Applied Analysis and Optimization
Volume 2, Number 3, 2018, 341–372

Yokohama Publishers
ISSN 2189-1664 Online Journal

INVERSE GENERALIZED VECTOR VARIATIONAL INEQUALITIES WITH RESPECT TO VARIABLE DOMINATION STRUCTURES AND APPLICATIONS TO VECTOR APPROXIMATION PROBLEMS

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Abstract. In this paper, we consider a generalized vector variational inequality with respect to a variable domination structure. We introduce two inverse set-valued vector variational inequalities and explore the relationships between the original vector variational inequality and the corresponding inverse vector variational inequalities. Further, we prove existence results for the generalized vector variational inequality and apply the inverse results to two different vector approximation problems with respect to a variable domination structure to justify the theoretical framework. We therefore show that generalized vector variational inequalities are closely related to vector optimization problems with a smooth objective function. For this reason, we compare the new inverse assertions with well-known duality results, using the conjugate/perturbation approach, for vector optimization problems.

1. Introduction

F. Giannessi introduced vector variational inequalities in 1980 in a finite-dimensional setting (see [13]). Within the last three decades, vector variational inequalities have been studied and extended extensively and have found numerous applications in various branches of pure and applied mathematics (see, e.g., [1], [7], [9], [14]). A significant extension of the vector variational inequalities introduced by F. Giannessi is to consider vector variational inequalities with variable domination structures (variable/moving ordering structures) which may vary dependently on the actual element in the linear space. While the solution concept for scalar-valued problems is entirely natural, in abstract spaces one can define a variable domination structure given by a family of convex cones which allows to describe different solution concepts for vector problems (see, e.g., [11], [18]). We remark that variable domination structures were already introduced in 1974 by Yu (see [30]) who presented different mathematical models with examples and applications. Recently, this topic has been studied by many authors, see [11] and the references therein for a detailed survey of this topic. Some popular applications of the notion of variable domination structures are, for example, in medical image registration, portfolio optimization, intensity modulated radiotherapy treatment and location problems (see, e.g., [12]).

2010 *Mathematics Subject Classification.* 46A20; 49J40; 90C29; 90C46.

Key words and phrases. Generalized vector variational inequality, inverse vector variational inequality, variable domination structure, duality, vector control approximation problem, intensity modulated radiotherapy treatment (IMRT).

In the context of (vector) optimization and variational inequalities, different dual problems have been introduced by using, for example, a scalarization technique, Wolfe and Mond-Weir duality concepts, a vector conjugate or a Lagrangian mapping (see, e.g., [2], [15], [16], [19]), among others. In essence, the dual problem gives a bound for the solution of the primal problem and further allows to attack the main (primal) problem differently. Dual problems for (vector) variational inequalities have been introduced in a finite-dimensional setting by Yang (see [29]).

We formulate the following assumption which will hold throughout:

($\mathcal{A}$) X and Y are real Banach spaces. $K : X \rightrightarrows Y$ is a set-valued mapping such that for every $x \in X$, $K(x)$ is a convex cone in Y with non-empty interior $\operatorname{int} K(x)$.

Let $\mathcal{F} : X \rightrightarrows \mathrm{L}(X,Y)$ and $\varphi : X \to Y \cup \{+\infty_Y\}$ be given mappings (see Section 2 for notations). In this paper, we consider the following *generalized vector variational inequality* w.r.t. the moving domination structure K: find an element $x \in \mathrm{Dom}(\mathcal{F}) \cap \mathrm{dom}(\varphi)$ such that for some operator $U \in \mathcal{F}(x)$ it holds

$$\text{(VVI)} \qquad \langle U, y-x\rangle_Y \not\leq_{\operatorname{int} K(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X.$$

In particular if $C \subseteq X$ is a non-empty convex set, $\varphi = \chi_C$ is the indicator mapping of C, that is, $\chi_C(x) := 0$ for $x \in C$ and $\chi_C(x) := +\infty_Y$ else and $\mathcal{F}$ is single-valued, denoted by F, then (VVI) recovers the following problem: find an element $x \in \mathrm{dom}(F) \cap C$ such that

$$\langle Fx, y-x\rangle_Y \not\leq_{\operatorname{int} K(x)} 0, \qquad \text{for every} \quad y \in X.$$

If we further define $Y := \mathbb{R}$ and $K(x) := \mathbb{R}_{\geq 0}$ for every $x \in X$, then the previous problem becomes the following well-known (scalar) variational inequality: find an element $x \in \mathrm{dom}(F) \cap C$ such that

$$\langle Fx, y-x\rangle \geq 0, \qquad \text{for every} \quad y \in X,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between X^* and X. Besides these two special cases, we consider the *first inverse vector variational inequality* of (VVI) w.r.t. the moving domination structure K, which consists in finding an operator $U_1 \in \mathrm{Dom}\,\mathcal{F}^{-1}(-\cdot)$ and an element $x_1 \in \mathcal{F}^{-1}(-U_1) \cap \mathrm{dom}(\varphi)$ such that it holds

$$\text{(IVVI}_1\text{)} \qquad \langle V - U_1, -x_1\rangle_Y \not\preccurlyeq^1_{\operatorname{int} K(x_1)} \varphi^*(U_1) - \varphi^*(V), \qquad \text{for every} \quad V \in \mathrm{L}(X,Y) \text{ with } \varphi^*(V) \neq \emptyset.$$

Here, φ^* denotes the weak vector conjugate of φ w.r.t. K, see Section 3.2. Note that $\mathcal{F}^{-1}(-\cdot)$ denotes the shifted set-valued mapping $x \mapsto \mathcal{F}^{-1}(-x)$. If we replace the binary set relation $\not\preccurlyeq^1_{\operatorname{int} K(\cdot)}$ by $\not\preccurlyeq^2_{\operatorname{int} K(\cdot)}$, then the *second inverse problem* is: find an operator $U_2 \in \mathrm{Dom}\,\mathcal{F}^{-1}(-\cdot)$ and an element $x_2 \in \mathcal{F}^{-1}(-U_2) \cap \mathrm{dom}(\varphi)$ such that

$$\text{(IVVI}_2\text{)} \quad \langle V - U_2, -x_2\rangle_Y \not\preccurlyeq^2_{\operatorname{int} K(x_2)} \varphi^*(U_2) - \varphi^*(V), \qquad \text{for every} \quad V \in \mathrm{L}(X,Y).$$

Note that the structure of the above problems is different from that in [5]. The idea of this paper is to nest the (primal) vector variational inequality into the two nverse problems in the sense, that, under appropriate conditions for the data of the problems, every solution of (IVVI$_2$) generates one of (VVI) and every solution of (VVI) generates one of (IVVI$_1$): The paper is organized as follows: In the next two

$$(\text{IVVI}_1) \xleftarrow{\text{direct assertion}} (\text{VVI}) \xleftarrow{\text{converse assertion}} (\text{IVVI}_2)$$

sections, we collect some notations, definitions and basic results for later use. We further prove a novel existence theorem for (VVI). The fourth section concentrates on inverse results for the generalized vector variational inequality (VVI). The purpose of Section 5 is to apply the duality results to vector control approximation problems with respect to a variable domination structure. The last section compares dual and inverse results for vector optimization problems with respect to a fixed domination structure.

2. Notations

In the following, let X be a real Banach space, and let X^* be its topological dual. If Y is another real Banach space, we denote by $\mathrm{L}(X,Y)$ the space of all linear continuous operators from X to Y and $\|\cdot\|_{\mathrm{L}(X,Y)}$ stands for the norm in $\mathrm{L}(X,Y)$. In the case that X and Y are real Euclidean spaces, say $X=\mathbb{R}^n$ and $Y=\mathbb{R}^m$, where $n,m\in\mathbb{N}$, then we use the identification $\mathrm{L}(\mathbb{R}^n,\mathbb{R}^m)\cong \mathrm{Mat}_{m\times n}(\mathbb{R})$. For $U,V\in \mathrm{L}(X,Y)$, $x\in X$, we define

$$\langle U,x\rangle_Y := U(x)\in Y \quad \text{and} \quad \langle V-U,x\rangle_Y := \langle V,x\rangle_Y - \langle U,x\rangle_Y \in Y.$$

If $X=\mathbb{R}^n$, $Y=\mathbb{R}^m$, $x\in\mathbb{R}^n$ and $A\in \mathrm{Mat}_{m\times n}(\mathbb{R})$ we write Ax instead of $\langle A,x\rangle_{\mathbb{R}^m}$. The domain and image of a mapping $F: X\to \mathrm{L}(X,Y)$ will be denoted by

$$\mathrm{dom}(F) := \{x\in X \mid Fx \text{ is well-defined}\}$$
$$\text{and} \quad \mathrm{im}(F) := \{A\in \mathrm{L}(X,Y) \mid \exists\, x\in X \text{ such that } A=Fx\},$$

respectively. If F is injective, then the adjoint mapping of F is defined by

$$F^{\#}: \mathrm{L}(X,Y)\to X, \quad F^{\#}U := F^{-1}(-U), \qquad \text{for every} \quad U\in \mathrm{dom}(F^{\#}).$$

Let Z be another real Banach space. Then, the domain, range and graph of a set-valued mapping $\mathcal{G}: X\rightrightarrows Z$ are defined by

$$\mathrm{Dom}(\mathcal{G}) := \{x\in X \mid \mathcal{G}(x)\neq\emptyset\}, \quad \mathrm{Im}(\mathcal{G}) := \bigcup_{x\in \mathrm{Dom}(\mathcal{G})} \mathcal{G}(x)$$
$$\text{and} \quad \mathrm{Gph}(\mathcal{G}) := \{(x,z)\in X\times Z \mid z\in\mathcal{G}(x)\}.$$

The inverse of $\mathcal{G}$, which always exists, is the set-valued mapping $\mathcal{G}^{-1}: Z\rightrightarrows X$ defined by $\mathcal{G}^{-1}(z)=\{x\in X \mid z\in\mathcal{G}(x)\}$. If for every $x\in\mathrm{Dom}(\mathcal{G})$, the set $\mathcal{G}(x)$ has the property P, then we say that $\mathcal{G}$ is P-valued.
We further denote the Minkowski sum and difference of two non-empty sets $A,B\subseteq Y$ by

$$A+B := \{a+b \mid a\in A,\ b\in B\} \quad \text{and} \quad A-B := \{a-b \mid a\in A,\ b\in B\}.$$

If $A=\{a\}$ or $B=\{b\}$ is a singleton, then we write $a+B$ and $A+b$ instead of $\{a\}+B$ and $A+\{b\}$, respectively. The Minkowski sum and difference of empty sets will be defined by the rules

$$A\pm\emptyset := \emptyset, \quad \emptyset\pm B := \emptyset \quad \text{and} \quad \emptyset\pm\emptyset := \emptyset$$

for (possibly empty) sets $A, B \subseteq Y$. In a similar way, we define the multiplication of a scalar $\alpha \in \mathbb{R}$ with a set by the rules

$$\alpha A := \{\alpha a \mid a \in A\}, \quad \text{for every non-empty set } A \subseteq Y, \quad \text{and} \quad \alpha\emptyset := \emptyset.$$

Let $\tilde{K}$ be a non-empty subset of Y. We call $\tilde{K}$ a cone if $\lambda\tilde{K} \subseteq \tilde{K}$ for every $\lambda \geq 0$. The cone $\tilde{K}$ is called convex if $\tilde{K} + \tilde{K} \subseteq \tilde{K}$, proper if $\tilde{K} \neq \{0\}$ and $\tilde{K} \neq Y$ and pointed if $\tilde{K} \cap (-\tilde{K}) = \{0\}$. Further, if $A \subseteq Y$ is a non-empty set, then the cone generated by A is defined as

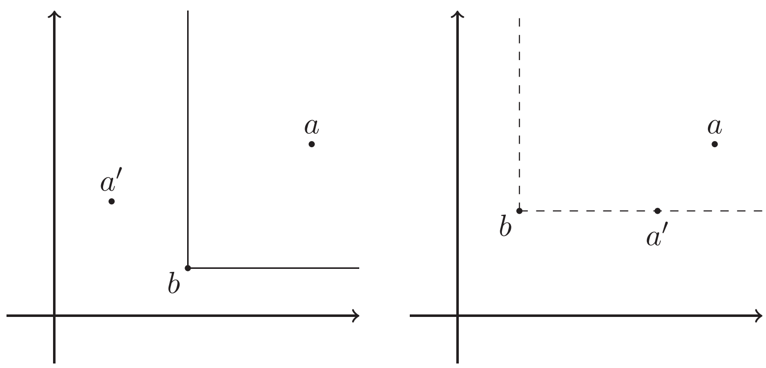
$$\operatorname{cone}(A) := \{y \in Y \mid y = \lambda a \text{ for some } \lambda \geq 0 \text{ and } a \in A\}.$$

In the Euclidean space $\mathbb{R}^m$, the so-called Pareto cone and its interior are given by

$$\mathbb{R}^m_{\geq 0} := \{y \in \mathbb{R}^m \mid y_j \geq 0 \text{ for } j = 1, \dots, m\}$$
$$\text{and} \quad \mathbb{R}^m_{>0} := \operatorname{int} \mathbb{R}^m_{\geq 0} = \{y \in \mathbb{R}^m \mid y_j > 0 \text{ for } j = 1, \dots, m\}.$$

Now, let $\tilde{K}$ be a convex cone in Y with non-empty interior. Then, we can define the following well-known binary relations for vectors $x, y \in Y$:

$$\begin{aligned} x \leq_{\tilde{K}} y \quad &:\Longleftrightarrow \quad y - x \in \tilde{K}, \\ x \not\leq_{\tilde{K}} y \quad &:\Longleftrightarrow \quad y - x \notin \tilde{K}, \\ x \leq_{\operatorname{int} \tilde{K}} y \quad &:\Longleftrightarrow \quad y - x \in \operatorname{int} \tilde{K}, \\ x \not\leq_{\operatorname{int} \tilde{K}} y \quad &:\Longleftrightarrow \quad y - x \notin \operatorname{int} \tilde{K}. \end{aligned}$$



(a) $a \leq_{\tilde{K}} b$ and $a' \not\leq_{\tilde{K}} b$ (b) $a \leq_{\operatorname{int} \tilde{K}} b$ and $a' \not\leq_{\operatorname{int} \tilde{K}} b$

FIGURE 1. Illustration of the different weak vector relations in $Y = \mathbb{R}^2$ for $\tilde{K} = \mathbb{R}^2_{\geq 0}$

The relations $\geq_{\tilde{K}}$, $\not\geq_{\tilde{K}}$, $\geq_{\operatorname{int} \tilde{K}}$ and $\not\geq_{\operatorname{int} \tilde{K}}$ are defined analogously. For non-empty sets $A, B \subseteq Y$, we introduce the following weak binary relations:

$$\begin{aligned} A \preccurlyeq^1_{\operatorname{int} \tilde{K}} B \quad &:\Longleftrightarrow \quad \exists a \in A,\ \forall b \in B: \ a \leq_{\operatorname{int} \tilde{K}} b, \\ A \not\preccurlyeq^1_{\operatorname{int} \tilde{K}} B \quad &:\Longleftrightarrow \quad \forall a \in A,\ \exists b \in B: \ a \not\leq_{\operatorname{int} \tilde{K}} b, \\ A \preccurlyeq^2_{\operatorname{int} \tilde{K}} B \quad &:\Longleftrightarrow \quad \forall a \in A,\ \exists b \in B: \ a \leq_{\operatorname{int} \tilde{K}} b, \\ A \not\preccurlyeq^2_{\operatorname{int} \tilde{K}} B \quad &:\Longleftrightarrow \quad \exists a \in A,\ \forall b \in B: \ a \not\leq_{\operatorname{int} \tilde{K}} b. \end{aligned}$$

If $A = \{a\}$ or $B = \{b\}$ is a singleton, then we write $a \preccurlyeq^1_{\operatorname{int}\tilde{K}} B$ and $A \preccurlyeq^1_{\operatorname{int}\tilde{K}} b$ instead of $\{a\} \preccurlyeq^1_{\operatorname{int}\tilde{K}} B$ and $A \preccurlyeq^1_{\operatorname{int}\tilde{K}} \{b\}$, respectively. The same convention holds for the other set relations. We further use the convention

$$A \not\preccurlyeq^2_{\operatorname{int}\tilde{K}} \emptyset, \qquad \text{for every non-empty subset } A \subseteq Y.$$

Of course, we have $A \preccurlyeq^2_{\operatorname{int}\tilde{K}} B$ if and only if $A \subseteq B - \operatorname{int}\tilde{K}$ provided A and B are non-empty. Notice that the relation $\preccurlyeq^2_{\operatorname{int}\tilde{K}}$ is known in the literature as upper set less order relation, see [19].

Let again $\tilde{K}$ be a convex cone in Y with non-empty interior. As usual, we attach to Y a smallest and greatest element with respect to $\tilde{K}$, denoted by $-\infty_Y$ and $+\infty_Y$, which do not belong to Y. Then for $y \in Y \cup \{\pm\infty_Y\}$, it holds $-\infty_Y \leq_{\tilde{K}} y \leq_{\tilde{K}} +\infty_Y$ and similar $-\infty_Y \leq_{\operatorname{int}\tilde{K}} y \leq_{\operatorname{int}\tilde{K}} +\infty_Y$ for $y \in Y$. On $Y \cup \{\pm\infty_Y\}$ we consider the following operations: $y + (+\infty_Y) = (+\infty_Y) + y := +\infty_Y$ for all $y \in Y \cup \{+\infty_Y\}$, $y + (-\infty_Y) = (-\infty_Y) + y := -\infty_Y$ for every $y \in Y \cup \{-\infty_Y\}$, $\lambda \cdot (+\infty_Y) := +\infty_Y$ for all $\lambda > 0$, $\lambda \cdot (+\infty_Y) := -\infty_Y$ for all $\lambda < 0$, $\lambda \cdot (-\infty_Y) := -\infty_Y$ for all $\lambda > 0$, $\lambda \cdot (-\infty_Y) := +\infty_Y$ for all $\lambda < 0$. Taking into account $-\infty_Y \leq_{\operatorname{int}\tilde{K}} y \leq_{\operatorname{int}\tilde{K}} +\infty_Y$ for every $y \in Y$, we have equivalently the following two conventions:

$$\text{(C1)} \qquad +\infty_Y \not\leq_{\operatorname{int}\tilde{K}} y, \qquad \text{for every} \quad y \in Y,$$
$$\text{(C2)} \qquad y \not\leq_{\operatorname{int}\tilde{K}} -\infty_Y, \qquad \text{for every} \quad y \in Y.$$

If $\varphi : X \to Y \cup \{+\infty_Y\}$ is a given mapping, then the effective domain of φ is given by

$$\operatorname{dom}(\varphi) := \{x \in X \mid \varphi(x) \neq +\infty_Y\}.$$

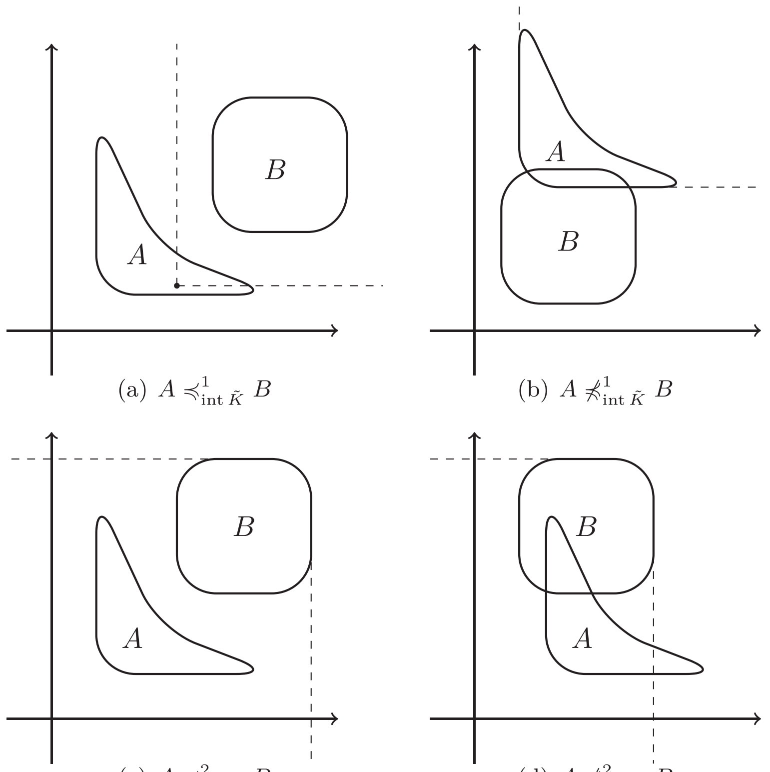
3. Preliminaries

3.1. **Solution concepts with respect to a variable domination structure.** In scalar-valued optimization the notion of (global) optimal solutions is very natural. Let X be some real linear space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a given mapping. Then, $x_0 \in \operatorname{dom}(f)$ is called a global minimum, that is, a solution of the problem

$$\min_{x \in X} f(x)$$

if $f(x_0) \leq f(x)$ for every $x \in X$. Here, the natural ordering $\leq$ in $\mathbb{R} \cup \{+\infty\}$ is a total ordering. However, if we replace the real numbers $\mathbb{R}$ by an other linear space Y, the ordering of elements in Y can be defined in different and non-obvious ways. If $\tilde{K} \subseteq Y$ is convex cone, then the binary relation $y^1 \leq_{\tilde{K}} y^2$ if and only if $y^2 - y^1 \in \tilde{K}$, $y^1, y^2 \in Y$, defines an order relation [19, Theorem 2.1.11] which is frequently used in vector optimization (see, e.g., [19], [23]). Binary relations which are not defined by a single cone but by a family of cones in Y will play an important role in the following.

Let us first give a precise definition of the notation of a variable domination structure, compare [11, Definition 1.8].



(a) $A \preccurlyeq^1_{\operatorname{int}\tilde{K}} B$ (b) $A \not\preccurlyeq^1_{\operatorname{int}\tilde{K}} B$

(c) $A \preccurlyeq^2_{\operatorname{int}\tilde{K}} B$ (d) $A \not\preccurlyeq^2_{\operatorname{int}\tilde{K}} B$

FIGURE 2. Illustration of the different weak set relations in $Y = \mathbb{R}^2$ for $\operatorname{int}\tilde{K} = \mathbb{R}^2_{>0}$

Definition 3.1. Let X and Y be real Banach spaces and $K : X \rightrightarrows Y$ a set-valued mapping with $K(x)$ a convex cone in Y for every $x \in X$. If elements in Y are compared using K, then K defines a *variable domination structure* on Y.

The next example demonstrates a variable domination structure in $\mathbb{R}^2$.

Example 3.2.

(a) Let $\ell : \mathbb{R}^2 \to \mathbb{R}^2$ be a given mapping and denote by $\|\cdot\|_2$ the Euclidean norm and by $\langle\cdot,\cdot\rangle$ the scalar product in $\mathbb{R}^2$. Define the set-valued mapping $K : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$K(x) := \left\{ y \in \mathbb{R}^2 \mid \|y\|_2 \le \langle \ell(x), y\rangle \right\}, \qquad \text{for every} \quad x \in \mathbb{R}^2.$$

For every $x \in \mathbb{R}^2$, $K(x)$ defines a cone in $\mathbb{R}^2$ and is called *Bishop-Phelps cone*, see [11] and Example 3.22. Hence, K defines a variable domination structure in $\mathbb{R}^2$. If we put $\ell(x) = (\ell_1(x), \ell_2(x)) := ((2 + \arctan x_1)/3, 2 + \sin x_2)$ for $x \in \mathbb{R}^2$, then the cone $K(x)$ can be visualized in the following way: denote by s and s' the intersection of the unit circle with the tangent line through the points $(0, 1/\ell_2(x))$ and $(1/\ell_1(x), 0)$. That is, $K(x)$ is given by the two half rays from the origin to s and s' respectively, compare [11, Example

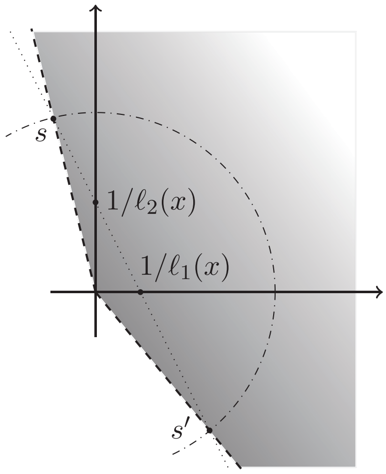


FIGURE 3. Visualization of the Bishop-Phelps cone $K(x)$, see [11]

1.28]. Notice that we have

$$K(\tan 1, 3\pi/2) = \mathbb{R}^2_{\geq 0} \quad \text{and} \quad \mathbb{R}^2_{\geq 0} \subseteq K(x), \quad \text{for every} \quad x \in \mathbb{R}^2.$$

(**b**) Some examples for variable domination structures and their applications can be found in [11] or Section 5.1 of this paper.

In the next definition, we describe solution concepts for vector optimization problems with respect to a fixed and variable domination structure, respectively (compare [2], [11], [19]).

Definition 3.3. Besides $(\mathcal{A})$, let $\psi : X \to Y \cup \{+\infty_Y\}$ be a given vector-valued mapping and define the set

$$\psi(\mathrm{dom}(\psi)) := \{\psi(x) \in Y \mid x \in \mathrm{dom}(\psi)\}.$$

(**a**) The element $\psi(x_0) \in Y$, where $x_0 \in \mathrm{dom}(\psi)$, is said to be a *weakly minimal element* of ψ w.r.t. the variable domination structure K, if we have

$$\psi(x) \not\leq_{\mathrm{int}\, K(x_0)} \psi(x_0), \qquad \text{for every} \quad x \in \mathrm{dom}(\psi),$$

that is, if no $y \in \psi(\mathrm{dom}(\psi))$ exists such that

$$\psi(x_0) \in y + \mathrm{int}\, K(x_0)$$

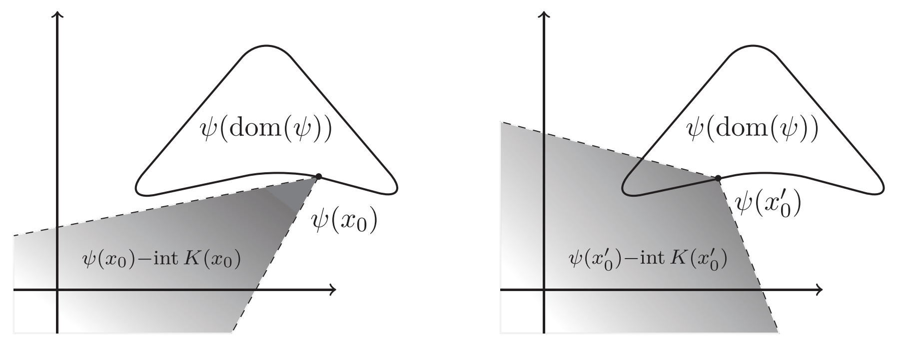
or equivalently

$$(\psi(\mathrm{dom}(\psi)) - \psi(x_0)) \cap (-\,\mathrm{int}\, K(x_0)) = \emptyset.$$

Similar, the element $\psi(x^0) \in Y$, where $x^0 \in \mathrm{dom}(\psi)$, is said to be a *weakly maximal element* of ψ w.r.t. the variable domination structure K, if we have $\psi(x^0) \not\leq_{\mathrm{int}\, K(x^0)} \psi(x)$ for all $x \in \mathrm{dom}(\psi)$.

(**b**) The set of weakly minimal and maximal elements of ψ w.r.t. the variable domination structure K will be denoted by the formulas $\mathrm{WMin}(\psi(\mathrm{dom}(\psi)), K(\cdot))$ and $\mathrm{WMax}(\psi(\mathrm{dom}(\psi)), K(\cdot))$, respectively.

(**c**) Assume now that the mapping $K : X \rightrightarrows Y$ is constant, that is, $K(x) = \tilde{K}$ for every $x \in X$, where $\tilde{K}$ is a convex cone in Y with non-empty interior. The element $\psi(x_0) \in Y$, where $x_0 \in \mathrm{dom}(\psi)$, is said to be a *weakly minimal element* of ψ w.r.t. the fixed domination structure $\tilde{K}$, if we have $\psi(x) \not\leq_{\mathrm{int}\,\tilde{K}} \psi(x_0)$ for all $x \in \mathrm{dom}(\psi)$. Similar $\psi(x^0) \in Y$, where $x^0 \in \mathrm{dom}(\psi)$, is said to be a *weakly maximal element* of ψ w.r.t. $\tilde{K}$, if we have $\psi(x^0) \not\leq_{\mathrm{int}\,\tilde{K}} \psi(x)$ for all $x \in \mathrm{dom}(\psi)$. The set of weakly minimal and weakly maximal elements of ψ w.r.t. the fixed cone $\tilde{K}$ will be denoted by $\mathrm{WMin}(\psi(\mathrm{dom}(\psi)), \tilde{K})$ and $\mathrm{WMax}(\psi(\mathrm{dom}(\psi)), \tilde{K})$, respectively.



(a) $\psi(x_0)$ is a weakly minimal element of ψ w.r.t. K

(b) $\psi(x_0')$ is not a weakly minimal element of ψ w.r.t. K

FIGURE 4. Illustration of the solution concept w.r.t. a variable domination structure K

Remark 3.4. Frequently, one looks for weakly minimal or maximal elements in a non-empty subset $C \subseteq X$ where the objective mapping is $\psi : C \to Y$. This can be reformulated in the form of Definition 3.1 by considering the new mapping $\widetilde{\psi} : X \to Y \cup \{+\infty_Y\}$, defined for $x \in X$ by

$$\widetilde{\psi}(x) = \begin{cases} \psi(x), & \text{for } x \in C, \\ +\infty_Y & \text{else.} \end{cases}$$

Let us prove the following very useful lemma. Notice that we do not assume the cone $\tilde{K}$ to be proper or pointed.

Lemma 3.5. *Besides* $(\mathcal{A})$*, let* $a, b \in Y$ *and* $\tilde{K}$ *be a convex cone in* Y *with non-empty interior. Then it holds that* $b - a \in \tilde{K}$ *and* $a \notin -\mathrm{int}\,\tilde{K}$ *implies* $b \notin -\mathrm{int}\,\tilde{K}$*.*

Proof. The proof of the statement follows from the useful identity

$$\tilde{K} + \mathrm{int}\,\tilde{K} = \mathrm{int}\,\tilde{K}. \tag{3.1}$$

Notice that $0 \in \tilde{K}$ and therefore it holds $\mathrm{int}\,\tilde{K} = \mathrm{int}\,\tilde{K} + 0 \subseteq \mathrm{int}\,\tilde{K} + \tilde{K}$. For the converse inclusion, let $x \in \mathrm{int}\,\tilde{K}$, $y \in \tilde{K}$ and $z \in Y$. Since $\tilde{K}$ is convex and $\mathrm{int}\,\tilde{K} \neq \emptyset$, it holds $\mathrm{int}\,\tilde{K} = \mathrm{cor}\,\tilde{K}$, see [23, Lemma 1.32], where $\mathrm{cor}\,\tilde{K} := \{k \in \tilde{K} \mid \forall\, y \in Y\ \exists\, \varepsilon' > 0,\ \forall\, \varepsilon \in [0, \varepsilon'],\ k + \varepsilon y \in \tilde{K}\}$ denotes the algebraic interior of $\tilde{K}$.

From $x \in \operatorname{int} \tilde{K}$ we therefore conclude that there is $\varepsilon' > 0$ such that $x + \varepsilon z \in \tilde{K}$ for every $\varepsilon \in [0, \varepsilon']$. The convexity of $\tilde{K}$ implies $x + y + \varepsilon z \in \tilde{K}$ for every $\varepsilon \in [0, \varepsilon']$, that is, $x + y$ is an interior point of $\operatorname{cor} \tilde{K}$. This shows (3.1).
Assume now that it holds $-b \in \operatorname{int} \tilde{K}$, where $b - a \in \tilde{K}$ and $a \notin -\operatorname{int} \tilde{K}$. Then, we deduce from (3.1) that $-a = b - a - b \in \operatorname{int} \tilde{K}$, which is a contradiction. The proof is complete. □

Remark 3.6. Obviously, we can rewrite the previous lemma in the following way: $0 \not\geq_{\operatorname{int} \tilde{K}} a \leq_{\tilde{K}} b$ implies $0 \not\geq_{\operatorname{int} \tilde{K}} b$. Some other important properties of the vector relations can be found in [1, Lemmas 2.2 and 2.3] and [15, Lemmas 7.1.1 and 7.1.2].

3.2. **Variational analysis.** We shall now collect some definitions and results for later use.

Definition 3.7. Besides $(\mathcal{A})$, let $\varphi : X \to Y \cup \{+\infty_Y\}$ be a given mapping. An operator $U \in \mathrm{L}(X, Y)$ is called a *weak subgradient* of φ at $x \in \operatorname{dom}(\varphi)$ w.r.t. the variable domination structure K if

$$\varphi(y) - \varphi(x) - \langle U, y - x \rangle_Y \not\leq_{\operatorname{int} K(x)} 0, \qquad \text{for every} \quad y \in X.$$

The set of *weak subgradients* of φ at x will be denoted by $\partial\varphi(x)$. If $\partial\varphi(x)$ is non-empty, then φ is said to be *weakly subdifferentiable at x*. If φ is weakly subdifferentiable at every point of its domain, then φ is said to be *weakly subdifferentiable.*

Remark 3.8. If the variable domination structure K is fixed and $\operatorname{dom}(\varphi) = X$, then the previous definition coincides with [26, Definition 6.1.2]. Unlike the scalar case, i.e., if we let $Y := \mathbb{R}$ and $K(x) := \mathbb{R}_{\geq 0}$ for every $x \in X$, the subdifferential is not necessarily a closed and convex set, see [26, Remark 6.1.3].

Definition 3.9. Besides $(\mathcal{A})$, let $\varphi : X \to Y \cup \{+\infty_Y\}$ be given. The set-valued mapping $\varphi^* : \mathrm{L}(X, Y) \rightrightarrows Y$, defined for every $U \in \mathrm{L}(X, Y)$ by

$$\varphi^*(U) := \operatorname{WMax}\left(\{\langle U, x \rangle_Y - \varphi(x) \mid x \in \operatorname{dom}(\varphi)\}, K(\cdot)\right),$$

is called the *weak conjugate* of φ w.r.t. the moving domination structure K. By the definition of WMax, we have $\langle U, x^0 \rangle_Y - \varphi(x^0) \in \varphi^*(U)$ for some $x^0 \in \operatorname{dom}(\varphi)$ if and only if

$$\langle U, x^0 \rangle_Y - \varphi(x^0) \not\leq_{\operatorname{int} K(x^0)} \langle U, x \rangle_Y - \varphi(x), \qquad \text{for every} \quad x \in X.$$

Example 3.10. Besides $(\mathcal{A})$, let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ and let $A \in \mathrm{Mat}_{m \times n}(\mathbb{R})$ be a given matrix. Then, the linear mapping $\varphi : \mathbb{R}^n \to \mathbb{R}^m$, defined by $\varphi(x) = Ax$ for every $x \in \mathbb{R}^n$, is subdifferentiable. Indeed, for every $x, y \in X$, it holds $0 = \varphi(y) - \varphi(x) - \langle A, y - x \rangle_Y \notin -\operatorname{int} K(x)$. This calculation further shows $0 \in \varphi^*(A)$. In comparison to the Fenchel conjugate of functions from X to $\mathbb{R} \cup \{+\infty\}$, the weak vector conjugate of a linear mapping is significantly larger. To be more precise, consider the fixed domination structure $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ given by $K(x) = \mathbb{R}^m_{\geq 0}$ for every $x \in \mathbb{R}^n$ and define $B := A + E$, where $E \in \mathrm{Mat}_{m \times n}(\mathbb{R})$ is a matrix with first entry 1 and 0 else. It then holds $(B - A)(y - x) = E(y - x) \in \operatorname{int} \mathbb{R}^m_{\geq 0}$ for all $x, y \in \mathbb{R}^n$, that is, $\varphi^*(B) = \{z \in \mathbb{R}^m \mid z_j = 0 \text{ for } j = 2, \ldots, m\}$ is a line in $\mathbb{R}^m$.

Lemma 3.11. *Besides* $(\mathcal{A})$*, let* $x \in \operatorname{dom}(\varphi)$ *and* $\varphi : X \to Y \cup \{+\infty_Y\}$*. We then have*

$$U \in \partial\varphi(x) \quad \textit{if and only if} \quad \langle U, x\rangle_Y - \varphi(x) \in \varphi^*(U).$$

Proof. By the definition of the weak conjugate of φ, we have $\langle U, x\rangle_Y - \varphi(x) \in \varphi^*(U)$ provided $x \in \operatorname{dom}(\varphi)$ if and only if $\langle U, x\rangle_Y - \varphi(x) \not\leq_{\operatorname{int} K(x)} \langle U, y\rangle_Y - \varphi(y)$ for every $y \in \operatorname{dom}(\varphi)$. The last inequality is obviously equivalent to $U \in \partial\varphi(x)$. The proof is complete. □

Definition 3.12. Besides $(\mathcal{A})$, let $\varphi : X \to Y \cup \{+\infty_Y\}$ be a given mapping and $\tilde{K}$ a convex cone in Y. φ is said to be $\tilde{K}$*-convex* if for all $x, y \in X$ and $t \in (0, 1)$

$$\varphi(tx + (1-t)y) \leq_{\tilde{K}} t\varphi(x) + (1-t)\varphi(y).$$

Analogously, we define the K_0*-convexity* of φ, where $K_0 := \bigcap_{y\in X} K(y)$, if we replace $\tilde{K}$ by K_0 in the previous relations.

Remark 3.13. Let $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping and $\tilde{K} := \mathbb{R}^m_{\geq 0}$. If the real-valued component functions $\varphi_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \ldots, m$ are convex, that is, for every $x, y \in \mathbb{R}^n$ and $t \in (0, 1)$

$$\varphi_j(tx + (1-t)y) \leq t\varphi_j(x) + (1-t)\varphi_j(y),$$

then it holds that φ is $\mathbb{R}^m_{\geq 0}$-convex.

Let us recall a basic concept of vector optimization (see [16, Section 3.2] and the references therein).

Definition 3.14. Besides $(\mathcal{A})$, let $\tilde{K}$ be a convex cone in Y with non-empty interior and let $A, B \subseteq Y \cup \{\pm\infty\}$ be non-empty sets. We say that A and B satisfy the *weak* (A, B)*-domination property* w.r.t. $\tilde{K}$ if for every $b \in B \setminus \{+\infty\}$ there exists $y^0 \in \operatorname{WMax}(A, \tilde{K})$ such that $y^0 \geq_{\tilde{K}} b$.

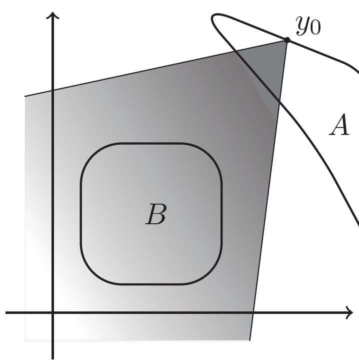


FIGURE 5. Illustration of the weak (A, B)-domination property of two sets A and B in $\mathbb{R}^2$

Remark 3.15.

(a) It is easy to see that the following assertions are equivalent:

(1) A and B satisfy the weak (A, B)-domination property with respect to $\tilde{K}$.

(2) It holds that $B \subseteq \mathrm{WMax}(A, \tilde{K}) - \tilde{K}$.

(b) We further have that $B \subseteq \mathrm{WMax}(A, \tilde{K}) - \tilde{K}$ and $B \not\subseteq \mathrm{WMax}(A, \tilde{K}) - \operatorname{int} \tilde{K}$ imply $B \subseteq \mathrm{WMax}(A, \tilde{K})$. Indeed, assume we have $B \not\subseteq \mathrm{WMax}(A, \tilde{K})$, that is, we can find $c \in \operatorname{int} \tilde{K}$ such that b is an element of $\mathrm{WMax}(A, \tilde{K}) - c$. But this contradicts the fact that we have $B \not\subseteq \mathrm{WMax}(A, \tilde{K}) - \operatorname{int} \tilde{K}$.

3.3. **An existence result for the generalized vector variational inequality.** The purpose of this section is to present an existence theorem for the generalized vector variational inequality (VVI). We first collect some useful definitions and lemmas. In the following, if $(\mathcal{A})$ holds, then we define

$$K_0 := \bigcap_{y \in X} K(y).$$

Definition 3.16. Let Z be a real normed vector space and denote by $\mathcal{P}_{\mathrm{cb}}(Z)$ the collection of all non-empty, closed and bounded subsets of Z. Then, the *Hausdorff metric* (distance) $\mathcal{H}_Z$ on $\mathcal{P}_{\mathrm{cb}}(Z)$ is for every $A, B \in \mathcal{P}_{\mathrm{cb}}(Z)$ defined by

$$\mathcal{H}_Z(A, B) := \max\left(\sup_{a \in A} \inf_{b \in B} \|a - b\|_Z,\ \sup_{b \in B} \inf_{a \in A} \|a - b\|_Z \right).$$

Notice that $\mathcal{H}_Z(\{a\}, \{b\}) = \|a - b\|_Z$ if $A = \{a\}$ and $B = \{b\}$ are singletons.

The following result can be found in [25].

Lemma 3.17. *Let Z be a real normed vector space and let $A, B \subseteq Z$ be non-empty compact sets. Then, for each $a \in A$, there exists an element $b \in B$ such that*

$$\|a - b\|_Z \leq \mathcal{H}_Z(A, B).$$

Definition 3.18. Besides $(\mathcal{A})$, let $\mathcal{F} : X \rightrightarrows \mathrm{L}(X, Y)$ be a set-valued mapping. $\mathcal{F}$ is said to be K_0*-monotone* if for every $x, y \in \mathrm{Dom}(\mathcal{F})$, $U \in \mathcal{F}(x)$ and $U' \in \mathcal{F}(y)$ it holds that

$$\langle U - U', x - y \rangle_Y \geq_{K_0} 0.$$

If in addition $\mathcal{F}$ is non-empty and compact-valued, i.e. we have in particular $\mathcal{F}(x) \in \mathcal{P}_{\mathrm{cb}}(\mathrm{L}(X, Y))$ for every $x \in X$, then $\mathcal{F}$ is said to be $\mathcal{H}$*-hemicontinuous* if for fixed elements $x, y \in X$, the mapping $\mathbb{R} \to \mathbb{R}$, $t \mapsto \mathcal{H}_{\mathrm{L}(X,Y)}(\mathcal{F}(x + t(y - x)), \mathcal{F}(x))$ is continuous at 0^+.

Remark 3.19. If the mapping $\mathcal{F}$ in the previous definition is single-valued and we denote this mapping by F, that is $F : X \to \mathrm{L}(X, Y)$, then the definition of K_0-monotonicity and $\mathcal{H}$-hemicontinuity become: F is K_0*-monotone* if $\langle Fx - Fy, x - y \rangle_Y \in K_0$ for every $x, y \in \mathrm{dom}(F)$. Further, F is said to be *hemicontinuous*, if the mapping $\mathbb{R} \to \mathrm{L}(X, Y)$, $t \mapsto F(x + t(y - x))$ is continuous at 0^+ for fixed elements $x, y \in X$.

Using the ideas in [4], we have the following result.

Lemma 3.20. *Besides $(\mathcal{A})$, let $\mathcal{F} : X \rightrightarrows \mathrm{L}(X, Y)$ and $\varphi : X \to Y \cup \{+\infty_Y\}$ be given mappings. Assume that the following conditions hold:*

(**a**) $\operatorname{int} K_0 \neq \emptyset$.
(**b**) *$\mathcal{F}$ is K_0-monotone, $\mathcal{H}$-hemicontinuous and has non-empty compact values.*
(**c**) *φ is K_0-convex and the effective domain $\operatorname{dom}(\varphi)$ is convex.*

Then, $x \in \operatorname{dom}(\varphi)$ and $U \in \mathcal{F}(x)$ satisfy

$$\langle U, y - x\rangle_Y \not\leq_{\operatorname{int} K(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X, \tag{3.2}$$

if and only if $x \in \operatorname{dom}(\varphi)$ satisfies

$$\langle U', y - x\rangle_Y \not\leq_{\operatorname{int} K(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X,\ U' \in \mathcal{F}(y). \tag{3.3}$$

Proof. Let $x \in \operatorname{dom}(\varphi)$ and $U \in \mathcal{F}(x)$ satisfy (3.2). Since it holds $K_0 \subseteq K(x)$, the K_0-monotonicity of $\mathcal{F}$ implies for every $y \in \operatorname{dom}(\varphi)$ and $U' \in \mathcal{F}(y)$

$$\langle U, y - x\rangle_Y + \varphi(y) - \varphi(x) \geq_{K(x)} \langle U', y - x\rangle_Y + \varphi(y) - \varphi(x).$$

By inequality (3.2) we have in particular

$$\langle U, y - x\rangle_Y \not\leq_{\operatorname{int} K(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in \operatorname{dom}(\varphi).$$

Using Lemma 3.5, we conclude from the previous inequalities that

$$\langle U', y - x\rangle_Y + \varphi(y) - \varphi(x) \not\leq_{\operatorname{int} K(x)} 0, \qquad \text{for every} \quad y \in \operatorname{dom}(\varphi),\ U' \in \mathcal{F}(y).$$

Since the previous inequality also holds for every $y \notin \operatorname{dom}(\varphi)$ in particular, see convention (C1) in Section 2, this shows (3.3).
Conversely, let $x \in \operatorname{dom}(\varphi)$ satisfy (3.3). Let $y \in \operatorname{dom}(\varphi)$ be arbitrarily chosen, put $y_t := (1-t)x + ty$ for $t \in (0,1)$ and let $U'_t \in \mathcal{F}(y_t)$. Notice that $y_t \in \operatorname{dom}(\varphi)$, see assumption (c). Inserting these elements into (3.3) yields

$$\langle U'_t, y_t - x\rangle_Y + \varphi(y_t) - \varphi(x) \not\leq_{\operatorname{int} K(x)} 0, \qquad \text{for every} \quad t \in (0,1).$$

By the K_0-convexity of φ we have, using again the fact that $K_0 \subseteq K(x)$, that it holds

$$\langle U'_t, y_t - x\rangle_Y + \varphi(y_t) - \varphi(x) \leq_{K(x)} t\ \left[\langle U'_t, y - x\rangle_Y + \varphi(y) - \varphi(x)\right].$$

From Lemma 3.5 we deduce that $x \in \operatorname{dom}(\varphi)$ satisfies

$$\langle U'_t, y - x\rangle_Y + \varphi(y) - \varphi(x) \not\leq_{\operatorname{int} K(x)} 0, \qquad \text{for every} \quad U'_t \in \mathcal{F}(y_t),\ t \in (0,1). \tag{3.4}$$

Since $\mathcal{F}$ is compact-valued, for each $U'_t \in \mathcal{F}(y_t)$ there exists $U_t \in \mathcal{F}(x)$ such that

$$\|U'_t - U_t\|_{\mathrm{L}(X,Y)} \leq \mathcal{H}_{\mathrm{L}(X,Y)}(\mathcal{F}(y_t), \mathcal{F}(x)),$$

see Lemma 3.17. Since $\mathcal{F}(x)$ is compact and the sequence $\{U_t\}$ lies in $\mathcal{F}(x)$, we can assume without loss of generality that $U_t \to U$ in $\mathrm{L}(X,Y)$ for $t \to 0^+$ and $U \in \mathcal{F}(x)$. Further, the inequality

$$\begin{aligned}\|U'_t - U\|_{\mathrm{L}(X,Y)} &\leq \|U'_t - U_t\|_{\mathrm{L}(X,Y)} + \|U_t - U\|_{\mathrm{L}(X,Y)}\\ &\leq \mathcal{H}_{\mathrm{L}(X,Y)}(\mathcal{F}(y_t), \mathcal{F}(x)) + \|U_t - U\|_{\mathrm{L}(X,Y)}\end{aligned}$$

shows $U'_t \to U$ in $\mathrm{L}(X,Y)$ for $t \to 0^+$, where we used the fact that $\mathcal{F}$ is $\mathcal{H}$-hemicontinuous, see (b). Since this convergence is strong and the set $Y\backslash(-\operatorname{int} K(x))$

is closed, we are able to pass in (3.4) to the limit $t \to 0^+$, that is, $x \in \operatorname{dom}(\varphi)$ and $U \in \mathcal{F}(x)$ satisfy

$$\langle U, y-x\rangle_Y \not\leq_{\operatorname{int} K(x)} \varphi(x)-\varphi(y), \qquad \text{for every} \quad y \in \operatorname{dom}(\varphi). \tag{3.5}$$

Since (3.5) holds for every $y \notin \operatorname{dom}(\varphi)$ in particular, see Convention (C2), the proof is complete. ☐

Definition 3.21. Besides $(\mathcal{A})$, let $\mathcal{W} : X \rightrightarrows Y$ be a set-valued mapping. $\mathcal{W}$ is said to be *closed* if $\operatorname{Gph}(\mathcal{W}) \subseteq X \times Y$ is closed, that is, for every sequence $\{(x_n, y_n)\} \subseteq \operatorname{Gph}(\mathcal{W})$ such that $(x_n, y_n) \to (x, y)$, we have $(x, y) \in \operatorname{Gph}(\mathcal{W})$.

Example 3.22. For the sake of simplicity let $X = Y$ in the previous definition and define a set-valued mapping $K : X \rightrightarrows X$ by

$$K(x) := \{y \in X \mid \|y\|_X \leq \langle \psi(x), y\rangle\}, \qquad \text{for every} \quad x \in X.$$

Here, $\psi : X \to X^*$ is a given operator such that $\|\psi(x)\|_{X^*} > 1$ for every $x \in X$. By this, it holds that $K(x)$ is a so-called Bishop-Phelps cone [11, Definition 1.14] for every $x \in X$ and from [11, Lemma 1.16] it follows that $K(x)$ is a proper, closed, convex and pointed cone in X, where the interior is non-empty and $\operatorname{int} K(x) = \{y \in X \mid \|y\|_X < \langle \psi(x), y\rangle\}$ in particular, see [11, Lemma 1.16]. If in addition $\psi : X \to X^*$ is continuous, then the set-valued mapping $\mathcal{W} : X \rightrightarrows X$, defined for every $x \in X$ by

$$\mathcal{W}(x) := X \setminus (-\operatorname{int} K(x)) = \{y \in X \mid \langle \psi(x), y\rangle \geq -\|y\|_X\},$$

is closed. Indeed, let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $x_n \to x$, $y_n \to y$ and $y_n \in \mathcal{W}(x_n)$. From $y_n \in \mathcal{W}(x_n)$, we have $\langle \psi(x_n), y_n\rangle \geq -\|y_n\|_X$ and the continuity of ψ and $\|\cdot\|_X$ imply $\langle \psi(x), y\rangle \geq -\|y\|_X$, that is, $y \in \mathcal{W}(x)$. This shows that $\mathcal{W}$ is closed.

Definition 3.23. Besides $(\mathcal{A})$, let $C \subseteq X$ be non-empty and a set-valued mapping $G : C \rightrightarrows X$ be given. G is said to be a *KKM mapping* if for any finite subset $\{y_1, \dots, y_k\} \subseteq C$, $k \in \mathbb{N}$, we have

$$\operatorname{conv}\{y_1, \dots, y_k\} \subseteq \bigcup_{j=1}^{k} G(y_j).$$

Theorem 3.24 (Fan-KKM)**.** *Besides $(\mathcal{A})$, let $C \subseteq X$ be non-empty and $G : C \rightrightarrows X$ a KKM mapping with non-empty and closed values. If there exists a point $y_0 \in C$ such that $G(y_0)$ is a compact subset of X, then it holds*

$$\bigcap_{y \in C} G(y) \neq \emptyset.$$

The next results uses the ideas in [4], [10] and [22].

Theorem 3.25. *Besides $(\mathcal{A})$, let $\mathcal{F} : X \rightrightarrows \mathrm{L}(X, Y)$ and $\varphi : X \to Y \cup \{+\infty_Y\}$ be given mappings. Assume that the following conditions hold:*

- **(a)** $\operatorname{int} K_0 \neq \emptyset$.
- **(b)** *The mapping $\mathcal{W} : X \rightrightarrows Y$, defined by $\mathcal{W}(x) := Y \setminus (-\operatorname{int} K(x))$ for every $x \in X$, is closed.*

(**c**) *φ is K_0-convex and continuous with convex domain.*
(**d**) *$\mathcal{F}$ is K_0-monotone, $\mathcal{H}$-hemicontinuous and has non-empty and compact values.*
(**e**) *$\mathcal{F}$ is coercive in the sense that there exists an element $y_0 \in \mathrm{dom}(\varphi)$ and a non-empty compact subset $B_0 \subseteq X$ such that*

$$\big\{x \in X \mid \langle U', y_0 - x\rangle_Y + \varphi(y_0) - \varphi(x) \not\leq_{\mathrm{int}\, K(x)} 0, \quad \textit{for every} \;\; U' \in \mathcal{F}(y_0)\big\}$$

is a subset of B_0.

Then, the generalized vector variational inequality (VVI) *has a solution, that is, there exists $x \in \mathrm{dom}(\varphi)$ and $U \in \mathcal{F}(x)$ such that*

$$\text{(VVI)} \qquad \langle U, y - x\rangle_Y \not\leq_{\mathrm{int}\, K(x)} \varphi(x) - \varphi(y), \qquad \textit{for every} \quad y \in X.$$

Proof. Let us define a set-valued mapping $G : \mathrm{dom}(\varphi) \rightrightarrows X$ for each $y \in X$ by

$$G(y) := \big\{x \in \mathrm{dom}(\varphi) \mid \langle U, y - x\rangle_Y + \varphi(y) - \varphi(x) \not\leq_{\mathrm{int}\, K(x)} 0, \quad \text{for some} \;\; U \in \mathcal{F}(x)\big\}.$$

The main tool of this proof is Theorem 3.24 which will ensure that

$$\bigcap_{y \in \mathrm{dom}(\varphi)} G(y) \neq \emptyset. \tag{3.6}$$

It is evident that every element in the intersection is obviously a solution of (VVI). For this purpose, we further define another set-valued mapping $G' : \mathrm{dom}(\varphi) \rightrightarrows X$ by

$$G'(y) := \big\{x \in \mathrm{dom}(\varphi) \mid \langle U', y - x\rangle_Y + \varphi(y) - \varphi(x) \not\leq_{\mathrm{int}\, K(x)} 0, \quad \text{for every} \;\; U' \in \mathcal{F}(y)\big\},$$

for every $y \in X$. Notice that we have in view of the K_0-monotonicity of $\mathcal{F}$ that it holds

$$G(y) \subseteq G'(y), \qquad \text{for every} \quad y \in X, \tag{3.7}$$

compare Lemma 3.20. Let us show that G is a KKM mapping. Indeed, assume by contradiction, there are $k \in \mathbb{N}$ and $y_1, \ldots, y_k \in \mathrm{dom}(\varphi)$ such that $\bar{y} := \sum_{j=1}^k \alpha_j y_j \notin \bigcup_{j=1}^k G(y_j)$, where $\sum_{j=1}^k \alpha_j = 1$ and $\alpha_j \geq 0$ for $j = 1, \ldots, k$. Notice that $\bar{y} \in \mathrm{dom}(\varphi)$, see (c). Since $\bar{y} \notin G(y_j)$ for $j = 1, \ldots, k$, for every $U' \in \mathcal{F}(\bar{y})$ it holds

$$\langle U', y_j - \bar{y}\rangle_Y + \varphi(y_j) - \varphi(\bar{y}) \leq_{\mathrm{int}\, K(\bar{y})} 0, \qquad \text{for} \quad j = 1, \ldots, k. \tag{3.8}$$

By the K_0-convexity of φ and (3.8) it holds for all $U' \in \mathcal{F}(\bar{y})$

$$\begin{aligned} 0 &= \langle U', \bar{y} - \bar{y}\rangle_Y + \varphi(\bar{y}) - \varphi(\bar{y}) \\ &\geq_{K(\bar{y})} \sum_{j=1}^k \alpha_j \left[\langle U', \bar{y} - y_j\rangle_Y + \varphi(\bar{y}) - \varphi(y_j)\right] \geq_{\mathrm{int}\, K(\bar{y})} 0, \end{aligned}$$

which is impossible since $0 \notin \mathrm{int}\, K(\bar{y})$. Hence, G is a KKM mapping and so is G', see relation (3.7). Let us show that for every $y \in \mathrm{dom}(\varphi)$, the set $G'(y)$ is closed. Indeed, fix $y \in \mathrm{dom}(\varphi)$ and let $\{x_n\} \subseteq G'(y)$ be a sequence such that $x_n \to x$.

We are going to show that the limit point x belongs to $G'(y)$. It holds for every $U' \in \mathcal{F}(y)$

$$\langle U', y - x_n\rangle_Y + \varphi(y) - \varphi(x_n) \notin -\operatorname{int} K(x_n). \tag{3.9}$$

Using the assumptions (b) and (c), we can pass in (3.9) to the limit such that we conclude $x \in G'(y)$. By (e), there exists $y_0 \in \operatorname{dom}(\varphi)$ such that $G'(y_0) \subseteq B_0$, where B_0 is a compact subset of X. Hence, $G'(y_0)$ is compact. From relation (3.7) and Theorem 3.24, we finally conclude $\bigcap_{y\in X} G(y) \subseteq \bigcap_{y\in X} G'(y) \neq \emptyset$ such that (3.6) holds. The proof is complete. □

The next corollary states the special case if $\mathcal{F}$ is single-valued.

Corollary 3.26. *Besides $(\mathcal{A})$, let $F : X \to \mathrm{L}(X, Y)$ and $\varphi : X \to Y \cup \{+\infty_Y\}$ be given mappings. Assume that the following conditions hold:*

- (**a**) $\operatorname{int} K_0 \neq \emptyset$.
- (**b**) *The mapping $\mathcal{W} : X \rightrightarrows Y$, defined by $\mathcal{W}(x) := Y \setminus (-\operatorname{int} K(x))$ for every $x \in X$, is closed.*
- (**c**) *φ is K_0-convex and continuous with convex domain.*
- (**d**) *F is K_0-monotone and hemicontinuous.*
- (**e**) *F is coercive in the sense that there exists an element $y_0 \in \operatorname{dom}(F)\cap\operatorname{dom}(\varphi)$ and a non-empty compact subset $B_0 \subseteq X$ such that*

$$\bigl\{x \in X \mid \langle Fy_0, y_0 - x\rangle_Y + \varphi(y_0) - \varphi(x) \not\leq_{\operatorname{int} K(x)} 0\bigr\} \subseteq B_0.$$

Then, there exists $x \in \operatorname{dom}(F) \cap \operatorname{dom}(\varphi)$ such that

$$\langle Fx, y - x\rangle_Y \not\leq_{\operatorname{int} K(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X.$$

Definition 3.27. Besides $(\mathcal{A})$, let $\psi : X \to Y$ be a given mapping. The directional derivative of ψ at $x \in X$ in the direction $h \in X$ is given by

$$\delta\psi(x; h) := \lim_{t\to 0} \frac{\psi(x + th) - \psi(x)}{t} \tag{3.10}$$

provided this limit exists. If $\delta\psi(x; h)$ exists for every $h \in X$, and if the mapping $D_G\psi(x) : X \to Y$ defined by

$$\langle D_G\psi\, x, h\rangle_Y := D_G(x)h := \delta\psi(x; h)$$

is linear and continuous, then we say that ψ is *Gâteaux-differentiable* at x, and we call $D_G\psi\, x$ the *Gâteaux-derivative* of ψ at x. If ψ is *Gâteaux-differen-tiable* at every point $x \in X$, then we say that ψ is *Gâteaux-differentiable.* Similar, we define the *right- and left-handed Gâteaux-derivative* of ψ at x which will be denoted by $D_G^+\psi\, x$ and $D_G^-\psi\, x$, respectively.

Example 3.28. Consider the mapping $\psi : \mathbb{R}^n \to \mathbb{R}$ defined by $\psi(x) = \|x - a\|_2^2$ for every $x \in \mathbb{R}^n$, where $a \in \mathbb{R}^n$ is a given element and $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^n$. To calculate the Gâteaux-derivative of ψ, we first notice that for every $t \in \mathbb{R}$ and $x, h \in \mathbb{R}^n$ it holds

$$\begin{aligned}\psi(x + th) - \psi(x) &= \langle x + th - a, x + th - a\rangle - \langle x - a, x - a\rangle \\ &= 2t\langle x - a, h\rangle + t^2\langle h, h\rangle\end{aligned}$$

from which we conclude $\delta\psi(x; h) = 2\langle x - a, h\rangle$ and $D_G\psi\, x = 2\langle x - a, \cdot\rangle$.

The following results connect vector variational and vector optimization problems.

Theorem 3.29. *Besides* $(\mathcal{A})$*, let* $C \subseteq X$ *be a non-empty convex subset and* $\psi : X \to Y$ *be a given mapping.*

(**a**) *If in addition* ψ *is right-handed Gâteaux-differentiable with derivative* $D_G^+\psi$ *and* $\psi(x) \in \mathrm{WMin}(\psi[C], K(\cdot))$ *for some* $x \in C$*, then* x *satisfies the vector variational inequality*

$$\langle D_G^+\psi\, x, y - x\rangle_Y \not\leq_{\operatorname{int} K(x)} 0, \qquad \text{for every} \quad y \in C. \tag{3.11}$$

(**b**) *Conversely, if* $x \in C$ *solves* (3.11) *and* ψ *is* $K(x)$*-convex in addition, then* $\psi(x) \in \mathrm{WMin}(\psi[C], K(\cdot))$.

Proof. (a). If $\psi(x) \in \mathrm{WMin}(\psi[C], K(\cdot))$ for some $x \in C$, then we have in particular

$$\frac{1}{t}\big[\psi(x + t(y - x)) - \psi(x)\big] \notin -\operatorname{int} K(x), \qquad \text{for every} \quad y \in C,\ t \in (0,1), \tag{3.12}$$

where we used the convexity of C. Since $Y \setminus (-\operatorname{int} K(x))$ is closed and ψ is right-handed Gâteaux-differentiable, passing in (3.12) to the limit $t \to 0^+$ yields $\langle D_G^+\psi\, x, y - x\rangle_Y \notin -\operatorname{int} K(x)$ for every $y \in C$.

(b). Conversely, if $x \in C$ solves (3.11) and ψ is $K(x)$-convex, then we have in particular $\frac{1}{t}\left[\psi(x + t(y - x)) - \psi(x)\right] \leq_{\operatorname{int} K(x)} \psi(y) - \psi(x)$ for every $y \in C$. The inequality further implies, using again the fact that ψ is right-handed Gâteaux-differentiable, that we have $\langle D_G^+\psi\, x, y - x\rangle + \psi(x) \leq_{K(x)} \psi(y)$ for every $y \in C$. Since it holds $\langle D_G^+\psi\, x, y - x\rangle \not\leq_{\operatorname{int} K(x)} 0$ for every $y \in C$, the previous inequalities imply $\psi(x) - \psi(y) \not\geq_{\operatorname{int} K(x)} 0$ for every $y \in C$, see Lemma 3.5. This confirms that $\psi(x) \in \mathrm{WMin}(\psi[C], K(\cdot))$ and the proof is complete. □

Remark 3.30. If the variable domination structure K is constant, i.e. $K(x) = \tilde{K}$ for every $x \in X$, then Theorem 3.29 recovers [15, Theorem 9.1.1].

4. Inverse generalized vector variational inequalities with respect to a variable domination structure

The aim of this section is to describe the so-called conjugate approach for generalized vector variational inequalities w.r.t. a moving domination structure.

4.1. **Inverse problems for** (VVI) **based on vector conjugate.** Besides our assumption $(\mathcal{A})$, let $\mathcal{F} : X \rightrightarrows \mathrm{L}(X, Y)$ and $\varphi : X \to Y \cup \{+\infty_Y\}$ be given mappings. Let us recall that the *generalized vector variational inequality* w.r.t. the moving domination structure K consists in finding an element $x \in \mathrm{Dom}(\mathcal{F}) \cap \mathrm{dom}(\varphi)$ such that for some operator $U \in \mathcal{F}(x)$ it holds

$$\langle U, y - x\rangle_Y \not\leq_{\operatorname{int} K(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X. \tag{VVI}$$

If $x \in \mathrm{Dom}(\mathcal{F}) \cap \mathrm{dom}(\varphi)$ and $U \in \mathcal{F}(x)$ satisfy (VVI), then we briefly say that the pair $(x, U) \in \mathrm{Gph}(\mathcal{F})$ solves (VVI). The *first inverse vector variational inequality* of

(VVI) w.r.t. the moving domination structure K reads as follows: find an operator $U_1 \in \mathrm{Dom}(\mathcal{F}^{-1}(-\cdot))$ and an element $x_1 \in \mathcal{F}^{-1}(-U_1) \cap \mathrm{dom}(\varphi)$ such that

$$(\mathrm{IVVI}_1) \quad \begin{aligned} &\langle V - U_1, -x_1\rangle_Y \not\preccurlyeq^1_{\mathrm{int}\,K(x_1)} \varphi^*(U_1) - \varphi^*(V), \\ &\qquad\qquad \text{for every} \quad V \in \mathrm{L}(X,Y) \text{ with } \varphi^*(V) \neq \emptyset. \end{aligned}$$

Notice that $\mathcal{F}^{-1}(-\cdot)$ denotes the shifted set-valued mapping $x \mapsto \mathcal{F}^{-1}(-x)$. If we replace the binary set relation $\not\preccurlyeq^1_{\mathrm{int}\,K(\cdot)}$ by $\not\preccurlyeq^2_{\mathrm{int}\,K(\cdot)}$ then the *second inverse vector variational inequality* becomes: find an operator $U_2 \in \mathrm{Dom}(\mathcal{F}^{-1}(-\cdot))$ and an element $x_2 \in \mathcal{F}^{-1}(-U_2) \cap \mathrm{dom}(\varphi)$ such that

$$(\mathrm{IVVI}_2) \quad \langle V - U_2, -x_2\rangle_Y \not\preccurlyeq^2_{\mathrm{int}\,K(x_2)} \varphi^*(U_2) - \varphi^*(V), \qquad \text{for every} \quad V \in \mathrm{L}(X,Y).$$

Again, if $U_i \in \mathrm{Dom}(\mathcal{F}^{-1}(-\cdot))$ and $x_i \in \mathcal{F}^{-1}(-U_i) \cap \mathrm{dom}(\varphi)$ satisfy (IVVI_i), then we briefly say that the pair $(U_i, x_i) \in \mathrm{Gph}(\mathcal{F}^{-1}(-\cdot))$ solves (IVVI_i), $i = 1, 2$.

The next theorem states a direct and converse assertion for (VVI).

Theorem 4.1 (Direct and converse assertion for (VVI))**.** *Besides* $(\mathcal{A})$*, let* $\mathcal{F} : X \rightrightarrows \mathrm{L}(X,Y)$ *and* $\varphi : X \to Y \cup \{+\infty_Y\}$ *be given mappings. Assume that* φ *is subdifferentiable w.r.t. the moving domination structure* K*. Then, the following statements hold:*

(a) *If the pair* $(x, U) \in \mathrm{Gph}(\mathcal{F})$ *is a solution of* (VVI) *and we have*

$$K(x) \subseteq K(y), \qquad \text{for every} \quad y \in X,$$

then the pair $(U_1, x_1) := (-U, x) \in \mathrm{Gph}(\mathcal{F}^{-1}(-\cdot))$ *solves* (IVVI_1).

(b) *Conversely, if the pair* $(U_2, x_2) \in \mathrm{Gph}(\mathcal{F}^{-1}(-\cdot))$ *is a solution of* (IVVI_2)*, it holds* $\partial\varphi(x_2) \neq \emptyset$ *and the sets*

$$A := \big\{\langle U_2, y\rangle_Y - \varphi(y) \mid y \in X\big\} \quad \text{and} \quad B := \big\{\langle U_2, x_2\rangle_Y - \varphi(x_2)\big\}$$

satisfy the weak (A, B)*-domination property w.r.t. the cone* $K(x_2)$*, then the pair* $(x, U) := (x_2, -U_2) \in \mathrm{Gph}(\mathcal{F})$ *is a solution of* (VVI).

Proof. (a). Let the pair $(x, U) \in \mathrm{Gph}(\mathcal{F})$ be a solution of (VVI), that is, we have after some rearrangement

$$(4.1) \qquad -\langle U, x\rangle_Y - \varphi(x) \not\leq_{\mathrm{int}\,K(x)} -\langle U, y\rangle_Y - \varphi(y), \qquad \text{for every} \quad y \in X.$$

This implies, using the definition of the weak conjugate of φ, that we have

$$(4.2) \quad -\langle U, x\rangle_Y - \varphi(x) \in \mathrm{WMax}\big(\big\{-\langle U, y\rangle_Y - \varphi(y) \mid y \in X\big\}, K(\cdot)\big) = \varphi^*(-U).$$

Suppose to the contrary that the pair $(U_1, x_1) := (-U, x) \in \mathrm{Gph}(\mathcal{F}^{-1}(-\cdot))$ does not solve (IVVI_1). Hence, there exists an operator $V_0 \in \mathrm{L}(X,Y)$ such that

$$\begin{aligned} -\langle V_0, x\rangle_Y - \langle U, x\rangle_Y &= \langle V_0 - U_1, -x_1\rangle_Y \\ &\preccurlyeq^1_{\mathrm{int}\,K(x)} \varphi^*(U_1) - \varphi^*(V_0) = \varphi^*(-U) - \varphi^*(V_0), \end{aligned}$$

which is in view of the definition of the set relation $\preccurlyeq^1_{\mathrm{int}\,K(x)}$ equivalent to

$$(4.3) \qquad -\langle V_0, x\rangle_Y - \langle U, x\rangle_Y \preccurlyeq^1_{\mathrm{int}\,K(x)} v - \varphi^*(V_0), \qquad \text{for every} \quad v \in \varphi^*(-U).$$

Since (4.3) holds in particular for the element $-\langle U, x\rangle_Y - \varphi(x) \in \varphi^*(-U)$, compare relation (4.2), we have after some rearrangement

$$\varphi(x) - \langle V_0, x\rangle_Y \preccurlyeq^1_{\operatorname{int} K(x)} -\varphi^*(V_0),$$

or equivalently

$$\varphi(x) - \langle V_0, x\rangle_Y \preccurlyeq^1_{\operatorname{int} K(x)} -w, \qquad \text{for every} \quad w \in \varphi^*(V_0). \tag{4.4}$$

Now, let $w_0 \in \varphi^*(V_0)$ be arbitrarily chosen, that is, $w_0 = \langle V_0, x_0\rangle_Y - \varphi(x_0)$ for some $x_0 \in \operatorname{dom}(\varphi)$. By the definition of the weak conjugate of φ, we conclude

$$\langle V_0, y\rangle_Y - \varphi(y) - \langle V_0, x_0\rangle_Y + \varphi(x_0) \notin \operatorname{int} K(x_0), \qquad \text{for every} \quad y \in X. \tag{4.5}$$

Since (4.4) holds in particular for the element $w_0 \in \varphi^*(V_0)$, we conclude

$$\varphi(x) - \langle V_0, x\rangle_Y \leq_{\operatorname{int} K(x)} \varphi(x_0) - \langle V_0, x_0\rangle_Y,$$

which implies in particular, using the assumption $K(x) \subseteq K(x_0)$, that

$$\langle V_0, x\rangle_Y - \varphi(x) - \langle V_0, x_0\rangle_Y + \varphi(x_0) \in \operatorname{int} K(x_0).$$

But the previous inequality leads to a contradiction if we insert the element x in inequality (4.5). This shows that the pair $(-U, x)$ is a solution of the inverse problem (IVVI_1).

(b). Let the pair $(U_2, x_2) \in \operatorname{Gph}(\mathcal{F}^{-1}(-\cdot))$ be a given solution of the inverse problem (IVVI_2) and put $U_2 := -U$ and $x_2 := x$. We are going to show that $(x, U) \in \operatorname{Gph}(\mathcal{F})$ is a solution of (VVI). Inserting $-U$ and x in the inverse problem (IVVI_2) yields

$$-\langle U, x\rangle_Y - \langle V, x\rangle_Y \not\preccurlyeq^2_{\operatorname{int} K(x)} \varphi^*(-U) - \varphi^*(V), \qquad \text{for every} \quad V \in \mathrm{L}(X, Y),$$

that is we have in view of the definition of the set relation $\not\preccurlyeq^2_{\operatorname{int} K(x)}$

$$-\langle U, x\rangle_Y - \langle V, x\rangle_Y \not\preccurlyeq^2_{\operatorname{int} K(x)} \varphi^*(-U) - w, \tag{4.6}$$

$$\text{for every} \quad w \in \varphi^*(V),\ V \in \mathrm{L}(X, Y). \tag{4.7}$$

Let $U_0 \in \partial\varphi(x)$ be arbitrarily chosen. Lemma 3.11 implies that we have equivalently $\langle U_0, x\rangle_Y - \varphi(x) \in \varphi^*(U_0)$. Inserting $V = U_0$ and $w = \langle U_0, x\rangle_Y - \varphi(x)$ in inequality (4.6) yields after some rearrangement

$$-\langle U, x\rangle_Y - \varphi(x) \not\preccurlyeq^2_{\operatorname{int} K(x)} \varphi^*(-U). \tag{4.8}$$

Using the definition of the set relation $\not\preccurlyeq^2_{\operatorname{int} K(x)}$, the previous inequality is equivalent to

$$-\langle U, x\rangle_Y - \varphi(x) \notin \varphi^*(-U) - \operatorname{int} K(x).$$

The weak (A, B)-domination property for the sets $\{-\langle U, y\rangle_Y - \varphi(y) \mid y \in X\}$ and $\{-\langle U, x\rangle_Y - \varphi(x)\}$ w.r.t. the fixed cone $K(x)$ implies

$$-\langle U, x\rangle_Y - \varphi(x) \in \varphi^*(-U), \tag{4.9}$$

see Remark 3.15. Now, suppose to the contrary that the pair $(x, U) = (x_2, -U_2) \in \operatorname{Gph}(\mathcal{F})$ is not a solution of (VVI). Then there exists an element $y_0 \in X$ such that

$$\langle U, y_0 - x\rangle_Y \leq_{\operatorname{int} K(x)} \varphi(x) - \varphi(y_0),$$

which is equivalent to (compare the first part of this proof)

$$-\langle U, x\rangle_Y - \varphi(x) \notin \varphi^*(-U).$$

This obviously contradicts (4.9). The proof is complete. □

Remark 4.2.

(**a**) Theorem 4.1 is motivated by the incorrect proof in [29]. We further relax the very restrictive condition $-\tilde{K} \cup \tilde{K} = Y$ in [29], where $\tilde{K}$ is a fixed cone, by the weak (A, B)-domination property of the sets A and B. Notice that the condition in [29] implies that $\tilde{K}$ is a half-space.

(**b**) The correctness of Theorem 4.1 holds, adapting the previous definitions and lemmas, if we replace $\not\leq_{\operatorname{int} K(\cdot)}$, $\not\preccurlyeq^1_{\operatorname{int} K(\cdot)}$ and $\not\preccurlyeq^2_{\operatorname{int} K(\cdot)}$ by $\not\leq_{K(\cdot)\setminus\{0\}}$, $\not\preccurlyeq^1_{K(\cdot)\setminus\{0\}}$ and $\not\preccurlyeq^2_{K(\cdot)\setminus\{0\}}$, respectively.

(**c**) The assumption $K(x) \subseteq K(y)$ for every $y \in X$ in part (a) of the previous theorem, where $x \in X$, can be rewritten as $K(x) = K_0$, where $K_0 := \bigcap_{y \in X} K(y)$.

4.2. **Inverse problems for** (VVI′) **based on a conjugate and perturbation approach.** Besides $(\mathcal{A})$, suppose now, that $\mathcal{F} : X \rightrightarrows \mathrm{L}(X, Y)$ is single-valued and denote this mapping by F. Assume further that $F : X \to \mathrm{L}(X, Y)$ is injective. Then the previous variational inequality reduces to the following problem: find an element $x \in \operatorname{dom}(F) \cap \operatorname{dom}(\varphi)$ such that

$$\text{(VVI}'\text{)} \qquad \langle Fx, y - x\rangle_Y \not\leq_{\operatorname{int} K(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X.$$

The first inverse vector variational inequality of (VVI′) w.r.t. the moving domination structure K reads as follows: find an operator $U_1 \in \operatorname{dom}(F^\#)$ such that

$$\text{(IVVI}'_1\text{)} \qquad \begin{aligned} &\langle V - U_1, -F^\# U_1\rangle_Y \not\preccurlyeq^1_{\operatorname{int} K(F^\# U_1)} \varphi^*(U_1) - \varphi^*(V), \\ &\qquad\qquad \text{for every} \quad V \in \mathrm{L}(X, Y) \text{ with } \varphi^*(V) \neq \emptyset. \end{aligned}$$

Here, $F^\#$ denotes the adjoint mapping of F, see Section 2. If we replace the binary set relation $\not\preccurlyeq^1_{\operatorname{int} K(\cdot)}$ by $\not\preccurlyeq^2_{\operatorname{int} K(\cdot)}$, then the second inverse vector variational inequality becomes: find an operator $U_2 \in \operatorname{dom}(F^\#)$ such that

$$\text{(IVVI}'_2\text{)} \qquad \begin{aligned} &\langle V - U_2, -F^\# U_2\rangle_Y \not\preccurlyeq^2_{\operatorname{int} K(F^\# U_2)} \varphi^*(U_2) - \varphi^*(V), \\ &\qquad\qquad \text{for every} \quad V \in \mathrm{L}(X, Y). \end{aligned}$$

We further have the following result.

Theorem 4.3 (Direct and converse assertion for (VVI′))**.** *Besides* $(\mathcal{A})$*, let* $F : X \to \mathrm{L}(X, Y)$ *be injective and* $\varphi : X \to Y \cup \{+\infty_Y\}$ *subdifferentiable w.r.t. the moving domination structure* K*. Then, the following statements hold:*

(**a**) *If* $x \in \operatorname{dom}(F) \cap \operatorname{dom}(\varphi)$ *is a solution of* (VVI′) *and we have*

$$K(x) \subseteq K(y), \qquad \text{for every} \quad y \in X,$$

then $-Fx \in \operatorname{dom}(F^\#)$ *solves* (IVVI′$_1$).

(**b**) *Conversely, if* $U_2 \in \mathrm{dom}(F^{\#})$ *is a solution of* (IVVI'_2)*, it holds* $x \in \mathrm{dom}(F) \cap \mathrm{dom}(\varphi)$*,* $\partial\varphi(x) \neq \emptyset$ *and the sets*

$$A := \big\{\langle -Fx, y\rangle_Y - \varphi(y) \mid y \in X\big\} \quad \textit{and} \quad B := \big\{ -\langle Fx, x\rangle_Y - \varphi(x)\big\}$$

satisfy the weak (A, B)*-domination property w.r.t. the cone* $K(x)$*, where* $x := F^{\#}U_2$*, then* x *solves* (VVI')*.*

Proof. The proof follows from Theorem 4.1. For (a) notice that $F^{\#}U_1 = F^{-1}Fx = x$ if we let $U_1 = -Fx$. Conversely, notice for (b) that $x = F^{\#}U_2 = F^{-1}(-U_2)$ implies $U_2 = -Fx$. □

Remark 4.4.

(**a**) The correctness of Theorem 4.3 holds, adapting the previous definitions and lemmas, if we replace $\not\leq_{\mathrm{int}\,K(\cdot)}$, $\not\preccurlyeq^1_{\mathrm{int}\,K(\cdot)}$ and $\not\preccurlyeq^2_{\mathrm{int}\,K(\cdot)}$ by $\not\leq_{K(\cdot)\setminus\{0\}}$, $\not\preccurlyeq^1_{K(\cdot)\setminus\{0\}}$ and $\not\preccurlyeq^2_{K(\cdot)\setminus\{0\}}$, respectively.

(**b**) If we let $Y := \mathbb{R}$ and $K(x) := \mathbb{R}_{\geq 0}$ for every $x \in X$, then (VVI') becomes the following (scalar) variational inequality: find $x \in \mathrm{dom}(F) \cap \mathrm{dom}(\varphi)$ such that

$$\langle Fx, y - x\rangle \geq \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in X,$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between X^* and X. Further, both the inverse problems (IVVI'_1) and (IVVI'_2) coincide to the (scalar) inverse problem: find a functional $u^* \in \mathrm{dom}(F^{\#})$ such that

$$\langle v^* - u^*, F^{\#}u^*\rangle \geq \varphi^*(u^*) - \varphi^*(v^*), \qquad \text{for every} \quad v^* \in X^*.$$

Here, φ^* denotes the well-known Fenchel conjugate of $\varphi : X \to \mathbb{R} \cup \{+\infty\}$, see [2]. By this special choice of the data, the previous theorem recovers the results for scalar variational inequalities in [15, Theorem 9.3.1], [24, Theorem 1].

Let the assumption $(\mathcal{A})$ be fulfilled, let $F : X \to \mathrm{L}(X, Y)$ be an injective mapping and $\varphi : X \to Y \cup \{+\infty_Y\}$. We define a *perturbation mapping* $\Psi : X \times X \times X \to Y \cup \{+\infty_Y\}$ such that

$$\Psi(x, y, 0) = \langle Fx, y\rangle_Y + \varphi(y), \qquad \text{for every} \quad x, y \in X.$$

In this way one can embed the problem (VVI') into a family of so-called *perturbed vector optimization problems* (VVI'_z) which are to find $x \in \mathrm{dom}(F) \cap \mathrm{dom}(\varphi)$ such that

$$(\mathrm{VVI}'_z) \qquad \Psi(x, x, z) \not\geq_{\mathrm{int}\,K(x)} \Psi(x, y, z), \qquad \text{for every} \quad y \in X,$$

for fixed $z \in X$. Obviously, if we let $z = 0$, then (VVI_0) is equivalent to the primal problem (VVI') since (VVI_0) then reads: find $x \in \mathrm{dom}(F) \cap \mathrm{dom}(\varphi)$ such that

$$\langle Fx, x\rangle_Y + \varphi(x) \not\geq_{\mathrm{int}\,K(x)} \langle Fx, y\rangle_Y + \varphi(y), \qquad \text{for every} \quad y \in X.$$

To formulate the inverse problem for (VVI'_z), we need the following definition.

Definition 4.5. Let $(\mathcal{A})$ hold. The *weak conjugate* of the perturbation mapping Ψ w.r.t. the moving domination structure K is the set-valued mapping $\Psi^* : \mathrm{L}(X,Y)\times \mathrm{L}(X,Y)\times \mathrm{L}(X,Y) \rightrightarrows Y$, where we have $\langle U,x\rangle_Y + \langle V,y\rangle_Y + \langle W,z\rangle_Y - \Psi(x,y,z) \in \Psi^*(U,V,W)$ for some $x^0,y^0,z^0 \in \mathrm{dom}(\varphi)$ and $U,V,W\in \mathrm{L}(X,Y)$ if and only if for every $x,y,z\in X$ it holds that

$$\langle U,x^0\rangle_Y + \langle V,y^0\rangle_Y + \langle W,z^0\rangle_Y - \Psi(x^0,y^0,z^0) \not\leq_{\mathrm{int}\,K(x^0)} \langle U,x\rangle_Y + \langle V,y\rangle_Y + \langle W,z\rangle_Y - \Psi(x,y,z).$$

Notice that the cone $K(x^0)$ depends on the element x^0 only.

Using this notation, we can state the following *inverse vector optimization problem* for (VVI$'_z$): find an operator $W_0 \in \mathrm{dom}(F^\#)$ such that

$$-\Psi^*(0,0,W_0)\cap \mathrm{WMax}\left(\bigcup_{W\in \mathrm{L}(X,Y)} -\Psi^*(0,0,W),\ K(F^\# W_0)\right) \neq \emptyset. \tag{4.10}$$

Obviously, $W_0\in \mathrm{L}(X,Y)$ is a solution of (4.10) if there are $x^0,y^0,z^0\in\mathrm{dom}(\varphi)$ and $W\in \mathrm{L}(X,Y)$ such that for every $x,y,z\in X$ it holds that

$$\langle W_0,z^0\rangle_Y - \Psi(x^0,y^0,z^0) \not\leq_{\mathrm{int}\,K(F^\# W_0)} \langle W,z\rangle_Y - \Psi(x,y,z). \tag{4.11}$$

Theorem 4.6 (Weak relationship between (VVI$'_z$) and (4.10))**.** *Besides* $(\mathcal{A})$*, let* $F: X\to \mathrm{L}(X,Y)$ *be injective and* $\varphi : X\to Y\cup\{+\infty_Y\}$*. Then, there exists* $\bar{x}\in X$ *such that*

$$\Psi^*(0,0,W) \not\leq^1_{\mathrm{int}\,K(\bar{x})} -\Psi(x,x,0), \qquad \textit{for every}\quad x\in\mathrm{dom}(\varphi),\ W\in \mathrm{L}(X,Y).$$

Proof. Let $x\in X$ and $W\in \mathrm{L}(X,Y)$ be arbitrarily chosen. Let $\langle W,\bar{z}\rangle_Y - \Psi(\bar{x},\bar{y},\bar{z}) \in \Psi^*(0,0,W)$ for some $\bar{x},\bar{y},\bar{z}\in X$. By the definition of the weak conjugate of Ψ, we have for every $x',y',z'\in X$

$$\langle W,\bar{z}\rangle_Y - \Psi(\bar{x},\bar{y},\bar{z}) \not\leq_{\mathrm{int}\,K(\bar{x})} \langle W,z'\rangle_Y - \Psi(x',y',z').$$

Inserting $x'=x$, $y'=x$ and $z'=0$ finishes the proof. □

Theorem 4.7 (Strong relationship between (VVI$'$) and (4.10))**.** *Besides* $(\mathcal{A})$*, let* $F: X\to \mathrm{L}(X,Y)$ *be injective and* $\varphi : X\to Y\cup\{+\infty_Y\}$*. Assume we have*

$$-\Psi(x,x,0)\in \Psi^*(0,0,-Fx) \tag{4.12}$$

for some $x\in\mathrm{dom}(\varphi)$*. Then,* x *is a solution of* (VVI$'$) *and* $-Fx\in \mathrm{L}(X,Y)$ *is a solution of* (4.10)*.*

Proof. By (4.12), using the definition of $\Psi^*(0,0,-Fx)$, we have

$$-\Psi(x,x,0) \not\leq_{\mathrm{int}\,K(x)} -\langle Fx,z'\rangle_Y - \Psi(x',y',z'), \qquad \text{for every}\quad x',y',z'\in X.$$

Inserting $x'=x$ and $z'=0$ yields $-\Psi(x,x,0)\not\leq_{\mathrm{int}\,K(x)} -\Psi(x,y',0)$ for every $y'\in X$ and consequently x solves (VVI$'$). Now, let us put $W_0=-Fx$. By this, we have $F^\# W_0 = F^{-1}Fx = x$ and inserting this element in (4.12) yields

$$-\Psi(F^\# W_0, F^\# W_0, 0) \not\leq_{\mathrm{int}\,K(F^\# W_0)} \langle W_0, z'\rangle_Y - \Psi(x',y',z')$$

for every $x', y', z' \in X$. Carefully taking into account (4.11), where we put $x = F^{\#}W_0$, $y = x$, $z = 0$ and $W = W_0$, we have that $W_0 = -Fx$ solves (4.10). The proof is complete. □

Remark 4.8. The previous results did not use any properties for the moving domination structure K.

5. Applications

The purpose of this section is to apply the inverse results of Section 4 to two vector approximation problems.

5.1. **Application to a vector control approximation problem with respect to a variable domination structure.** In this section, we focus on an application of the previous results for (VVI′). For this purpose, we consider a vector control approximation problem w.r.t. a moving domination structure K. One of the basic tools will be Theorem 3.29 which allows us to consider a variational problem instead, such that we can apply the previous results and derive new existence statements for finite-dimensional vector control approximation problems w.r.t. a moving domination structure.
To be precise, let X and Y be Euclidean spaces, that is, $X := \mathbb{R}^n$ and $Y := \mathbb{R}^m$, where $n, m \in \mathbb{N}$. We study the problem of determining elements of the set of weakly minimal elements w.r.t. the moving domination structure K, that is, we want to compute elements of the set

$$\mathrm{WMin}(\psi[\mathbb{R}^n], K(\cdot)), \tag{5.1}$$

where the objective vector mapping $\psi : \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$\psi(x) := \begin{pmatrix} \frac{1}{2}\|x - a^1\|_2^2 \\ \vdots \\ \frac{1}{2}\|x - a^m\|_2^2 \end{pmatrix}, \qquad \text{for every} \quad x \in \mathbb{R}^n.$$

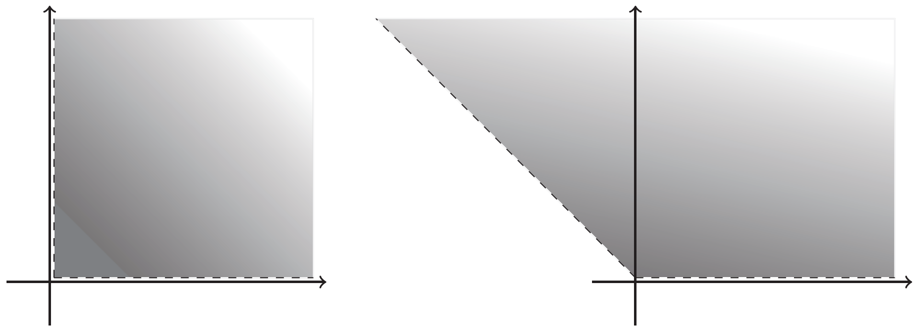
Here, $a^1, \dots, a^m$ are m given different points in $\mathbb{R}^n$ and $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^n$. The scalar product in $\mathbb{R}^n$ will be denoted by $\langle \cdot, \cdot \rangle$. The moving domination structure is given by the set-valued mapping $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ which is for every $x \in \mathbb{R}^n$ defined by

$$K(x) := \begin{cases} \mathbb{R}^m_{\geq 0} & \text{for } x_1 > 0, \\ \left\{ y \in \mathbb{R}^m \mid \sum_{j=1}^{m} y_j \geq 0 \text{ and } y_j \geq 0 \text{ for } j = 2, \dots, m \right\} & \text{else.} \end{cases} \tag{5.2}$$

In order to generate a reduced solution set of (5.1) corresponding to the preferences of the decision maker, one can chose larger (in the sense of inclusion) ordering cones for elements in certain regions of $\mathbb{R}^m$. It obviously holds that for every $x \in \mathbb{R}^n$, $K(x)$ is a proper, closed, convex and pointed cone in $\mathbb{R}^m$ with non-empty interior. Let us further define the set

$$\Lambda^n_{>0} := \{x \in \mathbb{R}^n \mid x_1 > 0\}.$$

Since ψ is $\mathbb{R}^m_{\geq 0}$-convex, see Remark 3.13, and the Gâteaux-derivative of the real-valued component function $x \mapsto \psi_i(x) := \|x - a^i\|_2^2 = \langle x - a^i, x - a^i \rangle$ at $x \in \mathbb{R}^n$



(a) $\operatorname{int} K(x)$ for $x_1 > 0$ (b) $\operatorname{int} K(x)$ for $x_1 \leq 0$

FIGURE 6. Illustration of the moving domination structure K for $m = 2$

is given by $D_G^+ \psi_i\, x = \langle x - a^i, \cdot\rangle$ for every $i = 1, \dots, m$, Theorem 3.29 implies that every solution $x \in \mathbb{R}^n$ of (5.1), which belongs to $\Lambda^n_{>0}$, fulfills the following vector variational inequality and vice versa:

$$\text{(5.3)}\quad \langle D_G^+ \psi\, x, y - x\rangle_{\mathbb{R}^m} = \begin{pmatrix} \langle x - a^1, y - x\rangle \\ \vdots \\ \langle x - a^m, y - x\rangle \end{pmatrix} \not\leq_{\operatorname{int} K(x)} 0, \qquad \text{for every} \quad y \in \mathbb{R}^n.$$

For further use, let define the set

$$\operatorname{conv}_{>0} := \operatorname{conv}\{a^1, \dots, a^m\} \cap \Lambda^n_{>0},$$

where conv denotes the convex hull.

Lemma 5.1. *Every element in* $\operatorname{conv}_{>0}$ *solves* (5.1) *and* (5.3)*, respectively.*

Proof. Let $x \in \operatorname{conv}_{>0}$ be arbitrarily chosen, that is, $x = \sum_{i=1}^m \lambda_i a^i$, where $\lambda_i \geq 0$ for $i = 1, \dots, m$ and $\sum_{i=1}^m \lambda_i = 1$. Suppose to the contrary that x does not solve (5.3), that is, there is $y_0 \in \mathbb{R}^n$ such that

$$\text{(5.4)}\qquad \langle D_G^+ \psi\, x, y_0 - x\rangle_{\mathbb{R}^m} \in -\operatorname{int} K(x)$$

Since $K(x) = \mathbb{R}^m_{\geq 0}$, inequality (5.4) further implies $\langle x - a^i, y_0 - x\rangle < 0$ for $i = 1, \dots, m$. Multiplying every inequality with $\lambda_i \geq 0$ and adding them yields to

$$0 = \sum_{i=1}^m \lambda_i \langle x, y_0 - x\rangle - \langle a^i, y_0 - x\rangle = \sum_{i=1}^m \lambda_i \langle x - a^i, y_0 - x\rangle < 0,$$

using the fact that $\sum_{i=1}^m \lambda_i = 1$. The previous inequality is obviously a contradiction. The proof is complete. □

Next, we are going to construct the inverse problem for (5.1) and (5.3), respectively. Using the identification $\mathrm{L}(\mathbb{R}^n, \mathbb{R}^m) \cong \mathrm{Mat}_{m\times n}(\mathbb{R})$, we define mappings $F : \mathbb{R}^n \to \mathrm{Mat}_{m\times n}(\mathbb{R})$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ by

$$Fx := (x, \dots, x)^\top \qquad \text{and} \qquad \varphi(x) := Ax, \qquad \text{for every} \quad x \in \mathbb{R}^n,$$

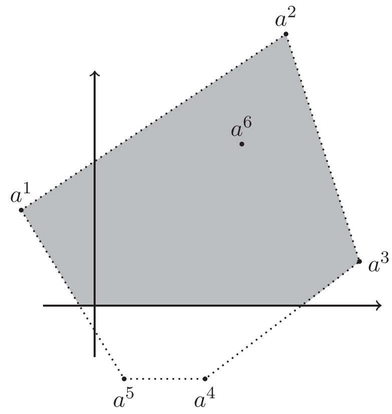


FIGURE 7. $\operatorname{conv}\{a^1,\dots,a^6\}$ (dotted) and $\operatorname{conv}_{>0}$ (gray) for $n=2$ and $m=6$

where $A := -(a^1,\dots,a^m)^\top \in \operatorname{Mat}_{m\times n}(\mathbb{R})$. Notice that $\operatorname{dom}(F) = \operatorname{dom}(\varphi) = \mathbb{R}^n$. Using this notation, the variational problem (5.3) becomes: find an element $x \in \mathbb{R}^n$ such that

$$\langle Fx, y-x\rangle \not\leq_{\operatorname{int} K(x)} \varphi(x) - \varphi(y), \qquad \text{for every} \quad y \in \mathbb{R}^n.$$

In the following, we use the notation $B = (\!|b|\!)$ instead of $B = (b,\dots,b)^\top$ for a matrix $B \in \operatorname{Mat}_{m\times n}(\mathbb{R})$ generated by $b \in \mathbb{R}^n$. Since F is linear and injective, the adjoint mapping $F^\# : \operatorname{Mat}_{m\times n}(\mathbb{R}) \to \mathbb{R}^n$ becomes $F^\# B = F^{-1}(-B) = -F^{-1}B = -b$ for every matrix $B = (\!|b|\!) \in \operatorname{Mat}_{m\times n}(\mathbb{R})$.
Using all the previous notations, the first inverse vector variational inequality for (5.1), respectively (5.3) is: find a matrix $-(\!|u|\!) \in \operatorname{Mat}_{m\times n}(\mathbb{R})$ such that

$$(V + (\!|u|\!))(-u) \not\leq^1_{\operatorname{int} K(u)} \varphi^*(-(\!|u|\!)) - \varphi^*(V), \qquad \text{for every} \quad V \in \operatorname{Mat}_{m\times n}(\mathbb{R}) \text{ with } \varphi^*(V) \neq \emptyset. \tag{5.5}$$

The second inverse vector variational inequality is to find a matrix $-(\!|w|\!) \in \operatorname{Mat}_{m\times n}(\mathbb{R})$ such that

$$(V + (\!|w|\!))(-w) \not\leq^2_{\operatorname{int} K(w)} \varphi^*(-(\!|w|\!)) - \varphi^*(V), \qquad \text{for every} \quad V \in \operatorname{Mat}_{m\times n}(\mathbb{R}). \tag{5.6}$$

Recall that the matrices $-(\!|u|\!)$ and $-(\!|w|\!)$ correspond to U_1 and U_2, respectively, compare Theorem 4.3.

Lemma 5.2. *Let $a \in \operatorname{conv}_{>0}$ and define*

$$A := \{-\langle Fa, y\rangle_{\mathbb{R}^m} - \varphi(y) \mid y \in \mathbb{R}^n\} \quad \text{and} \quad B := \{-\langle Fa, a\rangle_{\mathbb{R}^m} - \varphi(a)\}.$$

Then, the sets A and B satisfy the weak (A,B)-domination property w.r.t. to $\mathbb{R}^m_{\geq 0}$.

Proof. One can show that it holds $-\langle Fa, a\rangle_{\mathbb{R}^m} - \varphi(a) \in \operatorname{WMax}(A, \mathbb{R}^m_{\geq 0})$ similar to the proof of Lemma 5.1. Consequently, the weak (A,B)-domination property of A and B follows and the proof is complete. □

Theorem 5.3. *Using our previous observations and notations, we have:*

(**a**) *Every matrix* $-(\!|x|\!) \in \mathrm{Mat}_{m\times n}(\mathbb{R})$*, where* $x \in \mathrm{conv}_{>0}$*, is a solution of the first inverse problem* (5.5)*.*

(**b**) *Conversely, if* $a \in \mathrm{conv}_{>0} \setminus \{a^1, \dots, a^m\}$ *and* $\varphi^*(-(\!|a|\!)) = \emptyset$*, then the element* a *is a solution of* (5.1) *and* (5.3)*, respectively.*

Proof. The statements (a) and (b) follow from Theorem 4.3 and the previous observations. Recall that the mapping $F : \mathbb{R}^n \to \mathrm{Mat}_{m\times n}(\mathbb{R})$ is injective and $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ is subdifferentiable w.r.t. the moving domination structure given by (5.2). For the subdifferentiability of φ notice that for every $x \in \mathbb{R}^n$, it holds $A \in \partial\varphi(x)$, where $A = -(a^1, \dots, a^m)^\top$, see Example 3.10.

(a) Let $x \in \mathrm{conv}_{>0}$ be arbitrarily chosen. In particular, we have $x_1 > 0$ such that from the definition of K we conclude $K(x) \subseteq K(y)$ for every $y \in \mathbb{R}^n$. Since x solves (5.1) and (5.3), respectively, compare Lemma 5.1, Theorem 4.3 (a) states that the matrix $-Fx = -(\!|x|\!)$ solves the first inverse problem (5.5). This shows the first part of this theorem.

(b) Let $a \in \mathrm{conv}_{>0} \setminus \{a^1, \dots, a^m\}$ such that $\varphi^*(-(\!|a|\!)) = \emptyset$. We are going to show that the matrix $-(\!|a|\!) \in \mathrm{dom}(F^\#)$ is a solution of the second inverse vector variational inequality (5.6). Using the convention for the Minkowski difference of empty sets, we conclude $\varphi^*(-(\!|a|\!)) - \varphi^*(V) = \emptyset - \varphi^*(V) = \emptyset$ for every $V \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. Since it holds $(V + (\!|a|\!))(-a) \not\prec^2_{\mathrm{int}\,K(a)} \emptyset$ for every $V \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, see Section 2, the matrix $-(\!|a|\!) \in \mathrm{dom}(F^\#)$ solves (5.6). Finally, it holds $F^\#(-(\!|a|\!)) = F^{-1}(\!|a|\!) = a$ and the sets A and B satisfy the weak (A, B)-domination property w.r.t. $K(a) = \mathbb{R}^m_{\geq 0}$, see Lemma 5.2. Applying Theorem 4.3 and 3.29, we see that a solves (5.1) and (5.3). The proof is complete. □

Remark 5.4. The weak conjugate of the negative vertex matrix $-(\!|a^i|\!) \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ is non-empty for $i = 1, \dots, m$. Indeed fix $i \in \{1, \dots, m\}$ and let $x \in \Lambda^n_{>0}$ be arbitrarily chosen. Then, the inequality $-(\!|a^i|\!)x - \varphi(x) \not\leq_{\mathrm{int}\,K(x)} -(\!|a^i|\!)y - \varphi(y)$ for every $y \in \mathbb{R}^n$ holds since it is equivalent to

$$\begin{pmatrix} \langle a^i - a^1, y - x\rangle \\ \vdots \\ \langle a^i - a^m, y - x\rangle \end{pmatrix} \notin \mathbb{R}^m_{>0}, \qquad \text{for every} \quad y \in \mathbb{R}^n.$$

Notice that the ith component of the left hand side is zero.

5.2. **Application to a beam intensity optimization problem in radiotherapy treatment.** As a second application of our results, we present a vector optimization problem which arises in radio therapy treatment, see [8] and [21]. The intensity modulated radiotherapy treatment (IMRT) is currently used to treat cancer in prostate, head and neck, breast and many others, see [11], [12] and [21]. The main idea of IMRT is to apply to the patient a suitable radiation dose, that is, the intensity of rays going through sensitive critical structures is reduced while the dose in the infected structures is increased.

In [21], this problem is considered as a vector optimization problem with respect to a variable domination structure, which describes the dose of the beam intensity. To this aim, a threshold vector $\theta \in \mathbb{R}^n$ is given, where $\theta_1 := 0$ and every component θ_i, $i = 2, \dots, n$ is defined as the dose of radiation, below which the organism i

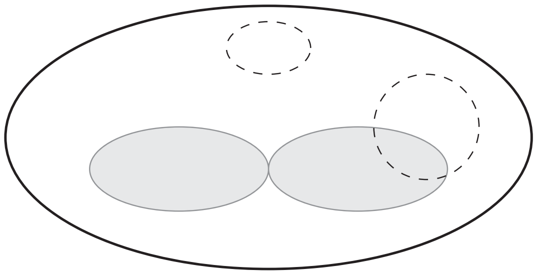


FIGURE 8. Schematic axial body cut: lunge cancer (gray) and critical organs spinal cord and heart (dashed)

does not suffer from any effect. In the following, we assume that the dose delivered to the tumor organ is given by $A_T x$, where $A_T \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is the regular dose deposition matrix and $x \in \mathbb{R}^n$ is the beam intensity. The dose delivered to the $n-1$ critical organs $C_1, \dots, C_{n-1}$ is given by $A_{C_1}x, \dots, A_{C_{n-1}}x$, where $A_{C_i} \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ are given regular matrices for $i = 1, \dots, n-1$. Notice that in [17] the author claims that the Moore–Penrose generalized inverse of the involved matrices exist. The composite matrix $A := (A_T, A_{C_1}, \dots, A_{C_{n-1}})^\top \in \mathrm{Mat}_{n^2\times n}$ is called the dose deposition matrix and we have the following relationship

$$d = Ax,$$

where $d \in \mathbb{R}^{n^2}$ is a dose vector. Since different tissues tolerate different amounts of radiation, the radiation oncologist needs to determine a target dose $a^{\mathrm{tar}} \in \mathbb{R}^n$ for the tumor, lower and upper bounds to tumor voxels $\underline{\alpha}, \overline{\alpha} \in \mathbb{R}^n$ and upper bounds on the dose to normal voxels which are divided into $\overline{\beta}_i \in \mathbb{R}^n$ for $i = 1, \dots, n-1$. The variable domination structure of this problem is constructed by using the following [21] practical perspective: the dose delivered to a critical organ i should be reduced when it exceeds the threshold θ_i. If not, one can increase this dose in favor of an improvement in the value of another critical organ. To precisely describe the variable domination structure $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, let the threshold vector $\theta \in \mathbb{R}^n$ be given and put $I^>(x) := \{i \in \{1, \dots, n\} \mid x_i > \theta_i\}$ for every $x \in \mathbb{R}^n$. The variable domination structure is for every $x \in \mathbb{R}^n$ defined by

$$K(x) := \{y \in \mathbb{R}^n \mid y_i \geq 0 \text{ for } i \in I^>(x)\}. \tag{5.7}$$

The next lemma states important properties of the domination structure given by (5.7), see [21].

Lemma 5.5. *Consider the variable domination structure $K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ determined by* (5.7)*. Then we have:*

(**a**) *For every $x \in \mathbb{R}^n$, $K(x)$ is a closed and convex cone in $\mathbb{R}^n$ with non-empty interior which satisfies $\mathbb{R}^n_{\geq 0} \subseteq K(x)$.*

(**b**) *The cone $K(x)$ is pointed if and only if $x_i > \theta_i$ for every $i = 1 \dots, n$.*

In order to describe the vector optimization problem, we need the convex set of bound conditions for beam intensity given by

$$C := \{x \in \mathbb{R}^n \mid 0 \leq_{\mathbb{R}^n_{\geq 0}\setminus\{0\}} x,\ \underline{\alpha} \leq_{\mathbb{R}^n_{\geq 0}\setminus\{0\}} A_T x \leq_{\mathbb{R}^n_{\geq 0}\setminus\{0\}} \overline{\alpha}$$

$$\text{and } A_{C_i}x \leq_{\mathbb{R}^n_{\geq 0}\setminus\{0\}} \overline{\beta}_i \text{ for } i = 1, \dots, n-1\}.$$

By using the variable domination structure given by (5.7), the problem of finding beam intensity in radiotherapy treatment can now get formulated as the vector optimization problem

$$\text{WMin}(\psi[C], K(\cdot)), \tag{5.8}$$

where the objective vector mapping $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is given by

$$\psi(x) := \begin{pmatrix} \frac{1}{2}\|A_T x - a^{\text{tar}}\|_2^2 \\ \frac{1}{2}\|A_{C_1}x\|_2^2 \\ \vdots \\ \frac{1}{2}\|A_{C_{n-1}}x\|_2^2 \end{pmatrix}, \qquad \text{for every} \quad x \in \mathbb{R}^n.$$

Recall that $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^n$. Here, the first component of ψ, that is $\frac{1}{2}\|A_T \cdot - a^{\text{tar}}\|_2^2$, can be interpreted as the deviation from the prescribed dose to the dose delivered to tumor, while $\frac{1}{2}\|A_{C_i}\cdot\|_2^2$ is the average dose delivered to the critical organ $i \in \{2, \dots, n\}$. In the following, we are going to rewrite (5.8) as a vector variational inequality with respect to the domination structure given by (5.7), such that we can apply the inverse assertions of Section 4. Calculating the (right-hand sided) Gâteaux-derivative of ψ and using Theorem 3.29, we have: if $\psi(x)$ solves (5.8) where $x \in C$ and $K(x) = \mathbb{R}^n_{\geq 0}$, then x is a solution of the following vector variational inequality, and vice versa: find an element $x \in C$ such that

$$\langle D^+_G\psi\, x, y - x\rangle_{\mathbb{R}^n} = \begin{pmatrix} \langle A_T x - a^{\text{tar}}, A_T(y-x)\rangle \\ \langle A_{C_1}x, A_{C_1}(y-x)\rangle \\ \vdots \\ \langle A_{C_{n-1}}x, A_{C_{n-1}}(y-x)\rangle \end{pmatrix} \not\leq_{\text{int}\, K(x)} 0, \tag{5.9}$$
$$\text{for every} \quad y \in C.$$

Now, we use the indicator mapping $\chi_C : \mathbb{R}^n \to \mathbb{R}^n \cup \{+\infty_{\mathbb{R}^n}\}$ defined by C, that is, $\chi_C(x) := 0$ for $x \in C$ and $\chi_C(x) := +\infty_{\mathbb{R}^n}$ else. Hence, we can rewrite (5.9) in the following way: find an element $x \in \mathbb{R}^n$ such that

$$\langle D^+_G\psi\, x, y - x\rangle_{\mathbb{R}^n} \not\leq_{\text{int}\, K(x)} \chi_C(x) - \chi_C(y), \qquad \text{for every} \quad y \in \mathbb{R}^n. \tag{5.10}$$

Notice that $D^+_G : \mathbb{R}^n \to \text{Mat}_{n\times n}(\mathbb{R})$ is injective and χ_C is subdifferentiable with respect to the variable domination structure given by (5.7). In order to derive inverse assertions for (5.10), we let $D^\#_G\psi := (D^+_G\psi)^\#$ and introduce the two following inverse problems: find a matrix $U_1 \in \text{dom}(D^\#_G\psi)$ such that

$$\langle V - U_1, -D^\#_G\psi\, U_1\rangle_Y \not\preccurlyeq^1_{\text{int}\, K(D^\#_G\psi\, U_1)} \chi^*_C(U_1) - \chi^*_C(V), \tag{5.11}$$
$$\text{for every} \quad V \in \text{Mat}_{n\times n}(\mathbb{R}) \text{ with } \varphi^*(V) \neq \emptyset.$$

Replacing the binary set relation $\not\preccurlyeq^1_{\text{int}\, K(\cdot)}$ by $\not\preccurlyeq^2_{\text{int}\, K(\cdot)}$, the second inverse vector variational inequality is to find a matrix $U_2 \in \text{dom}(D^\#_G\psi)$ such that

$$\langle V - U_2, -D^\#_G\psi\, U_2\rangle_Y \not\preccurlyeq^2_{\text{int}\, K(D^\#_G\psi\, U_2)} \chi^*_C(U_2) - \chi^*_C(V), \tag{5.12}$$
$$\text{for every} \quad V \in \text{Mat}_{n\times n}(\mathbb{R}).$$

Recall that χ_C^* denotes the weak conjugate of χ_C, see Definition 3.9. Let us put

$$C^\theta := C \cap \{x \in \mathbb{R}^n \mid x_i > \theta_i \text{ for } i = 1, \ldots, n\}.$$

Theorem 5.6. *Using our previous observations and notations, we have:*

(**a**) *If* $x \in C^\theta$ *is a solution of* (5.10), *then the matrix* $-D_G^+\psi\, x \in \mathrm{dom}(D_G^{\#}\psi)$ *is a solution of the first inverse problem* (5.11).

(**b**) *Conversely, if* $U_2 \in \mathrm{dom}(D_G^{\#}\psi)$ *is a solution of* (5.12), *it holds* $x \in C^\theta$ *and the sets*

$$A := \big\{\langle -D_G^+\psi\, x, y\rangle_Y - \chi_C(y) \mid y \in X\big\}$$
$$\text{and} \quad B := \big\{ -\langle D_G^+\psi\, x, x\rangle_Y - \chi_C(x)\big\}$$

satisfy the weak (A, B)*-domination property w.r.t.* $K(x)$*, where* $x := D_G^{\#}\psi\, U_2$*, then* $-D_G^{\#}\psi\, U_2 \in \mathrm{dom}(D_G^+)$ *is a solution of* (5.8) *and* (5.10)*, respectively.*

Proof. The proof follows from the Theorems 3.29 and 4.3. □

6. Comparison of dual and inverse assertions for vector optimization problems with respect to a fixed domination structure

Having a vector (minimum) optimization problem with fixed domination structure one can attach to it a conjugate dual (maximum) problem by using the so-called perturbation or conjugate approach. The main task then is to investigate existence of weak, strong and, sometimes, converse duality to connect both problems. The main reason to look for duality assertions is that one can conclude from the existence of solutions of the primal problem that the dual problem has a solution, and vice versa, or to get a lower bound for the solutions.
In the following, we will present the dual conjugate approach [2] for a vector optimization problem with respect to a fixed domination structure. To be precise, let X, Y and Z be real Banach spaces, let $\tilde{K} \subseteq Y$ be a convex cone with non-empty interior and $\psi : X \to Y$ a given mapping. The space Z can be interpreted as a parameter space. The main objective is to calculate the set of weakly minimal elements (primal problem)

$$\text{(P)} \qquad \mathrm{WMin}(\psi[X], \tilde{K}).$$

Let $\Psi : X \times Z \to Y$ be a so-called perturbation mapping such that $\Psi(x, 0) = \psi(x)$ for every $x \in X$. A widely used perturbation mapping in the framework of (constrained) vector optimization problems is $\Psi(x, z) := \psi(x + z)$ for $x \in X$ and $z \in Z$, see [2]. Further, define a set-valued mapping $\Phi : Z \rightrightarrows Y$ by $\Phi(z) := \mathrm{WMin}(\Psi[X, z], \tilde{K})$, where $\Psi[X, z] := \{\Psi(x, z) \in Y \mid x \in X\}$, for every $z \in Z$. Clearly, it holds $\Phi(0) = \mathrm{WMin}(\psi[X], \tilde{K})$. In order to derive duality results for (P), one considers the following equivalent vector optimization problem instead:

$$\text{(P)} \qquad \mathrm{WMin}(\Psi[X, 0], \tilde{K}).$$

To introduce the dual problem for (P), we need the definition of the weak conjugate of $\Psi : X \times Z \to Y$, which is the set-valued mapping $\Psi^* : \mathrm{L}(X, Y) \times \mathrm{L}(Z, Y) \rightrightarrows Y$ defined by $\Psi^*(U, V) := \mathrm{WMin}(\{\langle U, x\rangle_Y + \langle V, z\rangle_Y - \Psi(x, z) \mid x \in X \text{ and } z \in Z\}, \tilde{K})$

for $U \in \mathrm{L}(X,Y)$ and $V \in \mathrm{L}(Z,Y)$. Then, the conjugate dual problem is to find an operator $V^0 \in \mathrm{L}(Z,Y)$ such that

$$-\Psi^*(0,V^0) \cap \mathrm{WMax}\left(\bigcup_{V\in \mathrm{L}(Z,Y)} -\Psi^*(0,V), \tilde{K}\right) \neq \emptyset,$$

which will be formally written as

$$\text{(D)} \qquad \mathrm{WMax}(-\Psi^*[0,\mathrm{L}(Z,Y)],\tilde{K}).$$

As we have seen in Theorem 3.29, the primal problem (P) is closely related to the following vector variational inequality with respect to a fixed domination structure provided the mapping $\psi : X \to Y$ is right-handed Gâteaux-differentiable and $\tilde{K}$-convex: find an element $x \in X$ such that

$$\text{(P}'\text{)} \qquad \langle D^+_G\psi\, x, y-x\rangle_Y \notin -\operatorname{int}\tilde{K}, \qquad \text{for every} \quad y \in X.$$

Recall that $D^+_G\psi$ is a mapping from X to $\mathrm{L}(X,Y)$. In the following, to simplify the notation, we will put $D^\#_G\psi := (D^+_G\psi)^\#$ and $\varphi := 0$ denotes the zero-mapping from X to Y. In order to apply the results of the previous sections, we assume that $D^+_G\psi : X \to \mathrm{L}(X,Y)$ is injective. Recall, that given an operator $V \in \mathrm{L}(X,Y)$, the weak conjugate of φ is the set-valued mapping $\varphi^* : \mathrm{L}(X,Y) \rightrightarrows Y$, defined by $\varphi^*(V) := \mathrm{WMax}(\{\langle V,x\rangle_Y - \varphi(x) \mid x \in X\},\tilde{K})$ for every $V \in \mathrm{L}(X,Y)$. The first associated inverse problem to (P′) is to find an operator $U_1 \in \operatorname{dom}(D^\#_G)$ such that

$$\text{(D}'_1\text{)} \qquad \begin{aligned}&\langle V-U_1, -D^\#_G U_1\rangle_Y \not\leq^1_{\operatorname{int}\tilde{K}} \varphi^*(U_1)-\varphi^*(V),\\ &\qquad\qquad \text{for every} \quad V \in \mathrm{L}(X,Y) \text{ with } \varphi^*(V)\neq\emptyset.\end{aligned}$$

The second inverse problem is to find an operator $U_2 \in \operatorname{dom}(D^\#_G)$ such that

$$\text{(D}'_2\text{)} \qquad \langle V-U_2, -D^\#_G U_2\rangle_Y \not\leq^2_{\operatorname{int}\tilde{K}} \varphi^*(U_2)-\varphi^*(V), \qquad \text{for every} \quad V \in \mathrm{L}(X,Y).$$

In the following, we are going to present well-known duality results for (P), see [2] and [3]. Exploiting the equivalence of (P) and (P′), we will apply Theorem 4.1 to find new dual/inverse results for (P) and (P′). The following Theorem can be found in [3] and [26] for the finite-dimensional case.

Theorem 6.1 (Weak duality between (P) and (D))**.** *The problems* (P) *and* (D) *are weakly dual, that is,*

$$\Psi(x,0) \notin -\Psi^*(0,V) - \tilde{K}, \qquad \textit{for every} \quad x \in X,\ V \in \mathrm{L}(Z,Y).$$

In order to derive strong (direct and converse) duality results for (P), the following so-called domination property [28] (stability property [3]) has been used: It holds $\psi[X] \subseteq \mathrm{WMin}(\psi[X],\tilde{K}) + \tilde{K}$. This property is further crucial to derive duality results for (P) using a Lagrangian mapping, see [16]. The following theorem can be found in [3] and [26] for the finite-dimensional case.

Theorem 6.2 (Strong direct and converse duality between (P) and (D))**.** *Assume for every $z \in Z$ it holds*

$$\Psi[X,z] \subseteq \mathrm{WMin}(\Psi[X,z],\tilde{K}) + \tilde{K}.$$

Then we have:

(**a**) *The set-valued mapping* $\Phi : Z \rightrightarrows Y$*, defined before, is subdifferentiable at* 0 *if and only if, for each solution* $\psi(x) \in Y$ *of* (P)*, where* $x \in X$*, there exists a solution* $V^0 \in \mathrm{L}(Z,Y)$ *of* (D) *such that*

$$\Psi(x,0) \in -\Psi^*(0,V^0). \tag{6.1}$$

(**b**) *Conversely, if* $x \in X$ *and* $V^0 \in \mathrm{L}(Z,Y)$ *satisfy* (6.1)*, then* $\Psi(x,0) = \psi(x)$ *is a solution of* (P) *and* V^0 *is a solution of* (D)*.*

Similar to the well-known strong direct and converse duality result in Theorem 6.2, one can observe that one needs a modified weak domination property, see Definition 3.14, to derive direct and converse assertions for (P) and (P′), respectively. Using the previous notations, we have the following new result, see Theorem 3.29 and Theorem 4.3.

Theorem 6.3 (Strong direct and converse assertion for (P), (D_1') and (D_2'))**.**

(**a**) *Assume* $\psi(x)$ *is a solution of* (P)*, for some* $x \in X$*. Then, the operator* $-D_G^{+}\psi\, x \in \mathrm{dom}(D_G^{\#})$ *solves* (D_1')*.*

(**b**) *Conversely, if* $U_2 \in \mathrm{dom}(D_G^{\#})$ *solves* (D_2') *and it holds*

$$-\langle D_G^{+}\psi\, x, x\rangle_Y \in \mathrm{WMax}(\{-\langle D_G^{+}\psi\, x, y\rangle_Y \mid y \in X\}, \tilde{K}) - \tilde{K},$$

where $x := D_G^{\#}U_2$*, then* $\psi(x)$ *solves* (P)*.*

7. Conclusion

We have seen that vector variational inequalities with respect to variable domination structures can be used to solve approximation problems. These problems are of big interest in different fields of applications. In order to give necessary and sufficient existence results for the main and inverse problems, one needs to solve set-valued vector variational inequalities. In a future work, it would be important to prove existence results for the dual problems and to introduce a (non-linear) scalarization technique which allows to tackle the problems in a different way. Finally, from a practical perspective it would be useful to consider inverse problems for the vector variational inequality only using functionals instead of operators.

8. Acknowledgment

The authors are very grateful to Franco Giannessi for his valuable hints and fruitful discussions. We express our gratitude to China Medical University for the kind invitation and the possibility to present our results on the workshop. Akhtar A. Khan is supported by National Science Foundation grant under Award No. 1720067.

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Manuscript received November 21 2018
revised December 9 2018

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