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DYNAMICAL SYSTEMS WITH A LYAPUNOV FUNCTION ON BOUNDED METRIC SPACES

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Abstract. An algorithm for minimizing an objective function on a set can often be considered as a sequence of self-mappings of the set for which the objective function is a Lyapunov function. In this paper the set is a bounded metric space. We study the asymptotic behavior of trajectories of the dynamical system which is induced by the algorithm and generalize results which are known in the literature in the case when the set is a bounded, convex and closed subset of a Banach space.

1. Introduction and preliminaries

In this paper we study the asymptotic behavior of trajectories of a certain dynamical system which originates in a minimization problem. An algorithm for minimizing an objective function $f : K \to R^1$ on a set K can often be considered as a sequence of self-mappings $A_t : K \to K$, $t = 1, 2, \dots$ of the set K for which the objective function f is a Lyapunov function. More precisely,

$$f(A_t x) \leq f(x)$$

for all $x \in K$ and all natural number t. In this paper the set K is a bounded metric space. We introduce the notion of a normal set of mappings and show that if the sequence $\{A_t\}_{t=1}^{\infty}$ has a subsequence which is a normal set, then the sequence of values of the Lyapunov function f tends to the infimum of f along any trajectory generated by $\{A_t\}_{t=1}^{\infty}$. From the point of view of the theory of dynamical systems the sequence $\{A_t\}_{t=1}^{\infty}$ describes a nonstationary dynamical system with a Lyapunov function f. Also, some optimization procedures in Hilbert and Banach spaces can be represented in such a manner [8, 9, 11].

In the present paper we generalize the results which were obtained in [6] and presented in Chapter 4 of [13] in the case where K is a bounded, closed and convex set in a Banach space. In this case our dynamical system was also studied in [11, 12]. In contrast with our previous results, here we no longer assume that K is a subset of a Banach space. As a matter of fact, K a general bounded metric space. It should be mentioned that in [14] it was considered the case when K is an unbounded metric space but $A_t = A_1$ for all natural numbers t.

Assume that (K, d) is a bounded metric space and that $f : K \to R^1$ is a bounded uniformly continuous function. Set

$$\inf(f) = \inf\{f(x) : \ x \in K\}, \ \sup(f) = \sup\{f(x) : \ x \in K\}.$$

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Denote by $\mathcal{A}$ the set of all self-mappings $A : K \to K$ such that

$$f(Ax) \le f(x) \text{ for all } x \in K. \tag{1.1}$$

A nonempty set $E \subset \mathcal{A}$ is called normal if given $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for each $x \in K$ satisfying

$$f(x) \ge \inf(f) + \epsilon$$

and each $A \in E$ the inequality

$$f(Ax) \le f(x) - \delta(\epsilon)$$

holds.

Denote by $\mathrm{Card}(C)$ the cardinality of a set C. Our first result is proved in Section 2 while the second result is established in Section 3.

2. The first result

Theorem 2.1. *Let $\{A_t\}_{t=1}^\infty \subset \mathcal{A}$, $\{A_{t_k}\}_{k=1}^\infty$ be its subsequence such that the set $\{A_{t_k} : k = 1, 2, \dots\}$ is normal and $\epsilon > 0$. Then there exist a natural number N and $\delta > 0$ such that for each sequence $\{x_t\}_{t=0}^N \subset K$ satisfying*

$$d(A_t x_{t-1}, x_t) \le \delta,\ t = 1, \dots, N$$

the inequality

$$f(x_N) \le \inf(f) + \epsilon$$

holds.

Proof. We may assume without loss of generality that $t_1 > 2$ and $\epsilon < 1$. There exists $\delta_0 \in (0, \epsilon/4)$ such that the following property holds:

(a) for each $x \in K$ satisfying $\inf(f) + \epsilon/4 \le f(x)$ and each integer $k \ge 1$, we have

$$f(x) - f(A_{t_k} x) \ge 4\delta_0.$$

Choose a natural number $k_0 \ge 4$ such that

$$k_0 > \delta_0^{-1}(\sup(f) - \inf(f)) \tag{2.1}$$

and set

$$N = t_{k_0}. \tag{2.2}$$

There exists $\delta \in (0, \delta_0)$ such that for each $y_1, y_2 \in K$ satisfying

$$d(y_1, y_2) \le \delta$$

we have

$$|f(y_1) - f(y_2)| \le \delta_0/N. \tag{2.3}$$

Assume that $\{x_t\}_{t=0}^N \subset K$ satisfies

$$d(A_t x_{t-1}, x_t) \le \delta,\ t = 1, \dots, N. \tag{2.4}$$

We claim that

$$f(x_N) \le \inf(f) + \epsilon.$$

First we show that there exists $t \in \{0, \dots, N\}$ such that

$$f(x_t) \le \inf(f) + \epsilon/4.$$

Assume the contrary. Then

$$f(x_t) > \inf(f) + \epsilon/4,\ t = 0, \dots, N. \tag{2.5}$$

Property (a), (1.1) and (2.2)-(2.5) imply that

$$\begin{aligned}
\sup(f) - \inf(f) \ge f(x_0) - f(x_N) &= \sum_{t=1}^{N} (f(x_{t-1}) - f(x_t)) \\
&= \sum_{t=1}^{N} (f(x_{t-1}) - f(A_t x_{t-1}) + f(A_t x_{t-1}) - f(x_t)) \\
&= \sum_{t=1}^{N} (f(x_{t-1}) - f(A_t x_{t-1})) \\
&\quad + \sum_{t=1}^{N} (f(A_t x_{t-1}) - f(x_t)) \\
&\ge \sum_{t=1}^{N} (f(x_{t-1}) - f(A_t x_{t-1})) - N\delta_0/N \\
&\ge \sum_{k=1}^{k_0} (f(x_{t_k-1}) - f(A_{t_k} x_{t_k-1})) - \delta_0 \\
&\ge 4k_0\delta_0 - \delta_0 \ge k_0\delta_0
\end{aligned}$$

and

$$k_0 \le \delta_0^{-1}(\sup(f) - \inf(f)).$$

This contradicts (2.1). The contradiction we have reached proves that there exists

$$j_0 \in \{0, \dots, N\}$$

such that

$$f(x_{j_0}) \le \inf(f) + \epsilon. \tag{2.6}$$

We may assume without loss of generality that $j_0 < N$. It follows from (1.1), (2.3) and (2.4) that

$$\begin{aligned}
f(x_{j_0}) - f(x_N) &= \sum_{t=j_0+1}^{N} (f(x_{t-1}) - f(x_t)) \\
&= \sum_{t=j_0+1}^{N} (f(x_{t-1}) - f(A_t x_{t-1}) + f(A_t x_{t-1}) - f(x_t)) \\
&= \sum_{t=j_0+1}^{N} (f(x_{t-1}) - f(A_t x_{t-1})) \\
&\quad + \sum_{t=j_0+1}^{N} (f(A_t x_{t-1}) - f(x_t))
\end{aligned}$$

$$\geq \sum_{t=j_0+1}^{N} (f(A_t x_{t-1}) - f(x_t))$$
$$\geq N(-\delta_0/N) \geq -\delta_0.$$

By the relation above and (2.6),

$$f(x_N) \leq f(x_{j_0}) + \delta_0 \leq \inf(f) + \epsilon/4 + \epsilon/4.$$

Theorem 2.1 is proved. □

Theorem 2.1 extends a result of [6] obtained in the case when the whole sequence $\{A_t\}_{t=1}^{\infty}$ is normal and K is a bounded, closed and convex set in a Banach space.

If the minimization problem

$$f(x) \to \min$$
$$x \in K$$

is well-posed [16], then our result implies the convergence of infinite products $A_n \cdots A_1 x$ as $n \to \infty$ for all $x \in K$ to the unique point of minimum of f. Note that infinite products of operators find application in many areas of mathematics. See, for example, [1, 2, 3, 4, 5, 7, 10, 15].

3. The second result

Theorem 3.1. *Let $\{A_t\}_{t=1}^{\infty} \subset \mathcal{A}$, $\{A_{t_k}\}_{k=1}^{\infty}$ be its subsequence such that the set $\{A_{t_k} : k = 1, 2, \dots\}$ is normal and $\epsilon \in (0, 1)$. Assume that N_0 is a natural number such that for each integer $k \geq 1$,*

$$t_{k+1} - t_k \leq N_0. \tag{3.1}$$

Then there exist a natural number N and $\delta > 0$ such that for each sequence $\{x_t\}_{t=0}^{\infty} \subset K$ satisfying

$$d(A_t x_{t-1}, x_t) \leq \delta, \ t = 1, 2, \dots$$

the inequality

$$f(x_N) \leq \inf(f) + \epsilon$$

holds for all integers $t \geq N$.

Proof. There exists $\delta_0 \in (0, \epsilon/4)$ such that the following property holds:

(a) for each $x \in K$ satisfying $\inf(f) + \epsilon/4 \leq f(x)$ and each integer $k \geq 1$, we have

$$f(x) - f(A_{t_k} x) \geq 4\delta_0.$$

Choose a natural number $k_0 \geq 4$ such that

$$k_0 > \delta_0^{-1}(\sup(f) - \inf(f)) \tag{3.2}$$

and set

$$N = N_0 k_0. \tag{3.3}$$

There exists $\delta \in (0, \delta_0)$ such that for each $y_1, y_2 \in K$ satisfying

$$d(y_1, y_2) \leq \delta$$

we have

$$|f(y_1) - f(y_2)| \leq \delta_0/(8N). \tag{3.4}$$

Assume that $\{x_t\}_{t=0}^{\infty} \subset K$ satisfies

$$d(A_t x_{t-1}, x_t) \le \delta,\ t = 1, 2, \dots. \tag{3.5}$$

We claim that

$$f(x_t) \le \inf(f) + \epsilon \text{ for all integers } t \ge N. \tag{3.6}$$

Let $p \ge 0$ be an integer. First we show that there exists $t \in \{pN, \dots, (p+1)N\}$ such that

$$f(x_t) \le \inf(f) + \epsilon/4.$$

Assume the contrary. Then

$$f(x_t) > \inf(f) + \epsilon/4,\ t = pN, \dots, (p+1)N. \tag{3.7}$$

Property (a), (1.1), (3.3)-(3.5) and (3.7) imply that

$$\begin{aligned}
\sup(f) - \inf(f) &\ge f(x_{pN}) - f(x_{(p+1)N}) \\
&= \sum_{t=pN+1}^{(p+1)N} (f(x_{t-1}) - f(x_t)) \\
&= \sum_{t=pN+1}^{(p+1)N} (f(x_{t-1}) - f(A_t x_{t-1})) + \sum_{t=pN+1}^{(p+1)N} (f(A_t x_{t-1}) - f(x_t)) \\
&\ge \sum_{t=pN+1}^{(p+1)N} (f(x_{t-1}) - f(A_t x_{t-1})) - N(-\delta_0/(8N)) \\
&\ge \sum_{t=pN+1}^{(p+1)N} (f(x_{t-1}) - f(A_t x_{t-1})) - \delta_0/8 \\
&\ge 4\delta_0 \mathrm{Card}(\{t \in \{pN+1, \dots, (p+1)N\}\} \cap \{t_k :\ k = 1, 2, \dots\}) - \delta_0 \\
&\ge 4\delta_0 k_0 - \delta_0 \ge k_0 \delta_0
\end{aligned}$$

and

$$k_0 \le \delta_0^{-1}(\sup(f) - \inf(f)).$$

This contradicts (3.2). The contradiction we have reached proves that there exists

$$j \in \{pN, \dots, (p+1)N\}$$

such that

$$f(x_j) \le \inf(f) + \epsilon/4. \tag{3.8}$$

We claim that

$$f(x_{(p+1)N}) \le \inf(f) + \epsilon/2.$$

We may assume that $j < (p+1)N$. It follows from (1.1), (3.4) and (3.5) that

$$f(x_j) - f(x_{(p+1)N}) = \sum_{t=j+1}^{(p+1)N} (f(x_{t-1}) - f(x_t))$$

$$
\begin{aligned}
&= \sum_{t=j+1}^{(p+1)N} (f(x_{t-1}) - f(A_t x_{t-1}) + f(A_t x_{t-1}) - f(x_t)) \\
&= \sum_{t=j+1}^{(p+1)N} (f(x_{t-1}) - f(A_t x_{t-1})) \\
&\quad + \sum_{t=j+1}^{(p+1)N} (f(A_t x_{t-1}) - f(x_t)) \\
&\geq \sum_{t=j+1}^{(p+1)N} (f(A_t x_{t-1}) - f(x_t)) \\
&\geq N(-\delta_0/(8N)) \geq -\delta_0/8.
\end{aligned}
$$

By the relation above and (3.8),

$$f(x_{(p+1)N}) \leq f(x_j) + \delta_0/8 \leq \inf(f) + \epsilon/4 + \epsilon/4.$$

Therefore

$$f(x_{pN}) \leq \inf(f) + \epsilon/2 \text{ for all natural numbers } p. \tag{3.9}$$

Let $p \geq 1$ be an integer and

$$t \in \{pN+1, \dots, (p+1)N - 1\}.$$

In view of (1.1), (3.4) and (3.5),

$$
\begin{aligned}
f(x_t) - f(x_{pN}) &= \sum_{i=pN+1}^{t} (f(x_i) - f(x_{t-1})) \\
&= \sum_{i=pN+1}^{t} (f(x_i) - f(A_i x_{i-1})) + \sum_{i=pN+1}^{t} (f(A_i x_{i-1}) - f(x_{i-1})) \\
&\leq \sum_{i=pN+1}^{t} (f(x_i) - f(A_i x_{i-1})) \\
&\leq N(\delta_0/(8N)) \leq \delta_0/8.
\end{aligned}
$$

By the relation above and (3.9),

$$f(x_t) \leq f(x_{pN}) + \delta_0/8 \leq \inf(f) + \epsilon/2 + \delta_0 \leq \inf(f) + \epsilon.$$

Thus (3.6) holds and Theorem 3.1 is proved.

Theorem 3.1 generalizes a result of [6] obtained in the case when the whole sequence $\{A_t\}_{t=1}^{\infty}$ is normal and K is a bounded, closed and convex set in a Banach space.

$\square$

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