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OPERATORS ON MANIFOLDS WITH SINGULARITIES

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Abstract. Partial differential equations (PDEs) in models of applied sciences are often connected with geometric singularities, consisting of conical points, boundaries, interfaces, or edges. We establish here new elements of the corresponding pseudo-differential machinery by modifying the approach, where a specific degenerate behavior is formulated in terms of a parameter-dependent variant of edge calculus. With the Mellin-edge quantization in the background we express composition results on the level of involved degenerate symbols.

1. Introduction

We study here operators on configurations with singular geometry which contribute ideas for solving elliptic PDEs $Au = f$ and characterizing parametrices P. Although we intend to focus on "regular singularities", *i.e.,* stratified spaces where the strata "touch" each other in some transversal way, the analysis of PDEs, or more generally, of pseudo-differential operators (PDOs) requires sophisticated new techniques, both on the level of weighted distribution classes and of operators in terms of operator-valued symbols with twisted symbolic estimates. The corresponding new inventions seem to become more and more complex with increasing orders of singularities. Numerous articles and monographs are devoted to the problem of constructing parametrices in terms of suitable symbolic structures. In the present article we specify some assumptions and notation on the underlying stratified spaces ("singular manifolds") and we look at corresponding operators. In Section 2 we recall the shape of degenerate differential operators, motivated by geometric examples in the case of conical and edge singularities. In Section 3 we briefly outline some necessary tools on Fourier-and Mellin-based pseudo-differential operators. In Section 4 we consider operators on infinite straight cones with axial variable $r \in \mathbb{R}_+$ and associated notions appearing for $r \to 0$, *i.e.,* at the tip of the cone, and for $r \to \infty$, the conical exit to ∞. Section 5 is devoted to observations on parameter-dependent edge operators. Those are employed in Section 6 for studying the composition behavior of operator families where involved parameters give rise to new operator classes with a specific behavior in t. Degenerate operator-families have been investigated by many authors, often motivated by applications in physics or engineering with involved conical- or edge-singularities. These structures also cover results of [6] on Mellin operators on the half-axis in L^2 and on pseudo-differential boundary

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value problems, also the calculus of Boutet de Monvel [1], including Green operators, and boundary symbols in the framework of the transmission property. These are operator-valued, acting between spaces on the half-axis, endowed with certain simple rescaling groups, see also [12], [9]. A similar method works in the edge theory, first established in [16] for "abstract" pairs of Hilbert spaces with such groups and prepared by concrete Kegel spaces in [13], on infinite non-trivial cones, then continued in [11], [17], moreover, in [14], [15], [5], [18], [7], [8], [19], [20], [2], [3]. There are contributions by many other authors, but interesting problems remained open. Because of the extent of the program we limit ourselves to some specific aspects of recent research and focus on substructures of larger operator algebras. The results of the present paper can be subsumed under the keywords: observations on parameter-dependent edge operator-families, acquiring a new degenerate behavior under a natural "corner substitution" of parameters. In particular we characterize Leibniz products of holomorphic edge operator-valued Mellin symbols within the calculus.

2. Degenerate operators

Models of physics, geometry, or engineering often refer to PDEs on a manifold B with smooth edge Y, say, a wedge $X^\triangle \times \mathbb{R}^q$ with edge $\mathbb{R}^q$ and model cone

$$X^\triangle = (\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X) \tag{2.1}$$

for some smooth compact manifold X. In the quotient space (2.1) the vertex is represented by $\{0\} \times X$, collapsed to a point. Moreover, a fixed splitting $(r, x) \in \mathbb{R}_+ \times X =: X^\wedge$ of variables in the respective open stretched cone represents a regular singularity. This means, another choice $(\tilde{r}, \tilde{x}) \in \mathbb{R}_+ \times X$ is representing an equivalent cone structure, if the diffeomorphism $(r, x) \to (\tilde{r}, \tilde{x})$, $X^\wedge \to X^\wedge$, is the restriction of a diffeomorphism $\overline{\mathbb{R}}_+ \times X \to \overline{\mathbb{R}}_+ \times X$, *i.e.,* between smooth cylinders with boundary. In this sense we say, that $(\tilde{r}, \tilde{x})$ is regular with respect to (r, x). Other choices of $(\tilde{r}, \tilde{x})$ may describe cuspidal geometries which are ruled out here, and we keep the splitting (r, x) fixed. In a similar sense we also reflect the regular behavior of analytical objects over $X^\triangle \times \mathbb{R}^q$ in variables $(r, x, y) \in X^\wedge \times \mathbb{R}^q$ by asking, differential operators of order $\mu \in \mathbb{N} = \{0, 1, 2, \ldots\}$ to be as follows. A differential operator of the form

$$A_{\text{edge}} := r^{-\mu} \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r, y)(-r\frac{\partial}{\partial r})^j (rD_y)^\alpha \tag{2.2}$$

with coefficients $a_{j\alpha}(r, y) \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$ is called edge-degenerate. Here $\mathrm{Diff}^\nu(X)$ for $\nu \in \mathbb{N}$ denotes the Fréchet space of differential operators on X of order ν with smooth coefficients. The wedge is equipped with coordinates (r, x, y), and similarly to $X^\wedge$ we talk about a regular geometry over $X^\wedge \times \mathbb{R}^q$ when we keep in mind the class of equivalent choices of coordinates $(\tilde{r}, \tilde{x}, \tilde{y})$ which correspond to a bundle isomorphism $(\overline{\mathbb{R}}_+ \times X) \times \mathbb{R}^q \to (\overline{\mathbb{R}}_+ \times X) \times \mathbb{R}^q$ with base $\mathbb{R}^q$ and regular transition maps between fibers $\overline{\mathbb{R}}_+ \times X$. Globally a manifold B with edge Y of dimension q is described by the properties that $B \setminus Y$ is smooth and B is locally close to Y modeled on $X^\triangle \times Y$ with the structure of an $X^\triangle$-bundle over Y and the

indicated transition behavior of fibers $X^\triangle$, induced by regular transitions of $\overline{\mathbb{R}}_+ \times X$. For $\dim Y = 0$ we get a manifold with conical singularities. Close to the tip of the cone corresponding differential operators of order μ are assumed to be of the form

$$A_{\text{cone}} := r^{-\mu} \sum_{j=0}^{\mu} a_j(r)(r\frac{\partial}{\partial r})^j \tag{2.3}$$

with coefficients $a_j(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$. Examples for $\mu = 2$ of (2.2) and (2.3) are Laplace-Beltrami operators belonging to metrics

$$dr^2 + r^{-2}g_X + dy^2 \quad \text{and} \quad dr^2 + r^{-2}g_X,$$

respectively, for some Riemannian metric g_X on X.

Such operators also appear by considering differential operators in $\mathbb{R}^{1+n+q}$ for $n = \dim X$ by introducing polar coordinates (r, x) in $\mathbb{R}^{1+n}_{\tilde{x}} \setminus \{\tilde{x} = 0\}$, where in this case the unit sphere S^n plays the role of X.

3. Fourier- and Mellin- based pseudo-differential operators

The Fourier transform

$$Fu(\xi) = \int e^{-ix\xi}u(x)dx \quad \text{with its inverse} \quad F^{-1}g(x) = \int e^{ix\xi}g(\xi)đ\xi, \tag{3.1}$$

$đ\xi = (2\pi)^{-n}d\xi$, is the well-known background of pseudo-differential operators, using double symbols $a_{\mathrm{D}}(x, x', \xi)$, and oscillatory integrals

$$Au(x) = \iint e^{i(x-x')}a_{\mathrm{D}}(x, x', \xi)u(x')dx'đ\xi.$$

If $a_{\mathrm{D}}(x, x', \xi)$ is replaced by $a_{\mathrm{L}}(x, \xi)$ or $a_{\mathrm{R}}(x', \xi)$ the corresponding symbols are referred to as left or right symbols, respectively. While this is completely standard for scalar symbols of Hörmander's type $S^\mu(\Omega \times \mathbb{R}^n)$ or classical symbols $S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n)$ of order μ, and $\Omega \subseteq \mathbb{R}^n_x$ or $\Omega \subseteq \mathbb{R}^n_x \times \mathbb{R}^n_{x'}$ open, a novelty of singular analysis consists of using operator-valued symbol spaces

$$S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n; H, \tilde{H}) \tag{3.2}$$

for (separable) Hilbert spaces $H, \tilde{H}$. Recall that there is also a variant for Fréchet spaces, cf. [18]. By subscript "(cl)" we mean that the corresponding considerations are valid both for classical and general symbols. The terminology may be found, for instance, in [18]. The space H (and similarly also $\tilde{H}$) in (3.2) is assumed to be endowed with the action of a group $\kappa = \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$ of isomorphisms

$$\kappa_\delta : H \to H,$$

such that $\kappa_\delta \kappa_{\delta'} = \kappa_{\delta\delta'}$ for all $\delta, \delta' \in \mathbb{R}_+$, and $\kappa_1 = \mathrm{id}_H$. Moreover, for any $h \in H$ the function $h \to \kappa_\delta h, \delta \in \mathbb{R}_+$, is asked to belong to $C(\mathbb{R}_+, H)$. The corresponding group $\tilde{\kappa}$ for $\tilde{H}$ may be different from κ, but $\kappa, \tilde{\kappa}$ are kept fixed in our consideration. The space (3.2) (first in general meaning and, say, in the left-symbol case) is defined to be the set of all

$$a(x, \xi) \in C^\infty(\Omega \times \mathbb{R}^n, \mathcal{L}(H, \tilde{H}))$$

satisfying the symbol estimates

$$\|\tilde{\kappa}_{\langle\xi\rangle}^{-1}\{D_x^\alpha D_\xi^\beta a(x,\xi)\}\kappa_{\langle\xi\rangle}\|_{\mathcal{L}(H,\tilde{H})} \leq c\langle\xi\rangle^{\mu-|\beta|} \tag{3.3}$$

for all $(x,\xi) \in K \times \mathbb{R}^n$, and every $K \Subset \Omega$, for all $\alpha, \beta \in \mathbb{N}^n$ and constants $c = c(\alpha,\beta,K) > 0$. Here $\langle\xi\rangle := (1+|\xi|^2)^{1/2}$, and $\kappa_{\langle\xi\rangle}$ is defined by inserting $\delta = \langle\xi\rangle$ in κ_δ. Later on, in the operator-valued set-up we will write (y,η) rather than (x,ξ), except for the scalar case, *i.e.,* when $H = \tilde{H} = \mathbb{C}$ and $\kappa_\delta = \tilde{\kappa}_\delta = \mathrm{id}_{\mathbb{C}}$.

By definition classical amplitude functions $a(x,\xi)$ have a sequence of homogeneous components $a_{(\mu-j)}(x,\xi)$, $j \in \mathbb{N}$, belonging to

$$S^{(\mu-j)}(\Omega \times (\mathbb{R}^n \setminus \{0\}); H, \tilde{H})$$

for any $j \in \mathbb{R}$, where $S^{(\nu)}(\Omega \times (\mathbb{R}^n \setminus \{0\}); H, \tilde{H})$ is the space of all $a_{(\nu)}(x,\xi) \in C^\infty(\Omega \times (\mathbb{R}^n \setminus \{0\}), \mathcal{L}(H,\tilde{H}))$ such that

$$a_{(\nu)}(x,\delta\xi) = \delta^\nu \tilde{\kappa}_\delta a_{(\nu)}(x,\xi)\kappa_\delta^{-1} \tag{3.4}$$

for all $\delta \in \mathbb{R}_+$ and all $(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$. Then $a(x,\xi) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^n; H, \tilde{H})$ admits an asymptotic expansion

$$a(x,\xi) \sim \chi(\xi) \sum_{j=0}^\infty a_{(\mu-j)}(x,\xi) \tag{3.5}$$

where $\chi(\xi)$ denotes an excision function (*i.e.,* $\chi(\xi) \equiv 0$ for $0 < |\xi| < \varepsilon_0, \chi(\xi) \equiv 1$ for $|\xi| > \varepsilon_1$, for some $0 < \varepsilon_0 < \varepsilon_1$). For any $a(x,\xi)$ in $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n; H, \tilde{H})$ we write

$$\mathrm{Op}_x(a)u(x) = \iint e^{i(x-x')\xi} a(x,\xi)u(x')dx'đ\xi,$$

$đ\xi = (2\pi)^{-n}d\xi$, first for $u \in C_0^\infty(\Omega, H)$. Let

$$\mathcal{W}^s(\mathbb{R}^n, H), \tag{3.6}$$

$s \in \mathbb{R}$, defined to be the completion of $C_0^\infty(\mathbb{R}^n, H)$ with respect to the norm

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^n,H)} := \Big\{ \int \langle\xi\rangle^{2s} \|\kappa_{\langle\xi\rangle}^{-1} Fu(\xi)\|_H^2 \Big\}^{1/2}.$$

For $\Omega := \mathbb{R}^n$ we then have continuity

$$\mathrm{Op}_x(a) : \mathcal{W}^s(\mathbb{R}^n, H) \to \mathcal{W}^{s-\mu}(\mathbb{R}^n, \tilde{H}),\ s \in \mathbb{R},$$

provided that the symbol $a(x,\xi)$ satisfies some extra conditions with respect to x, *e.g.,* when it is independent on x for $|x| > c$ for some $c > 0$. More general criteria are given in [17, Theorem 6, page 283] and in [20]. Often it will be convenient to employ functions $[\xi] \in C^\infty(\mathbb{R}^n_\xi)$ rather than $\langle\xi\rangle$, where $[\xi] = |\xi|$ for $|\xi| > c$ for some $c > 0$. Clearly the calculu itself is equivalent to the former one. In singular analysis it is typical that the geometric configuration contains one (or more) half-axis direction(s) $r \in \mathbb{R}_+$ and we have to take into account the behavior of amplitude functions with respect to r and its covariable ρ. This does not concern only $r \to 0$

but, as we shall see below, for some intrinsic reasons also for $r \to \infty$. It turns out that the adequate context is the Mellin transform on $\mathbb{R}_+$

$$(3.7)\quad Mu(w) = \int_0^\infty r^w u(r)\frac{dr}{r} \quad \text{with its inverse} \quad M^{-1}g(r) = \int_{\Gamma_{\frac{1}{2}-\gamma}} r^{-w} g(w) đw$$

with $đw = (2\pi i)^{-1}dw$ and

$$\Gamma_\beta := \{w \in \mathbb{C} : \operatorname{Re} w = \beta\}.$$

The weight $\gamma \in \mathbb{R}$ in (3.7) will be specified below; for the moment we set $\gamma = 0$ and assume $u \in C_0^\infty(\mathbb{R}_+)$. Then $Mu(w) \in \mathcal{A}(\mathbb{C})$. Here $\mathcal{A}(U, E)$ for open $U \subseteq \mathbb{C}$ and a Fréchet space E means the space of all holomorphic functions in U with values in E, and E is omitted for $E = \mathbb{C}$. It is well-known that

$$M : C_0^\infty(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C})$$

is continuous, as well as the composition of M with the restriction to Γ_β

$$(3.8)\qquad C_0^\infty(\mathbb{R}_+) \to \mathcal{S}(\Gamma_\beta)$$

for any real β, where $\mathcal{S}(\Gamma_\beta)$ is the space of Schwartz functions on Γ_β. For any $\gamma \in \mathbb{R}$ the restriction

$$M_\gamma u := Mu|_{\Gamma_{\frac{1}{2}-\gamma}}$$

is well-defined and extends by continuity to an isomorphism

$$(3.9)\qquad M_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\gamma})$$

where the inverse is just the second formula in (3.7). Since $\mathcal{S}(\Gamma_{\frac{1}{2}-\gamma}) \subset L^2(\Gamma_{\frac{1}{2}-\gamma})$ we can give (3.8) for $\beta = \frac{1}{2} - \gamma$ a more precise meaning, namely to observe an isomorphism

$$M_\gamma : \mathcal{T}^\gamma(\mathbb{R}_+) \to \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma})$$

by introducing the weighted Mellin analogue of a weighted Schwartz space on $\mathbb{R}_+$, namely, $\mathcal{T}^\gamma(\mathbb{R}_+) := M_\gamma^{-1}\mathcal{S}(\Gamma_{\frac{1}{2}-\gamma})$, with the restriction of the inverse of (3.9) to $\mathcal{S}(\Gamma_{\frac{1}{2}-\gamma})$. Note that when we define the transformation

$$(3.10)\qquad S_\gamma : C_0^\infty(\mathbb{R}_+) \to C_0^\infty(\mathbb{R})$$

by the formula

$$(S_\gamma u)(\boldsymbol{t}) := e^{-(\frac{1}{2}-\gamma)\boldsymbol{t}} u(e^{-\boldsymbol{t}}),\ \boldsymbol{t} \in \mathbb{R},$$

we can write

$$M_{\gamma, r\to w} = \psi^*_{\frac{1}{2}-\gamma} \circ F_{\boldsymbol{t}\to\rho} \circ S_\gamma$$

for the Fourier transform $F_{\boldsymbol{t}\to\rho}$ on $\mathbb{R}_{\boldsymbol{t}}$ with covariable ρ and ψ^*_β the pull back under the bijection $\psi_\beta : \Gamma_\beta \to \mathbb{R}$, $\psi_\beta : w \mapsto \operatorname{Im} w$. For a (first) scalar Mellin symbol $f_{\mathrm{D}}(r, r', w) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}_+ \times \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma})$ where $w = \frac{1}{2} - \gamma + i\rho$, we set

$$\operatorname{Op}^\gamma_M(f_{\mathrm{D}})u(r) = \int_{\mathbb{R}}\int_{\mathbb{R}_+} (\frac{r}{r'})^{\frac{1}{2}-\gamma+i\rho} f_{\mathrm{D}}(r, r', \frac{1}{2} - \gamma + i\rho) u(r')\frac{dr'}{r'} đ\rho,$$

$đ\rho = (2\pi)^{-1}d\rho$, for $u \in C_0^\infty(\mathbb{R}_+)$ and then extended to larger distribution spaces. In the case of a left Mellin symbol $f_{\mathrm{L}}(r,w)$ we have

$$\mathrm{Op}_M^\gamma(f_{\mathrm{L}}) = M_\gamma^{-1} f_{\mathrm{L}} M_\gamma = r^\gamma \mathrm{Op}_M(T^{-\gamma} f_{\mathrm{L}}) r^{-\gamma} \tag{3.11}$$

for $\mathrm{Op}_M = \mathrm{Op}_M^0$, and $(T^\beta f_{\mathrm{L}})(r,w) := f_{\mathrm{L}}(r, w+\beta)$. A similar formalism will be used for operator-valued symbols, not only in the above-mentioned $\{H, \tilde{H}\}$-formalism but when Mellin symbols take values in Fourier-based pseudo-differential operators operating on some C^∞ manifold X. This belongs to the topics of the following section.

4. Operators on infinite straight cones

The shape of edge-degenerate differential operators (2.2) gives us a hint in which way pseudo-differential operators in the edge calculus should be organized. Let us first recall the notion of pseudo-differential operators on a (say, compact) smooth manifold X containing parameters $\lambda \in \mathbb{R}^d$. Those consist of spaces

$$A(\lambda) \in L^\mu_{(\mathrm{cl})}(X; \mathbb{R}^d_\lambda) \tag{4.1}$$

of operators of some order $\mu \in \mathbb{R}$. Here

$$L^{-\infty}(X; \mathbb{R}^d_\lambda) := \mathcal{S}(\mathbb{R}^d, L^{-\infty}(X))$$

with the identification $L^{-\infty}(X) \cong C^\infty(X \times X)$ via integral operators

$$Cu(x) = \int_X c(x,x')u(x')dx'$$

for $c(x,x') \in C^\infty(X \times X)$ and dx' associated with a Riemannian metric on X. Then (4.1) consists of all

$$A(\lambda) = A_{\mathrm{c}}(\lambda) + C(\lambda)$$

for $C(\lambda) \in L^{-\infty}(X; \mathbb{R}^d_\lambda)$ and

$$A_{\mathrm{c}}(\lambda) = \sum_{j=1}^N \varphi_j A_j(\lambda) \varphi_j'$$

for an open covering of X by coordinate neighbourhoods $(U_j)_{j=1,\dots,N}$ with a subordinate partition of unity by functions $(\varphi_j)_{j=1,\dots,N}$, $\varphi_j \prec \varphi_j$, $\varphi_j, \varphi_j' \in C_0^\infty(U_j)$ with $\varphi' \equiv 1$ in a neighbourhood of $\operatorname{supp}\varphi$. Here

$$A_j(\lambda) = (\chi_j^{-1})_* \mathrm{Op}_x(a_j)(\lambda)$$

stands for amplitude function $a_j(x,\xi,\lambda) \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^n_x \times \mathbb{R}^{n+d}_{\xi,\lambda})$ with $\lambda \in \mathbb{R}^d$ being treated as an extra covariable, and charts $\chi_j : U_j \to \mathbb{R}^n$; $j = 1, \dots, N$. The space (4.1) is equipped with its natural Fréchet topology. Now

$$M^\mu_{\mathcal{O}}(X; \mathbb{R}^d_\lambda) \tag{4.2}$$

is defined as the set of all $h(w,\lambda) \in \mathcal{A}(\mathbb{C}_w, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^d_\lambda))$ such that

$$h(w,\lambda)|_{\Gamma_\beta \times \mathbb{R}^l} \in \mathcal{A}(\mathbb{C}_w, L^\mu_{\mathrm{cl}}(X; \Gamma_\beta \times \mathbb{R}^l))$$

for every $\beta \in \mathbb{R}$, uniformly in compact β-intervals. In particular, functions

$$h(r,w) \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_{\mathcal{O}}(X))$$

will be interpreted as $L^\mu_{\mathrm{cl}}(X)$-valued Mellin symbols in expressions (3.11), where we set

$$M^\mu_{\mathcal{O}}(X) := M^\mu_{\mathcal{O}}(X;\mathbb{R}^d) \quad \text{for} \quad d=0.$$

Weighted Mellin operators make sense for arbitrary $h(r,w)$ as mappings

$$\mathrm{Op}^\beta_M(h) : C^\infty_0(X^\wedge) \to \mathcal{S}(\mathbb{R}_+, C^\infty(X))$$

for any real β. Recall that

$$\mathrm{Op}^\beta_M(h) = \mathrm{Op}^\gamma_M(h) \quad \text{for arbitrary} \quad h \in M^\mu_{\mathcal{O}}(X),\ \beta,\gamma \in \mathbb{R}, \tag{4.3}$$

as operators on $C^\infty_0(X^\wedge)$. For $\beta = \gamma - n/2$ they induce continuous operators

$$\mathrm{Op}^{\gamma-n/2}_M(h) : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma}(X^\wedge) \tag{4.4}$$

for arbitrary $s \in \mathbb{R}$. The convention with a translation of γ by $n = \dim X$ has the meaning of a normalization. The weighted Mellin Sobolev spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ over the infinite stretched cone $X^\wedge$ are defined in [17] or [18]. In particular, $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ is the completion of $C^\infty_0(\mathbb{R}_+)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+)} = \Big\{ \int_{\Gamma_{\frac{1}{2}-\gamma}} \langle w\rangle^{2s} |Mu(w)|^2 đw \Big\}^{1/2}.$$

for $đw := (2\pi i)^{-1} dw$. Recall that the definition for any closed smooth manifold X of dimension n can be reduced to the case $\mathbb{R}_+ \times \mathbb{R}^n$ and the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} := \Big\{ \int_{\Gamma_{\frac{n+1}{2}-\gamma}} \int_{\mathbb{R}^n} \langle \xi, w\rangle^{2s} |(F_{x\to\xi} M_{r\to w} u)(r,w)|^2 đw đ\xi \Big\}^{1/2} \tag{4.5}$$

Here $\langle \xi, w\rangle = (1+|\xi|^2+|w|^2)^{1/2}$. It will be necessary also to consider Kegel spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$, defined by

$$\mathcal{K}^{s,\gamma}(X^\wedge) := \{u = \omega u_0 + (1-\omega)u_\infty : u_0 \in \mathcal{H}^{s,\gamma}(X^\wedge), u_\infty \in H^s_{\mathrm{cone}}(X^\wedge)\}$$

for some fixed cut-off function ω. Here $H^s_{\mathrm{cone}}(X^\wedge)$ is defined to be the set of all

$$u(r,x) \in H^s_{\mathrm{loc}}(\mathbb{R}\times X)|_{X^\wedge}$$

such that

$$((1-\omega)\varphi u) \circ \chi^{-1}_{\mathrm{cone}} \in H^s(\mathbb{R}^{1+n})$$

for every chart

$$\chi_{\mathrm{cone}} : \mathbb{R}_+ \times U \to \mathbb{R}^{1+n}, \quad \chi_{\mathrm{cone}}(r,x) := (r, r\chi(x))$$

with $\chi : U \to \mathbb{R}^n$ being any chart on X. In particular, for $n=0$ we have

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) := \{u = \omega u_0 + (1-\omega)u_\infty : u_0 \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+), u_\infty \in H^s(\mathbb{R}_+)\}.$$

5. Parameter-dependent edge operators

Operators (2.2) belong to $\mathrm{Diff}^\mu(X^\wedge \times \mathbb{R}^q)$, and give rise to an operator space $\mathrm{Diff}^\mu_{\mathrm{deg}}(B \setminus Y)$ globally on a manifold B with edge Y with a typical edge-degenerate behavior close to Y. The elements of $\mathrm{Diff}^\mu_{\mathrm{deg}}(B \setminus Y)$ restrict to standard elements of

$$\mathrm{Diff}^\mu(U)$$

for any open $U \subset B \setminus Y$ such that $\overline{U}$ does not intersect Y. Setting

$$\tilde{h}(r, y, w, \tilde{\eta}) := \sum_{j+|\alpha| \le \mu} a_{j\alpha}(r, y) w^j \tilde{\eta}^\alpha \in C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}}))$$

for $\tilde{\eta} = r\eta$, and $h(r, y, w, \eta) := \tilde{h}(r, y, w, r\eta)$ then we have

$$A_{\mathrm{c}} u = \mathrm{Op}_y\{r^{-\mu} \mathrm{Op}^\gamma_M(h)(y, \eta)\}. \tag{5.1}$$

Recall that the structures of the edge calculus are motivated by the task to establish operator spaces over B containing $\mathrm{Diff}^\mu_{\mathrm{deg}}(B \setminus Y)$ for every $\mu \in \mathbb{N}$, together with parametrices of elliptic elements. Let us employ notation

$$L^\mu(B, \boldsymbol{g}) \tag{5.2}$$

for the respective class of operators of order $\mu \in \mathbb{R}$ and weight data $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ for $\Theta := (\vartheta, 0]$, $\vartheta < 0$. We will explain the meaning of $\boldsymbol{g}$ below. Recall that elements of (5.2) induce continuous operators

$$A : H^{s,\gamma}(B) \to H^{s-\mu,\gamma-\mu}(B) \tag{5.3}$$

for every $s \in \mathbb{R}$, cf. notation in [10, page 275]. In this context Fredholm property and ellipticity refer to the principal symbolic structure of operators, namely,

$$\sigma(A) := (\sigma_0(A)(\boldsymbol{x}, \boldsymbol{\xi}), \sigma_1(A)(y, \eta)), \tag{5.4}$$

associated to the stratification

$$s(B) := (s_0(B)), s_1(B)), \tag{5.5}$$

for

$$s_0(B) := B \setminus Y,\ s_1(B) := Y. \tag{5.6}$$

Here $(\boldsymbol{x}, \boldsymbol{\xi})$ are variables $\boldsymbol{x}$ and covariables $\boldsymbol{\xi} \neq 0$ on $s_0(B)$, where $\sigma_0(A)(\boldsymbol{x}, \boldsymbol{\xi})$ is the standard homogeneous principal symbol of order μ, according to the inclusion

$$L^\mu(B, \boldsymbol{g}) \subset L^\mu_{\mathrm{cl}}(B \setminus Y). \tag{5.7}$$

Moreover, $\sigma_1(A)$ is locally over Y in variables $y \in \mathbb{R}^q$ and covariables $\eta \in \mathbb{R}^q \setminus \{0\}$ associated with an operator-valued edge symbol

$$a(y, \eta) \in S^\mu(\mathbb{R}^q_y \times \mathbb{R}^q_\eta; H, \tilde{H})$$

for Kegel spaces

$$H := \mathcal{K}^{s,\gamma}(X^\wedge),\ \tilde{H} := \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge), \tag{5.8}$$

$s, \gamma \in \mathbb{R}$, where H, $\tilde{H}$ are endowed with group actions $\kappa := \{\kappa_\delta\}_{\delta \in \mathbb{R}_+}$, and $\tilde{\kappa} := \{\tilde{\kappa}_\delta\}_{\delta \in \mathbb{R}_+}$, respectively, namely,

$$(\kappa_\delta u)(r, x) := \delta^{(n+1)/2} u(\delta r, x),\ \delta \in \mathbb{R}_+, \tag{5.9}$$

for $n := \dim X$ and $\tilde{\kappa}$ is defined by the same expression (in this case). Then we write

$$\sigma_1(A)(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

for $\eta \neq 0$. We employ here notation from [10]. The edge calculus is much more specific than $L^\mu_{\mathrm{cl}}(B \setminus Y)$ and the symbols $\sigma_0(A)$ of operators $A \in L^\mu(B, \boldsymbol{g})$ are edge-degenerate near Y. Similarly to boundary value problems (BVPs) with the transmission property which may be subsumed under a variant of the edge calculus, cf. Boutet de Monvel [1], or the monographs of Rempel and Schulze [12], Grubb [9], operators do not only contain interior non-smoothing contributions in truncation quantization, but also Green operators (in the upper left corners when they are realized in form of 2×2 - operator block-matrices) but also operators of Mellin plus Green type, here denoted by $M + G$ and associated symbols by $m + g$. Those also appear when the transmission property is violated, cf. the joint article with Seiler [19]. Although they are very important for the edge calculus, we do not formulate them in explicit form here, since they are investigated in detail in other expositions, cf., *e.g.,* in [18].

Let $\mathbb{B}$ be the stretched manifold associated with B and $\partial\mathbb{B}$ its boundary which is a bundle over Y with fiber X. The space (5.2) consists of all operators

$$A_{\mathrm{edge}} := \omega_{\mathrm{glob}} A_{\mathrm{c}} \omega'_{\mathrm{glob}} + (1 - \omega_{\mathrm{glob}}) A_{\mathrm{int}} (1 - \omega''_{\mathrm{glob}}) + C \tag{5.10}$$

for global cut-off functions

$$\omega''_{\mathrm{glob}} \prec \omega_{\mathrm{glob}} \prec \omega'_{\mathrm{glob}}. \tag{5.11}$$

Such an ω_{glob} is any element of $C^\infty(\mathbb{B})$ which is equal to 1 in a small collar neighbourhood V_1 of $\partial\mathbb{B}$ and $\equiv 0$ off some other $V_0 \supset V_1$ of this kind.

The operator A_{int} is a standard pseudo-differential operator on $B \setminus Y$, cf. formula (5.7). Moreover,

$$C \in L^{-\infty}(B, \boldsymbol{g})$$

is smoothing, and A_{c} is a finite sum of operators of the form

$$\chi_*^{-1}(\psi \mathrm{Op}_y(a) \psi')$$

where $\chi : V \to \mathbb{R}^q$ is a coordinate diffeomorphism for open $V \subset Y$ and ψ runs over a partition of unity subordinate to the respective open covering of Y, and $\psi' \succ \psi$ for certain $\psi' \in C_0^\infty(V)$. Here we assume

$$a(y,\eta) := r^{-\mu} \mathrm{Op}_M^{\gamma - n/2}(h)(y,\eta) + (m+g)(y,\eta) \tag{5.12}$$

for

$$h(r,y,w,\eta) = \tilde{h}(r,y,w,r\eta) \tag{5.13}$$

for certain

$$\tilde{h}(r,y,w,\tilde{\eta}) := C^\infty_{[0,R]}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}})). \tag{5.14}$$

Here "$[0,R]$" indicates the subspace of all elements of

$$C^\infty(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^\mu_{\mathcal{O}}(X; \mathbb{R}^q_{\tilde{\eta}}))$$

which are independent of r for $r > R$. The above-mentioned $(m+g)(y,\eta)$, have the meaning of amplitude functions belonging to the asymptotic part of the edge calculus, where asymptotics refer to weight strips of width $\Theta = (\vartheta, 0]$ situated on the left of weight lines $\Gamma_{\frac{n+1}{2}-\gamma}$ and $\Gamma_{\frac{n+1}{2}-(\gamma-\mu)}$, $n = \dim X$, in the complex Mellin w-plane. Let us now generalize the spaces (5.2) to the case of dependence on extra parameters $\lambda \in \mathbb{R}^d$ and indicate corresponding spaces of symbols. In the following, for simplicity, we will ignore Θ and denote weight data by $(\gamma, \gamma-\mu)$. The space

$$R^{\mu}_{\mathrm{trad}}(\mathbb{R}^{2q}_{y,\eta}, \boldsymbol{g}; \mathbb{R}^d_\lambda) \quad \text{for weight data} \quad \boldsymbol{g} = (\gamma, \gamma-\mu)$$

is defined to be the set of all operator-functions of the form

$$\begin{aligned} a(y,\eta,\lambda) &= \sigma_1(r)\big(a_0(y,\eta,\lambda) + a_1(y,\eta,\lambda)\big)\sigma_0(r) \\ &\quad + (1-\sigma_1(r))P_0(y,\eta,\lambda)(1-\sigma_2(r)) + (m+g)(y,\eta,\lambda) \end{aligned} \tag{5.15}$$

with

$$a_0(y,\eta,\lambda) := \omega_1(r[\eta,\lambda])r^{-\mu}\mathrm{Op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta,\lambda)\omega_0(r[\eta,\lambda]) \tag{5.16}$$

$$a_1(y,\eta,\lambda) := (1-\omega_1(r[\eta,\lambda]))r^{-\mu}\mathrm{Op}_r(p)(y,\eta,\lambda)(1-\omega_2(r[\eta,\lambda])) \tag{5.17}$$

with

$$h(r,y,w,\eta,\lambda) \in C^\infty_{[0,\mathrm{R}]}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, M^\mu_{\mathcal{O}}(X; \mathbb{R}^{q+d}_{\tilde\eta,\tilde\lambda})|_{(\tilde\eta,\tilde\lambda)=(r\eta,r\lambda)}, \tag{5.18}$$

and

$$p(r,y,\rho,\eta,\lambda) \in C^\infty_{[0,\mathrm{R}]}(\overline{\mathbb{R}}_+ \times \mathbb{R}^q, L^\mu_{\mathrm{cl}}(X; \mathbb{R}^{1+q+d}_{\tilde\rho,\tilde\eta,\tilde\lambda}))|_{(\tilde\rho,\tilde\eta,\tilde\lambda)=(r\rho,r\eta,r\lambda)}, \tag{5.19}$$

moreover,

$$\begin{aligned} (m+g)(y,\eta,\lambda) &\in R^\mu_{\mathrm{M+G}}(\mathbb{R}^{2q}_{y,\eta}, \boldsymbol{g}; \mathbb{R}^d), \\ P_0(y,\eta,\lambda) &\in C^\infty(\mathbb{R}^q_y, L^\mu_{\mathrm{cl}}(X^\wedge; \mathbb{R}^{q+d}_{\eta,\lambda}))_0. \end{aligned} \tag{5.20}$$

Subscript "0" means that the kernels of operators vanish in r-direction off some compact interval of $\mathbb{R}_+$. The operators $A_{\mathrm{c}}(\lambda) = \mathrm{Op}(a)(\lambda)$ have symbols $a(y,\eta,\lambda)$ in

$$R^\mu_{\mathrm{trad}}(\mathbb{R}^{2q}_{y,\eta}, \boldsymbol{g}; \mathbb{R}^d_\lambda) \subset S^\mu(\mathbb{R}^{2q+d}_{y,\eta,\lambda}; H, \tilde H)$$

The space

$$R^\mu_{\mathrm{new}}(\mathbb{R}^{2q}_{y,\eta}, \boldsymbol{g}; \mathbb{R}^d_\lambda) \quad \text{for weight data} \quad \boldsymbol{g} = (\gamma, \gamma-\mu)$$

consists of all operator functions

$$\begin{aligned} a(y,\eta,\lambda) &:= \sigma_1(r)r^{-\mu}\mathrm{Op}_M^{\gamma-\frac{n}{2}}(h)(y,\eta,\lambda)\sigma_0(t) \\ &\quad + (1-\sigma_1(r))P_0(y,\eta,\lambda)(1-\sigma_2(r)) + (m+g)(y,\eta,\lambda). \end{aligned} \tag{5.21}$$

The Green symbols $g(y,\eta,\lambda)$ in (5.20) and (5.21) are described in [8], and the smoothing Mellin symbols $m(y,\eta,\lambda)$ do not occur in the proof of Theorem 5.1 below. Concerning the involved cut-off functions $\omega_i(r)$, $\sigma_j(r)$, $i,j = 0,1,2$, we assume

$$\omega_2 \prec \omega_1 \prec \omega_0, \quad \sigma_2 \prec \sigma_1 \prec \sigma_0. \tag{5.22}$$

Theorem 5.1 ([7]). *The classes R^μ_{trad} and R^μ_{new} are equivalent.*

Definition 5.2. *The space*

$$L^{\mu}(B,\boldsymbol{g};\mathbb{R}^d_\lambda). \tag{5.23}$$

consists of all families

$$A_{\mathrm{edge}}(\lambda) := \omega_{\mathrm{glob}} A_{\mathrm{c}}(\lambda)\omega'_{\mathrm{glob}} + (1-\omega_{\mathrm{glob}})A_{\mathrm{int}}(\lambda)(1-\omega'_{\mathrm{glob}}) + C(\lambda), \tag{5.24}$$

where A_{c} is locally close to the edge Y defined by symbols in $R^{\mu}_{\mathrm{new}}(\mathbb{R}^{2q}_{y,\eta},\boldsymbol{g};\mathbb{R}^d_\lambda)$ moreover, $C(\lambda)\in L^{-\infty}(B,\boldsymbol{g};\mathbb{R}^d) = \mathcal{S}(\mathbb{R}^d, L^{-\infty}(B,\boldsymbol{g}))$ and $A_{\mathrm{int}}(\lambda)$ is determined by $L^{\mu}_{\mathrm{cl}}(B\setminus Y;\mathbb{R}^d)$ in terms of parameter-dependent symbols in $(\boldsymbol{x},\boldsymbol{\xi},\lambda)$. Finally, the smoothing Mellin plus Green symbol now takes the form $(m+g)(y,\eta,\lambda)$.

The principal symbolic hierarchy of the parameter-dependent calculus (5.23) in this case consists of pairs

$$\sigma(A)(\lambda) = (\sigma_0(A)(\boldsymbol{x},\boldsymbol{\xi},\lambda), \sigma_1(A)(y,\eta,\lambda)),$$

cf. also formula (5.4), here for $(\boldsymbol{\xi},\lambda)\neq 0$, and $(\eta,\lambda)\neq 0$, respectively. The extension of corresponding results to full block-matrices with extra entries of trace and potential type works as for $d=0$ which is a special case, see, *e.g.,* [18]. Recall that the specific features of the edge-calculus concerning operator families in (5.23) come from the first summands on the right-hand side of (5.24). Here we systematically employ the edge quantization from [8] for

$$\tilde{p}(r,y,\tilde{\rho},\tilde{\eta},\tilde{\lambda}) \in C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+\times\mathbb{R}^q, L^{\mu}_{\mathrm{cl}}(X;\mathbb{R}^{1+q+d}_{\tilde{\rho},\tilde{\eta},\tilde{\lambda}})) \tag{5.25}$$

belonging to

$$p(r,y,\rho,\eta,\lambda) = \tilde{p}(r,y,r\rho,r\eta,r\lambda). \tag{5.26}$$

The corresponding parameter-dependent Mellin-edge quantization which produces the same edge calculus (5.23) may be found in [18, Theorem 3.2.7] and in convenient form also in [7]. The main information consists of comparing the operator functions

$$\mathrm{Op}_r(p)(y,\eta,\lambda) \quad\text{and}\quad \mathrm{Op}^{\gamma}_M(h)(y,\eta,\lambda). \tag{5.27}$$

The corresponding relationship will be also referred to as Mellin-edge quantization.

Theorem 5.3 ([7, 18])**.** *For every operator function $p(r,y,\rho,\eta,\lambda)$ indicated in* (5.26) *there exists an $h(r,y,w,\eta,\lambda)=\tilde{h}(r,y,w,r\eta,r\lambda)$ with*

$$\tilde{h}(r,y,w,\tilde{\eta},\tilde{\lambda}) \in C^{\infty}_{[0,R]}(\overline{\mathbb{R}}_+\times\mathbb{R}^q, M^{\mu}_{\mathcal{O}}(X;\mathbb{R}^q_{\tilde{\eta},\tilde{\lambda}})) \tag{5.28}$$

such that the remainder

$$c(y,\eta,\lambda) := \mathrm{Op}_r(p)(y,\eta,\lambda) - \mathrm{Op}^{\gamma}_M(h)(y,\eta,\lambda) \tag{5.29}$$

is C^{∞} with respect to $y\in\mathbb{R}^q_y$ with values in $L^{-\infty}(X^{\wedge};\mathbb{R}^{q+d})$ for every $\gamma\in\mathbb{R}$.

For a more precise description of smoothing remainders we refer to the above-mentioned papers, see also [8, Theorem 3.2].

6. Degenerate families with extra corner-direction

We now study parameter-dependent edge operators from the point of view of corner symbols. In the following discussion we introduce another axial variable $t \in \mathbb{R}_+$ with parameters $(\hat{\tau}, \hat{\zeta}) \in \mathbb{R}^{1+e}$. In order to indicate a future application, continued in another project, we call the correspondence

$$(\hat{\tau}, \hat{\zeta}) \mapsto (t\tau, t\zeta)$$

a corner-substitution of parameters. The former λ has been first replaced by $(\hat{\tau}, \hat{\zeta})$ which turns to $(t\tau, t\zeta)$ locally close to Y corresponds to $(rt\tau, rt\zeta)$. Here $\tau \in \mathbb{R}$ is the covariable of t, and $\zeta \in \mathbb{R}^e$ a further parameter which is motivated as the covariable of another edge $\mathbb{R}^e$ in the variable z. Here, for simplicity we ignore z but we single out a special aspect of calculus in t, namely, the Leibniz compositions of corresponding degenerate families of operators. Some procedures are similar to the preceding computations in Section 5. We focus here on operators in t with respect to the Fourier transform in t, though the Mellin version under a Mellin quantization also makes sense. This would require extra weights in t, Mellin symbols could be smooth up to $t = 0$. Therefore, in order to preserve formal similarities to this aspect, we assume

$$b(t, \tau, \zeta) = \hat{b}(t, t\tau, t\zeta) \tag{6.1}$$

for some

$$\hat{b}(t, \hat{\tau}, \hat{\zeta}) \in C^\infty\big(\overline{\mathbb{R}}_+, L^\nu(B, \boldsymbol{l}; \mathbb{R}^{1+e}_{\hat{\tau},\hat{\zeta}})\big) \tag{6.2}$$

for $\boldsymbol{l} := (\beta, \beta - \nu)$.

Theorem 6.1. *Let $\hat{b}_{\nu-j}(\hat{\tau}, \hat{\zeta})$ be arbitrary elements of $L^{\nu-j}(B, \boldsymbol{l}; \mathbb{R}^{1+e}_{\hat{\tau},\hat{\zeta}})$, $j \in \mathbb{N}$, where the weight data $\boldsymbol{l} = (\lambda, \lambda - \nu, \Theta)$ are independent of j as well as the asymptotic types in possibly involved Green summands. Then there is a $\hat{b}(\hat{\tau}, \hat{\zeta}) \in L^\nu(B, \boldsymbol{l}; \mathbb{R}^{1+e}_{\hat{\tau},\hat{\zeta}})$ such that*

$$\hat{b}(\hat{\tau}, \hat{\zeta}) - \sum_{j=0}^{N} \hat{b}_{\nu-j}(\hat{\tau}, \hat{\zeta}) \in L^{\nu-(N+1)}(B, \boldsymbol{l}; \mathbb{R}^{1+e}_{\hat{\tau},\hat{\zeta}})$$

for every N, and $\hat{b}$ is uniquely determined mod $L^{-\infty}(B, \boldsymbol{l}; \mathbb{R}^{1+e}_{\hat{\tau},\hat{\zeta}})$.

The result is an analogue of the a corresponding theorem from the edge calculus without parameters, see, *e.g.,* [18]. Note that an analogue of Theorem 6.1 remains valid if we admit smooth dependence of operator functions on $t \in \overline{\mathbb{R}}_+$.

The idea of the corner-edge calculus with corner axis variable $t \in \mathbb{R}_+$ and second order edge variable $z \in \mathbb{R}^e$ is to employ elements

$$\hat{b}(t, z, \hat{\tau}, \hat{\zeta}) \in C^\infty_{[0,T]}(\overline{\mathbb{R}}_+ \times \mathbb{R}^e, L^\nu(B, \boldsymbol{l}; \mathbb{R}^{1+e}_{\hat{\tau},\hat{\zeta}}))$$

in (6.2) to create corner symbols

$$t^{-\nu} b(t, z, \tau, \zeta) := t^{-\nu} \hat{b}(t, z, t\tau, t\zeta). \tag{6.3}$$

As noted before we drop the variable z and work with the condition that

$$\hat{b}(t, \hat{\tau}, \hat{\zeta}) \in C^\infty_{[0,T]}(\overline{\mathbb{R}}_+, L^\nu(B, \boldsymbol{l}; \mathbb{R}^{1+e}_{\hat{\tau},\hat{\zeta}})) \tag{6.4}$$

vanishes in a neighbourhood of $t = 0$. This corresponds to a localization of the Fourier pseudo-differential action off $t = 0$. In connection with Mellin quantizations it makes sense to admit the class (6.4) including smoothness in t up to $t = 0$, where vanishing at $t = 0$ is a natural property in traditional quantization with respect to t. Then the Fourier-action near t=0 is replaced by a corresponding Mellin-action, see analogous expressions (5.16), (5.17) in the variable r.

The associated operators $t^{-\nu}\mathrm{Op}_t(b)(\zeta)$ induce continuous maps

$$t^{-\nu}\mathrm{Op}_t(b)(\zeta) : C_0^\infty(\mathbb{R}_+, H^{s,\beta}(B)) \to C_0^\infty(\mathbb{R}_+, H^{s-\nu,\beta-\nu}(B)), \tag{6.5}$$

cf., relation (5.3).

Theorem 6.2. *Let $t^{-\mu}\mathrm{Op}_t(a)(\zeta)$ be another such operator family of the indicated structure* (6.5)*, of order $\mu \in \mathbb{R}$ and associated with the edge-weight data $\boldsymbol{g} := (\gamma, \gamma - \mu)$. Then the ζ-wise composition*

$$t^{-\nu}\mathrm{Op}_t(b)(\zeta) \circ t^{-\mu}\mathrm{Op}_t(a)(\zeta)$$

is of the same nature, of order $\mu + \nu$, modulo $C_0^\infty(\mathbb{R}_+, L^{-\infty}(B, \boldsymbol{l} \circ \boldsymbol{g}; \mathbb{R}^{1+e}_{\hat\tau,\hat\zeta}))$, and associated with the weight data $\boldsymbol{l} \circ \boldsymbol{g} := (\gamma, \gamma - (\mu + \nu))$.

Proof. We formally apply the Leibniz rule for pseudo-differential operators. It is valid for the occurring operator-valued amplitude functions and then we may pass to an asymptotic summation by applying Theorem 6.1 which also works with extra dependence of the variable t. So the main issue is to characterize the summands in the Leibniz product

$$t^{-\nu}b(t, t\tau, t\zeta)\# t^{-\mu}a(t, t\tau, t\zeta) \sim \sum_{k=0}^{\infty} \frac{1}{k!}\partial_\tau^k\Big(t^{-\nu}b(t, t\tau, t\zeta)\Big)\, D_t^k\Big(t^{-\mu}a(t, t\tau, t\zeta)\Big). \tag{6.6}$$

For convenience in this proof we assume $e = 1$. Moreover, observe that the term with $k = 0$ in the asymptotic sum on the right-hand side of (6.6) is equal to the point wise product

$$\begin{aligned} &t^{-(\nu+\mu)}a(t, t\tau, t\zeta)b(t, t\tau, t\zeta) \\ &\qquad\qquad \in t^{-(\nu+\mu)}C^\infty_{[0,T]}(\overline{\mathbb{R}}_+, L^{\nu+\mu}(B, \boldsymbol{l} \circ \boldsymbol{g}; \mathbb{R}^{1+e}_{\hat\tau,\hat\zeta}))|_{(\hat\tau,\hat\zeta)=(t\tau,t\zeta)}. \end{aligned} \tag{6.7}$$

which can be characterized as in the paper [8]. For brevity we also employ notation

$$L^{\mu-m}(B, \boldsymbol{g}; \mathbb{R}^{1+e}_{t\tau,t\zeta}) := L^{\mu-m}(B, \boldsymbol{g}; \mathbb{R}^{1+e}_{\hat\tau,\hat\zeta}))|_{(\hat\tau,\hat\zeta)=(t\tau,t\zeta)} \tag{6.8}$$

for any $m \in \mathbb{N}$. Next we check the derivatives of first order in formula (6.6). The corresponding summand to be evaluated is of the form

$$\frac{1}{i}\partial_\tau(t^{-\nu}b(t, t\tau, t\zeta))\partial_t(t^{-\mu}a(t, t\tau, t\zeta)) \tag{6.9}$$

Let us indicate derivatives in $\hat\tau$ by subscripts $(\hat\tau)$ and with respect to $\hat\zeta$ by subscripts $(\hat\zeta)$. Then after omitting factors with obvious meaning we may consider the product of

$$\partial_\tau(t^{-\nu}b(t, t\tau, t\zeta)) = t^{-\nu+1}b_{(\hat\tau)}(t, t\tau, t\zeta) \tag{6.10}$$

and

$$\begin{aligned}\partial_t(t^{-\mu}a(t,t\tau,t\zeta)) &= t^{-\mu-1}a(t,t\tau,t\zeta)+t^{-\mu}\partial_t a(t,t\tau,t\zeta)\\ &= t^{-\mu-1}a(t,t\tau,t\zeta)+t^{-\mu}[a'+\tau a_{(\hat\tau)}+\zeta a_{(\hat\zeta)}](t,t\tau,t\zeta),\end{aligned} \tag{6.11}$$

where a' indicates the derivative in the first t-argument of a. Now the product of expressions (6.10) and (6.11) has the form

$$\begin{aligned}&t^{-\nu+1}b_{(\hat\tau)}\{t^{-\mu-1}a+t^{-\mu}[a'+\tau a_{(\hat\tau)}+\zeta a_{(\hat\zeta)}]\}\\ &\qquad\in t^{-(\nu+\mu)}C^\infty_{[0,T]}(\overline{\mathbb{R}}_+,L^{\nu+\mu-1}(B,\boldsymbol{l}\circ\boldsymbol{g};\mathbb{R}^{1+e}_{t\tau,t\zeta})).\end{aligned} \tag{6.12}$$

Relation (6.12) can be verified separately for the occurring products. In fact, both

$$t^{-\nu+1}b_{(\hat\tau)}\{t^{-\mu-1}a+t^{-\mu}a'\}=t^{-(\nu+\mu)}b_{(\hat\tau)}a+t^{-(\nu+\mu)+1}b_{(\hat\tau)}a' \tag{6.13}$$

and

$$t^{-\nu+1}b_{(\hat\tau)}\{t^{-\mu}[\tau a_{(\hat\tau)}+\zeta a_{(\hat\zeta)}]\}, \tag{6.14}$$

since $b_{(\hat\tau)}a\in L^{\nu+\mu-1}\subset L^{\nu+\mu-1}$, and $t^{-(\nu+\mu)}L^{\nu+\mu-1}\subset t^{-(\nu+\mu)}L^{\nu+\mu-1}$ the extra t-power appearing on the left does not affect the final result, because the relevant asymptotic sum is carried out by Theorem 6.1 without such factors, and a' in relation (6.13) vanishes for large t such that the additional t can be absorbed in the dependence of symbols with respect to the first t-variable. Moreover, internal weight data which concern the local variable r are not destroyed under differentiations of functions, but "improved" under the process the resulting weight data $\boldsymbol{g}\circ\boldsymbol{l}$ over B are valid for all summands. For (6.14) we can apply similar arguments, since we may multiply the L-spaces by powers of $\hat\tau$ or $\hat\zeta$ which raises orders, while differentiations in $\hat\tau$ or $\hat\zeta$ diminish orders by the corresponding magnitude.

The summands for $k>1$ can be treated by induction. Moreover, the arguments remain valid when we admit symbols with covariable $\hat\zeta$ involved as $t\zeta$. This completes the proof. □

Remark 6.3. *There is an analogue of the relationship* (5.27) *comparing*

$$\mathrm{Op}_t(b)(z,\zeta)\quad\text{and}\quad\mathrm{Op}^\alpha_M(k)(z,\zeta) \tag{6.15}$$

for a corresponding edge operator-valued Mellin symbol k of order ν for some weights α and b is an element of

$$C^\infty_{[0,\mathrm{T}]}(\overline{\mathbb{R}}_+,L^\nu(B,\boldsymbol{g};\mathbb{R}^{1+e}_{\hat\tau,\hat\zeta}))|_{(\hat\tau,\hat\zeta)=(t\tau,t\zeta)}$$

The weight data $(\beta,\beta-\nu)$ concern the weights involved in the Melllin action of $k(t,v)$ along $t\in\mathbb{R}_+$ in the complex Mellin covariable v.

The arguments are similar as in [18, Theorem 3.2.7]. The Mellin quantization in the present exposition refers to the degenerate cone axis variable t and first produces from the Fourier-symbol in (t,τ) a Mellin symbol in in (t,v) with v being the complex Mellin covariable with respect to $t\in\mathbb{R}_+$, where v is running as imaginary part on a weight line, say, $\Gamma_{1/2}$. There is involved an asymptotic summation analogously as Theorem 6.1. Then a kernel cut-off argument produces a holomorphic representative of the desired quality. These constructions, carried out in [18] on the

level of degenerate symbols of first order extend to the case of order 2 which we have here, *i.e.,* we replace X by B. This gives us a map

$$\begin{aligned} C^\infty_{[0,\mathrm{T}]}(\overline{\mathbb{R}}_+, L^\nu(B,\boldsymbol{l};\mathbb{R}^{1+e}_{\hat\tau,\hat\zeta}))|_{(\hat\tau,\hat\zeta)=(t\tau,t\zeta)} \\ \to C^\infty_{[0,\mathrm{T}]}(\overline{\mathbb{R}}_+, M^\nu_{\mathcal{O}}(B,\boldsymbol{l};\mathbb{R}^e_{\hat\zeta}))|_{\hat\zeta=t\zeta} \end{aligned} \tag{6.16}$$

which yields the claimed correspondence $b \mapsto k$.

For an analogue of Theorem 6.2 for Mellin operators in contrast to the Fourier pseudo-differential arguments we may now apply the Mellin-Leibniz product, but the terms also give us information on the behavior of summands for $t \to \infty$.

7. Rinal remark

Let us make some final remarks on future problems in PDEs on manifolds with singularities. We are considering here stratified spaces where singularities are of some regular behavior, *i.e.,* the strata "touch each other " in some transversal way, and Riemannian metrics are non-complete, like over manifolds with smooth boundaries. Since the pseudo-differential analysis in classical context is aimed at establishing the principles of solvability of equations in terms of operator algebras in scales of weighted Sobolev spaces with multiple weights and substructures of operators which reflect asymptotics close to singularities of the respective underlying spaces, the main issues will focus on analyzing individual subalgebras coming from the involved hierarchies of operator-valued symbols. It seems that even most simple questions concerning analogues of index theorems in elliptic cases and the evaluation of analogues of K- groups, generated by difference constructions for principal symbol tuples, in order to express the index in geometric or topological terms, are to a large extent open. Also spectral properties and the evaluation of concrete positions of poles of meromorphic Mellin symbols seem to be far from being completely understood in operators motivated by several applications. Other important questions concern operators which are not elliptic, *e.g.,* parabolic, and which should induce isomorphisms between scales of anisotropic weighted Sobolev spaces. All this induces an enormous abundance of new investigations which in singular cases requires elucidation of operator algebra structures which are to some extent analogous to those from the standard pseudo-differential calculus in connection with the construction of parametrices on the level of symbols. The present investigation has analyzed necessary composition constructions in corner geometries which are modeled by corner substitutions in parameter-dependent theories on a lower level of singularities, and other investigations in this direction may follow, as they have been foreshadowed by the approach in the articles [2] and [3].

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