangian submanifold is given as intersections of particular constant hypersurfaces of linearly independent first integrals. In this case Lagrangian submanifolds are tori always.*

5. Eigenvalue theorem by Weinstein and the main theorems

In this section we recall a theorem called Eigenvalue Theorem (cf. [22]). There are various version of this type existence theorem of eigenvalues of the Laplacian. The theorem here is interesting from the point of the Fourier integral operator theory (cf. [5, 6, 10, 13]). There is a similar such existence theorem which is more intuitive from a point of view of physics (cf. [18], and [19] in relation to the history of such type theorems) (= correspondence of eigenstates and closed geodesics).

We assume that there is a compact Lagrangian submanifold $L \subset T_0^*(X)$ satisfying the conditions Mas[1] $\sim$ Mas[3]. As was noted in Remark 4.6, let d_L be the smallest integer in $\{1, 2, 4\}$ such that $\frac{d_L}{2\pi} \cdot L$ is integral. Then the Eigenvalue Theorem says that

Theorem 5.1. *There exists such a sequence* $\{\lambda_k\}_{k=0}^{\infty}$ *of eigenvalues of the Laplacian* Δ^X *that*

$$\left|\lambda_k - E_L(d_L k + 1)^2\right| \le C \;:\; \text{bounded.}$$

The proof is given by constructing a *Fourier integral operator* $A : L_2(U(1)) \to L_2(X)$.

An outline of the proof is as follows:

Step(1): By the condition Mas[2], the Lagrangian submanifold $\frac{d_L}{2\pi} \cdot L$ is integral, so that we can construct a conic Lagrangian submanifold $\widehat{\frac{d_L}{2\pi} \cdot L}$ in $T_0^*(X) \times T_0^*(U(1))$. This construction was explained in Corollary 2.6 and Proposition 2.7.

Step(2): The operator $A : C^{\infty}(U(1)) \to C^{\infty}(X)$, or rather its kernel distribution K_A we are going to construct, is a Lagrangian distribution with respect to the conic Lagrangian submanifold $\widehat{\frac{d_L}{2\pi} \cdot L}$, which satisfies the condition that the Lagrangian distribution

$$\mathcal{D}(K_A) := \left(\Delta^X \otimes Id + Id \otimes E_L \frac{d^2}{dt^2}\right) K_A$$

is of order 0 *"mod half density factor"*. So we take a Lagrangian distribution whose principal symbol on $\widehat{\frac{d_L}{2\pi} \cdot L}$ is $\sqrt{d\mu_L} \otimes \sqrt{d\tau} \otimes \mathbf{s}_{\mathrm{Pft}(\hat{L})}$, where $d\mu_L$ is the invariant measure assumed in Mas[3] (note that we may regard that $\hat{L}$ is more or less $L \times \mathbb{R}_+$. Also see Propositions 4.1 and 4.2 and the proof of Proposition 4.2 explained in the Appendix for the relation of global sections $\mathbf{s}_{\mathrm{Pft}(L)}$ and $\mathbf{s}_{\mathrm{Pft}(\hat{L})}$.

By applying the operator $\mathcal{D} = \Delta^X \otimes Id + Id \otimes E_L \frac{d^2}{dt^2}$ to such a distribution K_A, we know that the principal symbol of the distribution $\mathcal{D}(K_A)$ (as a Lagrangian distribution) vanishes, since on the conic Lagrangian submanifold $\widehat{\frac{d_L}{2\pi} \cdot L}$ the principal symbol of the operator $\mathcal{D}$ vanishes by the condition Mas[1]. Then as the $1st$ order Lagrangian distribution, the principal symbol of the distribution $\mathcal{D}(K_A)$ again vanishes, because the subprincipal symbol of the operator $\mathcal{D}$ vanishes and the principal symbol of our distribution K_A is invariant under the geodesic flow action of the Laplacian by Mas[1] (precisely to say, its lift to $T_0^*(X) \times T_0^*(U(1))$, cf. [10]). So the operator $\mathcal{D} = \Delta^X \circ A + E_L A \circ \frac{d^2}{dt^2} : C^{\infty}(U(1)) \to C^{\infty}(X)$ is bounded as an operator : $L_2(U(1)) \to L_2(X)$. Since the action of $\mathbb{Z}_{d_L}$ on $\widehat{\frac{d_L}{2\pi} \cdot L}$ comes from the natural action on the base space $U(1)$ and by the assumption Mas[1] and Mas[3] we may find a candidate of such an operator A always $\mathbb{Z}_{d_L}$ action *equivariant*, so that the operator A is descended to an operator acting on the space of sections $\Gamma\big(\mathcal{E} \otimes |\bigwedge|^{1/2}\big(U(1)/\mathbb{Z}_{d_L}\big)\big)$ of the line bundle $\mathcal{E} \otimes |\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})$ on $U(1)/\mathbb{Z}_{d_L}$, where $\mathcal{E}$ is the line bundle on $U(1)/\mathbb{Z}_{d_L}$ associated to the principal bundle $U(1) \to U(1)/\mathbb{Z}_{d_L}$ via the natural representation of the structure group $\mathbb{Z}_{d_L}$ ($\cong \{1\}$ or $\{\pm 1\}$ or $\{\pm 1, e^{\pm \frac{\pi}{2}\sqrt{-1}}\}$) to $U(1)$ and $|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})$ is the half density line bundle of $U(1)/\mathbb{Z}_{d_L}$ (this is the reason why the eigenvalues of the form $(d_L \cdot k + 1)^2$ are appearing in this space as eigenvalues of the operator $-\frac{d^2}{dt^2}$, see Proposition A.6).

Step(3) By some non-trivial modification (based on clean intersection theorem) of A in the lower order terms, we can make the operator A maps isometrically the subspace consisting of eigenspaces of $-\frac{d^2}{dt^2}$ with the eigenvalues $(d_Lk+1)^2$ ($k=0,1,\ldots$) in $L_2(U(1))$ to $L_2(X)$ with the bounded commutator. Note that we can define an inner product on the space $\Gamma\big(\mathcal{E}\otimes|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_l})\big)$ in an intrinsic way. We also note that if we consider the operator A as $\Gamma\big(\mathcal{E}\otimes|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})\big)\to\Gamma\big(|\bigwedge|^{1/2}(X)\big)$, the operator

$$ {}^{t}\!A\circ A:\Gamma\big(\mathcal{E}\otimes|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})\big)\to\Gamma\big(\mathcal{E}\otimes|\bigwedge|^{1/2}(U(1)/\mathbb{Z}_{d_L})\big) $$

is a pseudo-differential operator of order zero whose principal symbol can be identified in a natural way with the Heaviside function on each fiber of $T_0^*(U(1))\cong U(1)\times\mathbb{R}\backslash\{0\}$.

Then by using the spectral decomposition of the Laplacian we can prove the existence of eigenvalues of the Laplacian Δ^X described in the statement in the Theorem.

Now let $\varphi:M\to N$ be a Riemannian submersion and we assume the existence of a compact Lagrangian submanifold in $T_0^*(N)$ satisfying the conditions Mas[1] $\sim$ Mas[3]. Then as was discussed in §3 the Lagrangian submanifold $\mathfrak{p}_\varphi{}^{-1}(L)=\varphi^*(L)$ also satisfies the conditions Mas[1] and Mas[2]. Then we assume that a measure (4.19) is invariant under the geodesic flow $\{\Phi_t^M\}$ action, or the same thing that a section $f\cdot dF$ is invariant under the action of $\{\Phi_t^M\}$, then associated to the conic Lagrangian submanifold $\widehat{\mathfrak{p}_\varphi{}^{-1}(L)}=\mathfrak{p}_{\hat\varphi}{}^{-1}(\hat L)$ there is a Fourier integral operator $\hat A$ which satisfies the properties by the Eigenvalue Theorem which is quasi-commuting with the Laplacian on M and $-E_L\frac{d^2}{dt^2}$. Hence as a corollary of the Eigenvalue Theorem

Theorem 5.2. *There is a sequence $\{\lambda_k\}$ of eigenvalues of the Laplacian on M such that $|\lambda_k-E_L(d_Lk+1)^2|$ is bounded.*

The Fourier integral operator $A:C^\infty(U(1))\to C^\infty(N)$ which is isometric and quasi-commuting with the Laplacian on N whose existence is guarantied by the assumptions Mas[1] $\sim$ Mas[3] (the main part of the Eigenvalue Theorem [22]) and the operator $\hat A$ will not be quasi-commuting through the map φ^* (that is $\hat A\circ\varphi^*-\varphi^*\circ A$ is not bounded), since the principal symbol of $\hat A$ included an ambiguity from the pull-back of the principal symbol of A by the map $\mathfrak{p}_{\hat\varphi}$. Off course, if $\Delta^M\circ\varphi^*=\varphi^*\circ\Delta^N$, then to know the existence of particular series of eigenvalues of the Laplacian Δ^M from that of Δ^N, we need not depend on the operator $\hat A$.

Next we assume there exists a compact Lagrangian submanifold $\tilde L$ in $\varphi^*(T_0^*(N))$ satisfying the conditions Mas[1] $\sim$ Mas[3] with an invariant measure $d\tilde\mu_{\tilde L}$ of Mas[3] and additionally we assume that $\mathfrak{p}_\varphi{}^{-1}(\mathfrak{p}_\varphi(\tilde L))=\tilde L$. Then in this case the Lagrangian submanifold $\mathfrak{p}_\varphi(\tilde L)$ in $T_0^*(N)$ satisfies the conditions Mas[1] $\sim$ Mas[3] with the invariant measure $(\mathfrak{p}_\varphi)_*(d\tilde\mu_{\tilde L})$ ((4.9)). Hence we can construct a Fourier integral operator $A:C^\infty(U(1))\to C^\infty(N)$ quasi commuting with the Laplacian on N and by applying the Eigenvalue Theorem we have

Theorem 5.3. *There exists a sequence* $\{\lambda_k\}_{k=0}^{\infty}$ *of eigenvalues of the Laplacian on* N *such that*

$$|\lambda_k - E_L(d_L k+1)^2| \text{ is bounded.}$$

Here the Fourier integral operators $A : C^\infty(U(1)) \to C^\infty(N)$ and $\hat{A} : C^\infty(U(1)) \to C^\infty(M)$, those existence are guaranteed by the assumptions Mas[1] $\sim$ Mas[3], again need not be quasi-commuting through the map φ^*, since the pull-back of the measure $(\mathfrak{p}_\varphi)_*(d\tilde{\mu}_{\tilde{L}})$ may not be possible to make it to coincide with the original measure $d\tilde{\mu}_{\tilde{L}}$.

6. Eigenvalue theorem for a sub-Laplacian

In this last section, we consider the most simple case of a sub-Riemannian structure and the possible Eigenvalue Theorem for its intrinsic sub-Laplacian.

Let X be a compact orientable manifold with a non-trivial subbundle $\mathcal{H}$ in the tangent bundle $T(X)$, which is equipped with an inner product $<\cdot, \cdot>^{\mathcal{H}}$.

We assume

$$\Gamma(\mathcal{H}) + [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] = \Gamma(T(X)), \tag{6.1}$$

that is, all the vector fields $\in \Gamma(T(X))$ *mod* $\Gamma(\mathcal{H})$ are generated from the vector fields taking values in $\mathcal{H}$ by the bracket operation. This property is said to be bracket generating of 2-step and is equivalent to the property that

$$\left\{\begin{array}{l}\text{the induced bundle map } \mathfrak{B} : \mathcal{H}\otimes\mathcal{H} \longrightarrow T(X)/\mathcal{H} \text{ from the} \\ \text{bracket operation} \\ \Gamma(\mathcal{H})\times\Gamma(\mathcal{H}) \ni (Z, Z') \longmapsto [Z, Z'] \in \Gamma(T(X)) \\ \text{is surjective at any point in } X.\end{array}\right. \tag{6.2}$$

Then we can install an inner product on the quotient bundle $T(X)/\mathcal{H}$ by assuming that it is isometric with the orthogonal complement of the kernel of the map $\mathfrak{B}$,

$$(\operatorname{Ker}\mathfrak{B})^\perp \overset{\text{isometric}}{\cong} T(X)/\mathcal{H}.$$

Hence by the natural isomorphism

$$\bigwedge^{\dim\mathcal{H}} \mathcal{H} \otimes \bigwedge^{\dim(T(X)/\mathcal{H})} (T(H)/\mathcal{H}) \cong \bigwedge^{\dim X} T(X)$$

and the isomorphism of their duals

$$\bigwedge^{\dim\mathcal{H}} \mathcal{H}^* \otimes \bigwedge^{\dim\big(T(X)/\mathcal{H}\big)} \big(T(H)/\mathcal{H}\big)^* \cong \bigwedge^{\dim X} T^*(X)$$

we can define a measure, called Poppe's measure (cf. [14], [2]) which we denote by $d\mathcal{P}$ and can be locally expressed as

$$d\mathcal{P} = \alpha_1\wedge\cdots\wedge\alpha_{\dim\mathcal{H}} \otimes \beta_1\wedge\cdots\wedge\beta_{\dim X-\dim\mathcal{H}},$$

where $\{\alpha_i\}$ and $\{\beta_j\}$ are local orthonormal frames of the bundles $\mathcal{H}^*$ and $\big(T(X)/\mathcal{H}\big)^*$ with respect to their dual inner products.

Then we can define the divergence of the vector field $Z \in \Gamma(T(X))$ with respect to the measure $d\mathcal{P}$ by

$$div_{d\mathcal{P}}(Z)\cdot d\mathcal{P} := \mathcal{L}_Z\,(d\mathcal{P}) = (d\circ i_Z + i_Z\circ d)(d\mathcal{P}) = d(i_Z(d\mathcal{P})),$$

where $\mathcal{L}_Z$ is the Lie derivative and i_Z is the interior product.

We also define the gradient vector field $grad_{\mathcal{H}}(f) \in \Gamma(\mathcal{H})$ for $f \in C^\infty(X)$ along $\mathcal{H}$ by

$$< Z,\, grad_{\mathcal{H}}(f) >^{\mathcal{H}} := Z(f),\ {}^\forall Z \in \Gamma(\mathcal{H}).$$

Now we define a second order differential operator Δ_{sub} in the similar way to define the Laplacian:

$$\Delta_{sub}(f) := \ div_{d\mathcal{P}}(grad_{\mathcal{H}}(f)).$$

Then it is sub-elliptic (cf. [12]) and so has compact resolvents.

In this case, we assume that there exists a compact Lagrangian submanifold L in $T_0^*(X)$ satisfying the conditions

$\mathrm{Mas}[1]_{sub}$: the principal symbol $\sigma(\Delta_{sub}) \equiv E_L$ (positive constant) on L,

and

$\mathrm{Mas}[3]_{sub}$: there exists an invariant measure on L under the action of the bi-characteristic flow $\{\phi_t^X\}$ of the sub-Laplacian Δ_{sub}.

Then,

Theorem 6.1. *Together with the condition* Mas[2] *the same conclusion of the Eigenvalue Theorem 5.1 holds.*

Example 6.2. *As an example of the above sub-Riemannian structure, we consider a submersion $\varphi : M \to N$ between compact orientable manifolds. We assume that there is a splitting of the tangent bundle $T(M) = \mathcal{H} \oplus \operatorname{Ker} d\varphi$ (we put $\mathcal{V} := \operatorname{Ker} d\varphi$) and the subbundle $\mathcal{H}$ satisfies the two conditions that*

(1) *it is a sub-Riemannian structure in the above sense. Hence the brackets of vector fields $[Z, Z']$, $Z, Z' \in \Gamma(\mathcal{H})$, generate $\Gamma(\mathcal{V})$.*

(2) *the inner product $< \cdot, \cdot >^{\mathcal{H}}$ satisfies the Riemannian submersion condition, that is, if $\varphi(p) = \varphi(p') = q$ then*

$$d\varphi_p : \mathcal{H}_p \to T_q(N) \text{ and } d\varphi_{p'} : \mathcal{H}_{q'} \to T_q(N)$$

define the same inner product on $T_q(N)$. Hence we can install a Riemannian metric on the base manifold N, which we denote by g^N. Through the isomorphism between $\mathcal{V} \cong T(M)/\mathcal{H}$ given by the sequence $\mathcal{V} \to T(M) \to T(M)/\mathcal{H}$, we can also introduce an inner product on $\mathcal{V}$. Then it will be natural to install a Riemannian metric on $T(M)$ by assuming that the bundles $\mathcal{V}$ and $\mathcal{H}$ are orthogonal. We denote the resulting Riemannian metric on M by g^M. Then $\varphi : M \to N$ is a Riemannian submersion with the property that the horizontal subbundle $\mathcal{V}^\perp \cong \mathcal{H}$ has the property (6.2) *and the inner product on $\mathcal{V}$ comes from the inner product on $\mathcal{H}$.*

In this case the Popp's measure $d\mathcal{P}$ on M and the Riemannian volume form dv^M coincide.

We note a relation of $grad_{\mathcal{H}}(\varphi^*(f))$, $grad(\varphi^*(f))$ for $f \in C^\infty(N)$. So let $f \in C^\infty(N)$ and denote by $grad(f)$ (respectively $grad(\varphi^*(f))$) the gradient vector field on N (or on M) with respect to the Riemannian volume form dv^N (or with respect to the Popp's measure = the Riemannian volume form dv^M). Then

$$grad_{\mathcal{H}}(\varphi^*(f)) = H_{grad(\varphi^*(f))} = \widetilde{grad(f)},$$

where $H_{grad(\varphi^*(f))}$ is the $\mathcal{H}$ component of $grad(\varphi^*(f))$ according to the orthogonal decomposition $T(M) = \mathcal{V} \oplus \mathcal{H}$ and $\widetilde{grad(f)}$ is the horizontal lift of the gradient vector field $grad(f)$.

Let (x, y) be local coordinates expressing the submersion map φ as the projection map $(x, y) \mapsto x$ (see §3). The vector field $grad(f)$ is expressed as $\sum_{i\,j} g^{ij}(x)\frac{\partial f}{\partial x_i}$, where we put $(G^N(x)^{-1})_{i\,j} = g^{i\,j}(x)$. Hence

$$\widetilde{grad(f)} = \sum_{i,\,j\,\leq \dim N} g^{ij}(x)\frac{\partial f(x)}{\partial x_j}\left(\frac{\partial}{\partial x_i} + \sum_{\alpha} b_{i\,\alpha}(x, y)\frac{\partial}{\partial y_\alpha}\right),$$

where the matrix elements are denoted as in (4.13).

There will be some conditions for the commutativity $\Delta_{sub} \circ \varphi^* = \varphi^* \circ \Delta^N$ and will be studied in a forth coming paper.

In this case of a Riemannian submersion $\varphi : M \to N$ such that the horizontal subbundle is a 2 step bracket generating, we can also consider the spectrum of the sub-Laplacian Δ_{sub} on M instead of the Laplacian on M, since we have the same commutative diagram to (4.17) with respect to the geodesic flow on N and the bi-characteristic flow $\{\phi_t^M\}$ of the sub-Laplacian

$$\begin{array}{ccccc} T_0^*(M) & \xleftarrow{\chi_\varphi} & \varphi^*(T_0^*(N)) & \xrightarrow{\mathfrak{p}_\varphi} & T_0^*(N) \\ \downarrow{\scriptstyle \phi_t^M} & & \downarrow{\scriptstyle \phi_t^M} & & {\scriptstyle \Phi_t^N}\downarrow \\ T_0^*(M) & \xleftarrow{\chi_\varphi} & \varphi^*(T_0^*(N)) & \xrightarrow{\mathfrak{p}_\varphi} & T_0^*(N). \end{array} \tag{6.3}$$

Hence

Theorem 6.3. *Under the same assumptions* Mas[1] $\sim$Mas[3] *for a Lagrangian submanifold L in $T_0^*(N)$ and with an invariant measure* (4.19) *under the action of the bi-characteristic flow $\{\phi_t^M\}$ of the sub-Laplacian, we may conclude the same conclusion as Theorem 5.2 for the sub-Laplacian Δ_{sub} on M.*

Also under the assumptions Mas$[1]_{sub}$, Mas[2], Mas$[3]_{sub}$ *and $\mathfrak{p}_\varphi{}^{-1}\big(\mathfrak{p}_\varphi(\tilde{L})\big) = \tilde{L}$ for a Lagrangian submanifold $\tilde{L}$ in $\mathfrak{p}_\varphi{}^{-1}\big(T_0^*(N)\big)$ we have the same conclusion with Theorem 5.3 for the Laplacian on N.*

Remark 6.4. *The sub-ellipsity of the sub-Laplacian guarantees the compact resolvents of the sub-Laplacian, which property is used in the third step* Step(3) *in* §5 *of the Eigenvalue Theorem.*

Appendix A. Maslov index and Maslov class

In this appendix we recall a definition of the Maslov class for symplectic vector bundles with two Lagrangian subbundles based on the Maslov index defined for arbitrary paths (cf. [7]) and prove the invariance of the Maslov class for a compact Lagrangian submanifold in the punctured cotangent bundle under dilation (Proposition 4.5). Also we remark two facts used in the outline for the proof of the Eigenvalue Theorem 5.1 and the coincidence of the Maslov class defined here and "α-construction" in [13].

A.1. **Maslov index and Maslov class.** We consider $\mathbb{C}^n$ as a typical symplectic vector space with the anti-symmetric and non-degenerate bilinear form $\omega_0(z,w)$

$$\omega_0(z,w) := \mathrm{Im}\left(\sum z_i\overline{w}_i\right) = \sum x_{n+i}y_i - x_iy_{n+i},$$

where $z = (z_1,\dots,z_n) = (x_1,x_2,\dots,x_n\,;x_{n+1},\dots,x_{2n})$, $z_i = x_i + x_{n+i}\sqrt{-1}$ and $w = (w_1,\dots,w_n) = (y_1,y_2,\dots,y_n\,;y_{n+1},\dots,y_{2n})$, $w_i = y_i + y_{n+i}\sqrt{-1}$.

A subspace in $\mathbb{C}^n$ is said to be a Lagrangian subspace, if it is a real subspace of the dimension n and on which the anti-symmetric bilinear form ω_0 vanishes.

More generally if the anti-symmetric bilinear form vanishes on a real subspace, it is call an isotropic subspace. Then it dimension is less than n.

For h a subspace (real vector space) in $\mathbb{C}^n$, we denote by h° the subspace defined by

$$h^\circ = \{z \in \mathbb{C}^n \mid \omega_0(z,v) = 0 \text{ for any } v \in h\}.$$

So, h is isotropic, if and only if $h \subset h^\circ$ and h is a Lagrangian subspace, if and only if $h = h^\circ$.

The subspaces

$$\lambda_{\mathrm{Re}} := \{\ (x_1,\dots,x_n\,;0,\dots,0)\ \}$$

and

$$\lambda_{\mathrm{Im}} := \{\ (0,\dots,0\,;x_{n+1},\dots,x_{2n})\ \}$$

are typical Lagrangian subspaces and $\mathbb{C}^n = \lambda_{\mathrm{Re}} \oplus \lambda_{\mathrm{Im}}$.

We denote the space of all Lagrangian subspaces in $\mathbb{C}^n$ by $\Lambda(n)$, which as is well known isomorphic to the quotient space $U(n)/O(n)$ and is called the Lagrangian-Grassmaniann and together with the projection map

$$\pi_{\mathcal{F}} : U(n) \to \Lambda(n),\ U(n) \ni U \longmapsto U(\lambda_{\mathrm{Re}})$$

it is a principal bundle with the structure group $O(n)$.

Let $\lambda \in \Lambda(n)$ and denote by $\mathcal{P}_\lambda$ the orthogonal projection operator $\mathbb{C}^n \to \lambda \subset \mathbb{C}^n$. Then the operator $\tau_\lambda := 2\mathcal{P}_\lambda - Id$ is an involution with λ as the 1-eigenspace and the orthogonal complement $\lambda^\perp$ as the -1-eigenspace. Also for $U \in U(n)$ let's denote the operator $\tau_\lambda \circ U^* \circ \tau_\lambda$ by $\theta_\lambda(U)$. In particular, if $\lambda = \lambda_{\mathrm{Re}}$ and we express the matrix $U = \left(u_{ij}\right)$ with the standard orthonormal basis $\{e_i\}$ of $\mathbb{C}^n$, then $\theta_{\lambda_{\mathrm{Re}}}(U) = \overline{U}$, that is $\overline{U} = \left(\overline{u}_{ij}\right)$.

For each $\lambda \in \Lambda(n)$, let $\mathcal{S}_\lambda : \Lambda(n) \to U(n)$ be a map, called Souriou map, defined by

$$\mathcal{S}_\lambda : \Lambda(n) \ni \mu \longmapsto U \circ \theta_\lambda(U) \in U(n),$$

where $\mu = U(\lambda_{\mathrm{Im}})$. In fact this does not depend on the operator U for $\mu = U(\lambda^\perp)$, since we have an expression

$$\mathcal{S}_\lambda(\mu) = -\tau_\mu \circ \tau_\lambda.$$

Let $U_{\mathfrak{M}}$ be a subset in $U(n)$ defined by

$$U_{\mathfrak{M}} = \{\,U \in U(n) \mid U + Id \text{ is not invertible}\,\}. \tag{A.1}$$

Then we call the subset defined by

$$\mathcal{M}_\lambda := {\mathcal{S}_\lambda}^{-1}(U_{\mathfrak{M}}) = \left\{\mu \in \Lambda(n) \;\middle|\; \mu \bigcap \lambda \neq \{0\}\right\} \tag{A.2}$$

the "*Maslov cycle*" passing through a Lagrangian subspace $\lambda \in \Lambda(n)$.

Let $\gamma : [0,\, 1] \to \Lambda(n)$ be a continuous curve. We define an intersection number of γ and $\mathcal{M}_\lambda$ in the following way (cf. [7]):

We can find a partition $\{0 = t_0 < t_1 < t_2 < \cdots < t_\ell = 1\}$ of the interval $[0,\, 1]$ and a set of small positive numbers $\{\, 0 < \varepsilon_j \ll 1\,\}_{j=0}^{\ell-1}$ satisfying the condition that for $j = 0, \ldots, \ell - 1$

$$\text{(A.3)} \qquad \begin{cases} \text{the values } e^{\sqrt{-1}(\pi \pm \varepsilon_j)} \text{ are not eigenvalues of the operators} \\ \mathcal{S}_\lambda(\gamma(t)) \text{ for } t_j \le t \le t_{j+1}. \end{cases}$$

This condition means that the eigenvalues of the operators $\{\mathcal{S}_\lambda(\gamma(t))\}_{t_j \le t \le t_{j+1}}$ included in the arc $\{e^{\sqrt{-1}s} \mid \pi - \varepsilon_j \le s \le \pi + \varepsilon_j\}$ stay there when the parameter $t_j \le t \le t_{j+1}$. Then we define an integer $Mas(\{\gamma\}, \lambda)$, and call it Maslov index for path $\{\gamma\}$ with respect to the Maslov cycle $\mathcal{M}_\lambda$ by

Definition A.1.

$$\begin{aligned} Mas(\{\gamma\}, \lambda) := \sum_{j=0}^{\ell-1} \\ &\text{the number of the eigenvalues of the operator } \mathcal{S}_\lambda(\gamma(t_{j+1})) \\ &\qquad\qquad \text{in the sector } \left\{e^{\sqrt{-1}s} \;\middle|\; \pi \le s \le \pi + \varepsilon_j\right\} \\ &- \text{the number of the eigenvalues of the operator } \mathcal{S}_\lambda(\gamma(t_j)) \\ &\qquad\qquad \text{in the sector } \left\{e^{\sqrt{-1}s} \;\middle|\; \pi \le s \le \pi + \varepsilon_j\right\}. \end{aligned}$$

Then,

M-ind(1): The integer $Mas(\{\gamma\},\, \lambda)$ does not depend on the partition $\{t_j\}$ of the interval $[0,\, 1]$ and the small positive numbers $\{\varepsilon_j\}$ satisfying the condition (A.3),

M-ind(2): It is a homotopy invariant for the paths with the fixed end points,

M-ind(3): It satisfies the additivity under catenations of paths.

Let $\Psi : E \to X$ be a symplectic vector bundle over a space X (we put the fiber dimension $= n$). The space X will have suitable properties satisfied by manifolds. We denote the anti-symmetric non-degenerate bilinear form on E by ω^E, then we can install an inner product $< \cdot\,, \cdot >$ on E "*compatible*" with the symplectic structure ω^E in such a sense that there exists an almost complex structure $J : E \to E$, $J^2 = -Id$, $\Psi \circ J = \Psi$ such that

$$\omega^E(u, v) =< J(u), v >,\ < J(u), J(v) >=< u, v >,\ u,\ v \in E_x.$$

We assume that there exist two Lagrangian sub-bundles F and G in E, that is their fibers at each point x are Lagrangian subspaces in E_x.

Let $\{\gamma(t)\}$ be a continuous curve, $\gamma : [0,\, 1] \to X$. We divide it into small segments $\left\{\{\gamma(t)\}_{t_i \le t \le t_{i+1}}\right\}$ in such a way that there exist a finite open covering $\{O_i\}_i$ around the curve $\{\gamma(t)\}$ and $\gamma([t_i,\, t_{i+1}]) \subset O_i$, such that the vector bundle E has local trivializations

$$\psi_i : O_i \times \mathbb{C}^n \cong \Psi^{-1}(O_i)$$

satisfying the property that by this trivialization for each $x \in O_i$, $(x, \lambda_{\mathrm{Im}})$ is mapped to $\psi_i(x, \lambda_{\mathrm{Im}}) = F_x = \Psi^{-1}(x) \bigcap F$. Then we can assign an integer $Mas_{(F,G)}(\{\gamma(t)\})$ for an arbitrary continuous path $\gamma : [0, 1] \to X$ as the sum

$$Mas_{(F,G)}(\{\gamma(t)\}) = \sum_i Mas(\{{\psi_i}^{-1}(G_{\gamma(t)})\}_{t_i \leq t \leq t_{i+1}}, \lambda_{\mathrm{Im}}). \tag{A.4}$$

This quantity can be defined for all paths and has the properties:

$\mathcal{M}(0)$: The definition does not depend on the partition of the interval [0, 1], nor the local trivializations of the symplectic vector bundle E satisfying the conditions above nor does not depend on the inner product installed which satisfies the "*compatibility properties*",

$\mathcal{M}(1)$: Homotopy invariance for paths with fixed end points,

$\mathcal{M}(2)$: Additivity under catenations.

Hence, let $\pi : \tilde{X} \to X$ be the universal covering space of X consisting of homotopy classes of paths starting from a fixed initial point $x_0 \in X$. Then we can define a function

$$Mas_{(F,G)} : \tilde{X} \longrightarrow \mathbb{Z},\ \tilde{X} \ni \{\gamma\} \longmapsto Mas_{(F,G)}(\{\gamma(t)\}). \tag{A.5}$$

Especially its restriction to the fiber $\pi^{-1}(x_0)$ defines a homomorphism:

$$Mas_{(F,G)} : \pi^{-1}(x_0) \cong \pi_1(X) \to \mathbb{Z}.$$

Consequently, we have a cohomology class $\in H^1(X, \mathbb{Z})$, which we denote by $\mathfrak{m}_{(F,G)}$ and is called the "*Maslov class*" of the pair of Lagrangian subbundles F and G. Note that $\mathfrak{m}_{(F,G)} = -\mathfrak{m}_{(G,F)}$.

Proposition A.2. *It will be apparent if the intersection $F \cap G$ on a curve $\{\gamma(t)\}$ is trivial bundle, then $Mas_{(F,G)}(\{\gamma\}) = 0$*

Definition A.3. Let $\chi_{\pi/2}$ be the representation $\chi_{\pi/2} : \mathbb{Z} \to U(1)$, $n \mapsto e^{\pi/2\sqrt{-1}n}$ and we define an associated complex line bundle $L_{\mathfrak{m}_{(F,G)}}$ on X to the principal bundle $\pi : \tilde{X} \to X$ through the representation $\pi_1(X) \overset{Mas_{(F,G)}}{\longrightarrow} \mathbb{Z} \overset{\chi_{\pi/2}}{\longrightarrow} U(1)$. It is called *Maslov line bundle.*

Let E be symplectic a vector bundle on a space X with two Lagrangian subbundle F and G. Let $\mathfrak{f} : Y \to X$ be a continuous map, then we can define the symplectic vector bundle $\mathfrak{f}^*(E)$ on Y with two Lagrangian subbundles $\mathfrak{f}^*(F)$ and $\mathfrak{f}^*(G)$. Let $\tilde{\mathfrak{f}} : \tilde{Y} \to \tilde{X}$ be the map between their universal covering spaces $\tilde{Y}$ and $\tilde{X}$. Then

$$Mas_{(F,G)} \circ \tilde{\mathfrak{f}} = Mas_{(\mathfrak{f}^*(F), \mathfrak{f}^*(G))}. \tag{A.6}$$

Now let L be a Lagrangian submanifold in the cotangent bundle $T^*(X)$. Then the restriction of the tangent bundle $T(T^*(X))$ to L is a symplectic vector bundle together with two Lagrangian subbundles, the tangent bundle of L, $T(L)$, and the restriction of $\mathrm{Ker}\, d\pi^X$ on L, the vertical subbundle with respect to the projection map $\pi^X : T^*(X) \to X$.

Hence we have a cohomology class $\mathfrak{m}_{\left(\mathrm{Ker}\, d\pi^X, T(L)\right)}$ as a homomorphism

$$\mathfrak{m}_{\left(\mathrm{Ker}\, d\pi^X, T(L)\right)} : \pi_1(L) \to \mathbb{Z},$$

which we will denote simply by $\mathfrak{m}_L$.

Remark A.4. *The definition of Maslov index for arbitrary paths given in* [20] *has a modification term at the end points and is not natural one. In* [9] *it was noticed for the first time without any modification term and in* [4] *and* [7]*, it was given based on the arguments by* [17] *including the infinite dimensional symplectic Hilbert space case.*

A.2. **Three remarks.** Here we notice two properties used in the outline of the proof of Eigenvalue Theorem 5.1 and the α-construction given in [13] based on our definition of Malsov index and Maslov class.

First, we prove Proposition 4.5.

Proposition A.5. *Let L be a compact Lagrangian submanifold in $T_0^*(X)$. Then for any positive real number $A > 0$ and any closed curve $\{\gamma\}$ in L,*

$$< \mathfrak{m}_L,\, \gamma >=< \mathfrak{m}_{A\cdot L},\, A\cdot\gamma > . \tag{A.7}$$

Proof. Since the Maslov index $< \mathfrak{m}_L, \gamma >$ for a path $\{\gamma\}$ is defined based on the data

$$\left\{\dim\left(T_{\gamma(t)}(L)\bigcap\left(\operatorname{Ker} d\pi^X\right)_{\gamma(t)}\right)\right\}_{t\in[0,\,1]}$$

and it holds that

$$\dim\left(T_{\gamma(t)}(\lambda)\bigcap\left(\operatorname{Ker} d\pi^X\right)_{\gamma(t)}\right) = \dim\left(T_{A\cdot\gamma(t)}(A\cdot\lambda)\bigcap\left(\operatorname{Ker} d\pi^X\right)_{A\cdot\gamma(t)}\right)$$

for any t, since the dilation $A\cdot : T_0^*(X) \longrightarrow T_0^*(X)$, $(x;\,\xi) \longmapsto A\cdot(x;\,\xi) = (x;\,A\cdot\xi)$, $A > 0$, is a diffeomorphism. Hence (A.7) holds. □

Next, we prove Proposition 4.2: By our definition of the Maslov class based on the Maslov index for arbitrary paths, the proof of Proposition 4.2 is now almost clear by the data given in Propositions 4.3 and 4.4, since it is enough to prove the coincidence of the intersection $T(L) \cap \mathcal{V}^X$ and $T(\hat{L}) \cap \mathcal{V}^{X\times U(1)}$ after a suitable reduction of the symplectic vector bundle. In fact through the map $p_L : \hat{L} \to L$ we know that the symplectic vector bundle $T(T^*(X))_{|L}$ on L can be seen as a symplectic quotient bundle of the symplectic vector bundle $T(T^*(X \times U(1)))_{|\hat{L}}$ and Propositions 4.3 and 4.4 guarantee the coincidence of the Maslov index for arbitrary curve in L, and it is enough to consider such a curve also in $L \times \{1\}$, since $\hat{L}$ and L are homotopic. □

Let L be a compact Lagrangian submanifold appearing in the Eigenvalue Theorem 5.1 and $\hat{L}$ the corresponding conic Lagrangian submanifold in $T_0^*(X)\times T_0^*(U(1))$. We also note the obvious free action of the group $\mathbb{Z}_k$ (the cyclic group of order k) on the space $U(1)$ is lifted to $T_0^*(U(1))$ and the lifted action leaves invariant the Lagrangian submanifold $\hat{L}$. Moreover the Maslov class $\mathfrak{m}_{\hat{L}}$ is invariant under this action. Hence

Proposition A.6. *The conic Lagrangian submanifold $\hat{L}$ is descended to the conic Lagrangian submanifold $\hat{L}/\mathbb{Z}_k$ in $T_0^*(X) \times T_0^*(U(1)/\mathbb{Z}_k)$, and the pull-back of the Maslov class $\mathfrak{m}_{\hat{L}/\mathbb{Z}_k}$ to $\hat{L}$ coincides with $\mathfrak{m}_{\hat{L}}$.*

At the end of the Appendix, we remark a construction called "α-construction" given in [13] in relation to our definition of the Maslov index for arbitrary paths.

First we recall the construction of the universal covering space $\Phi : \tilde{X} \to X$ from the very beginning. The space $\tilde{X}$ consists of homotopy classes of paths $\{\gamma\}$ starting from a fixed common point $\gamma(0) = x_0 \in X$. So, for each homotopy class $[\gamma] \in \tilde{X}$, $\Phi([\gamma]) = \gamma(1)$, the end point. Then $\Phi : \tilde{X} \to X$ is a principal bundle with the structure group $\pi_1(X) = \pi_1(X,\, x_0)$ (homotopy classes of loops with the base point x_0) so that there is an open covering $\{U_\ell\}$ of X and homeomorphisms $\{\phi_\ell\}$

$$\phi_\ell : U_\ell \times \pi_1(X) \overset{\sim}{\to} \Phi^{-1}(U_\ell)$$

which we define as follows:

For any point $x \in X$ we take a sufficiently small "*simply*" connected open neighborhood U_x (existence of such neighborhoods is assumed) and fix a path $\{\sigma_x\}$ connecting x_0 and x, $\sigma_x(0) = x_0$, $\sigma_x(1) = x$. Let $y \in U_x$ and we connect x and y by an arbitrary fixed path $s(U_x, y)$ in U_x. Since U_x is simply connected, the homotopy class of the path $s(U_x, y)$ is uniquely determined. Then, let $\phi_x : U_x \times \pi_1(X,\, x_0) \ni (y, [\gamma]) = [\gamma * \sigma_x * s(U_x, y)]$, where we mean by $[\gamma * \sigma_x * s(U_x, y)]$ the homotopy class of catenations of the loop γ and the paths σ_x and $s(U_x, y)$ with the end point y.

Let $y \in U_x \cap U_{x'}$. Then,

$$\phi_x(y, [\gamma]) = \phi_{x'}(y, [\mu]) \tag{A.8}$$

implies that there exists a unique element $[C_{x,x'}] \in \pi_1(X,\, x_0)$ such that the paths

$$\{C_{x,x'} * \gamma * \sigma_x * s(U_x, y)\} \text{ and } \{\mu * \sigma_{x'} * s(U_{x'}, y)\} \text{ are homotopic} \tag{A.9}$$

and the correspondence $U_x \cap U_{x'} \ni y \longmapsto C_{x,x'}(y)$ gives the transition functions of the principal bundle $\Phi : \tilde{X} \to X$. If we can connect $y_1, y_2 \in U_x \cap U_{x'}$ by a path in $U_x \cap U_{x'}$, then the loops $\{\sigma_x * s(U_x, y_1) * s(U_{x'}, y_1)^{-1} * {\sigma_{x'}}^{-1}\}$ and $\{\sigma_x * s(U_x, y_2) * s(U_{x'}, y_2)^{-1} * {\sigma_{x'}}^{-1}\}$ are homotopic. Hence we see that the correspondence $C_{x,x'} : U_x \cap U_{x'} \ni y \mapsto C_{x,x'} = C_{x,x'}(y)$ is a locally constant map on $U_x \cap U_{x'}$ taking values in $\pi_1(X) = \pi_1(X,\, x_0)$.

Now let $\Psi : E \to X$ be a symplectic vector bundle over a suitable space X as before with two Lagrangian subbundles F and G.

Then the integer valued locally constant functions $\{Mas_{F,G}(\{C_{x,x'}\})\}$,

$$Mas_{F,G} \circ C_{x,x'} : U_x \cap U_{x'} \longrightarrow \pi_1(X) \overset{Mas_{F,G}}{\longrightarrow} \mathbb{Z}$$

define a 1-Čech cocycle which cohomology class in $\check{H}^1(X, \mathbb{Z})$ corresponds to the Maslov class $\mathfrak{m}_{F,G}$ (cf. [5]).

Here we explain a realization of a set of the transition functions $\{C_{x,x'}\}$ given in terms of "*Hörmander index*".

Let λ_1 and λ_2 be two Lagrangian subspaces in $\mathbb{C}^n$ and we consider two Lagrangian subspaces μ, ν such that each of μ and ν is transversal to both of λ_1 and λ_2. Then the index called "Hörmander index" (cf. [13]) can be defined as

$$\sigma(\lambda_1, \lambda_2; \mu, \nu) := Mas(\{\gamma\}, \lambda_2) - Mas(\{\gamma\}, \lambda_1), \tag{A.10}$$

where $\{\gamma\}$ is a path connecting μ and ν. Then by the fact that the Maslov index for loop does not depend on the particular Maslov cycle $\mathcal{M}_\lambda$, the integer (A.10) is well-defined. In fact, for two paths $\{\gamma\}$ and $\{\gamma'\}$ connecting μ and ν we have

$$Mas([\gamma * {\gamma'}^{-1}]) = Mas(\{\gamma\}, \lambda_1) - Mas(\{\gamma'\}, \lambda_1) = Mas(\{\gamma\}, \lambda_2) - Mas(\{\gamma'\}, \lambda_2).$$

Let fix a point $x, x' \in X$, and take simply connected open neighborhoods $U_x \ni x$ and $U_{x'} \ni x'$ such that the principal bundle $\Phi : \tilde{X} \to X$ is trivial on each of them as before. Then for $y \in U_x \cap U_{x'}$, the difference

$$Mas_{F,G}(\{\mu * \sigma_{x'} * s(U_{x'}, y)\}) - Mas_{F,G}(\{\gamma * \sigma_x * s(U_x, y)\})$$

coincides with the Maslov index $Mas_{F,G}(\{C_{x,x'}\})$:

$$\begin{aligned} &Mas_{F,G}(\{\mu * \sigma_{x'} * s(U_{x'}, y)\}) - Mas_{F,G}(\{\gamma * \sigma_x * s(U_x, y)\}) \\ &= Mas_{F,G}(\{C_{x,x'}\}) = \mathfrak{m}_{F,G}([C_{x,x'}]), \end{aligned}$$

where we assumed (A.9).

Then for $y \in U_x \cap U_{x'}$, we can find two Lagrangian subspaces μ, ν in E_y such that each of the Lagrangian subspaces F_y and G_y is transversal to μ and ν, and moreover the Hörmander index $\sigma(F_y, G_y; \mu, \nu) = \mathfrak{m}_{F,G}([C_{x,x'}])$, since for any pair of Lagrangian subspaces F_y and G_y the values $\sigma(F_y, G_y; \mu, \nu)$ can take any integer by taking the suitable Lagrangian subspaces μ, ν in E_y transversal to F_y and G_y. Then under the local trivialization of the associated the Lagrangian-Grassmannian bundle $\Psi_\Lambda : \Lambda(E) \to X$ (the fibers $\Lambda(E)_x = {\Psi_\Lambda}^{-1}(x)$ are the Lagrangian-Grassmannian $\cong \Lambda(n)$) the transversality condition for μ and ν at y allows us to leave the value $\sigma(F_z, G_z; \mu, \nu)$ invariant around the point y and coincides with the value $\mathfrak{m}_{F,G}([C_{x,x'}])$. Hence a collection of Hörmander index $\{\sigma(F_z, G_z; \mu, \nu)\}$ is a realization of a set of the transition functions of the $\check{C}$eck cohomology class $\in \check{H}^1(X, \mathbb{Z}_X)$ corresponding to the Maslov class $\mathfrak{m}_{F,G} \in H^1(X)$.

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