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GRADIENT METHODS FOR SOLVING ZERO-SUM LINEAR-QUADRATIC DIFFERENTIAL GAMES

AVIV GIBALI* AND OLEG KELIS

Abstract. In this paper we focus on a zero-sum linear-quadratic differential game. The main feature of this game is that the weight matrix of the minimizer's control cost in the cost functional is singular. Due to this singularity, the game cannot be solved either by applying the Isaacs MinMax principle, or the Bellman-Isaacs equation approach.

Glizer and Kelis [22] studied appropriate diagonal singular form of the weight matrix in the cost functional. In this paper we study the case where the weight matrix has general singular form. This means that only a part of coordinates of the minimizer's control is singular, while the rest of coordinates are regular.

As application, we introduce a pursuit-evasion differential game and propose two gradient methods for solving this game, the Arrow-Hurwicz-Uzawa and Korpelevich's Extragradient method. We present numerical illustrations which demonstrate the procedures performances.

1. Introduction

In this paper, a zero-sum differential game with linear dynamics and quadratic cost functional is considered. A weight matrix of the minimizer's control cost in this cost functional is singular, meaning that the game is singular. Namely, the considered game can be solved neither by application of the Isaacs MinMax principle [30], nor by using the Bellman-Isaacs equation approach [6, 9, 30, 36]. This occurs, because the problem of minimization of its variational Hamiltonian with respect to the minimizer control either has no solution, or has infinitely many solutions.

Singular differential games appears in such topics of control theory and applications as: pursuit-evasion [14, 24, 44, 46]; robust controllability [51]; robust interception of a maneuvering target [45, 52]; robust tracking [53]; robust investment [29].

In some cases, higher order optimality conditions can be helpful in solving singular games (see, e.g., [14, 46]). However, such conditions are useless for analysis of singular games having no an optimal control of at least one of the players in the class of regular functions, even if the cost functional has finite infsup and supinf in this class of functions. To the best authors' knowledge, this case was analyzed only in few works. Thus, in the works of [48, 49], the existence of an almost equilibria of an infinite-horizon differential game with no minimizer's control cost in the cost functional was established by using a quadratic matrix inequality. In [1], a finite-horizon differential game with a singular minimizer's control cost and no terminal

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state cost in the cost functional was studied. Using an auxiliary game with an impulsive dynamics, a new game of reduced order, having the same value as the original game, was constructed. The open-loop solution of the original singular game was derived in some class of generalized functions. In the work of [45], a finite-horizon differential game was analyzed by a regularization approach i.e., by its approximate converting to an auxiliary regular game, in the case where the cost functional does not contain the minimizer's control cost, while contains the terminal state cost.

The paper of [22] deals with a more general case of the singular game than in the work of [45]. Namely, the cost functional of the considered differential game contains the minimizer's control cost. The weight matrix of this cost is singular but, in general, nonzero. This game does not have, in general, an optimal minimizer's control in the class of regular functions. The considered game is treated also by a regularization. This regularization yields an auxiliary differential game with partial cheap control of the minimizer.

Differential games with total cheap control of at least one of the players were studied in the literature, see for example [17, 40, 47, 51–53]. However, to the best knowledge of the authors, differential games with partial cheap control of at least one of the players, have been studied only in a few works, for example [22].

It should be noted that the regularization approach was applied in the literature to studying singular optimal control problems, see e.g. [7, 18–20, 37] and references therein. In the papers of [45] and [22], this approach was applied to rigorous analysis of a singular differential game.

In the present paper, we considered a slightly more general case of the singular game than in the work of [22], where the weight matrix of the minimizer's control cost has general singular form. As a particular case of such a problem we consider an pursuit-evasion game and applied appropriate gradient methods under certain assumptions to solution of this game. So, our aim is to find a saddle-point of a special case of zero-sum singular differential game by using gradient methods and not a regularization approach that was used in [53] and [22].

In Subsection 1.1, we survey the results of the Gibali, Censor and Reich [10–12] and of Glizer and Kelis [21–23] which pour light on the two different approaches that will be tested in this paper for finding saddle points.

Our paper is organized as follows. In Section 2 the problem formulation of a zero-sum differential game with linear dynamics and quadratic cost functional is considered. Then in Section 3 we study the *Singular Differential Game* (SDG) which follows by a special case of a singular problem, an pursuit-evasion game, in Section 4. In Section 5 we present numerical illustrations by applying two gradient methods, the Arrow-Hurwicz-Uzawa algorithm [4] and Korpelevich's extragradient method [35]. Finally in Sections 6 our conclusions are presented.

1.1. **Relation to previous work.**

1.1.1. *The works of Gibali, Censor and Reich.*

Lets us first recall the *Variational Inequality Problem* (VIP) in the Euclidean space $\mathbb{E}^n$. Let $C \subset \mathbb{E}^n$ be a non-empty, closed and convex set and let $F : \mathbb{E}^n \to \mathbb{E}^n$. The

VIP consists in finding a point $x^* \in C$, such that

$$x^* \in C \text{ and } \langle F(x^*), x - x^* \rangle \geq 0, \text{ for all } x \in C. \tag{1.1}$$

This problem, which is fundamental in Optimization Theory was introduced by Hartman and Stampacchia in [28]. Many algorithms for solving the VIP are projection algorithms that employ projections onto C or onto some related set in order to reach iteratively a solution. For an excellent treatise on variational inequality problems in finite-dimensional spaces, see the two-volume book by Facchinei and Pang [13]. The books by Konnov [34] and Patriksson [39] contain extensive studies of VIPs including applications, algorithms and numerical results. For a wide range of applications of VIPs, see, e.g., the book by Kinderlehrer and Stampacchia [32]. See also Auslender and Teboulle [5] and the recent book of Zaslavski [54].

The importance of VIPs stems from the fact that several fundamental problems in Optimization Theory can be formulated as VIPs, as the following few examples show.

Example 1.1. *Constrained minimization.* Let $C \subset \mathbb{E}^n$ be a nonempty, closed and convex subset and let $g : \mathbb{E}^n \to \mathbb{E}$ be a continuously differentiable function which is convex on C. Then x^* is a minimizer of g over C if and only if x^* solves the VIP

$$\langle \nabla g(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C, \tag{1.2}$$

where ∇g is the gradient of g (see, e.g., [8, Proposition 3.1, p. 210]). When g is not differentiable, we get the VIP

$$\langle u^*, x - x^* \rangle \geq 0 \text{ for all } x \in C, \tag{1.3}$$

where $u^* \in \partial g(x^*)$ and ∂g is the (set-valued) subdifferential of g (see, e.g., [26, Chapter 4, Subsection 3.5]).

Example 1.2. Let $\mathbb{E}^n$ and $\mathbb{E}^n$ be two Euclidean spaces, and let C_1 and C_2 be two convex subsets of $\mathbb{E}^n$ and $\mathbb{E}^n$, respectively. Given a bifunction $g : \mathbb{E}^n \times \mathbb{E}^m \to \mathbb{E}$, the *Saddle-Point Problem* (SDP) is to find a point $(u_1^*, u_2^*) \in C_1 \times C_2$ such that

$$g(u_1^*, u_2) \leq g(u_1^*, u_2^*) \leq g(u_1, u_2^*) \text{ for all } (u_1, u_2) \in C_1 \times C_2. \tag{1.4}$$

This problem can be written as the VIP of finding $(u_1^*, u_2^*) \in C_1 \times C_2$ such that

$$\left\langle \begin{pmatrix} \nabla g_{u_1}(u_1^*, u_2^*) \\ -\nabla g_{u_2}(u_1^*, u_2^*) \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \right\rangle \geq 0 \text{ for all } (u_1, u_2) \in C_1 \times C_2. \tag{1.5}$$

Many algorithms for solving the VIP are projection algorithms that employ projections onto C (the feasible set of the VIP) or onto some related set in order to reach iteratively a solution. Korpelevich method for example [35] (see also Antipin [2]) which is known as the *Extragradient Method* uses two orthogonal projections onto C per each iteration, according to the following rule. Given the current iterate x^k, calculate

$$y^k = P_C(x^k - \tau F(x^k)), \tag{1.6}$$

$$x^{k+1} = P_C(x^k - \tau F(y^k)), \tag{1.7}$$

where τ is some positive number and P_C denotes the Euclidean nearest point projection onto C. Although the convergence of the algorithm is guaranteed under the assumptions of Lipschitz continuity and pseudo-monotonicity of the involved mapping F, there is still the need to calculate two projections onto C, which might seriously affect the efficiency of the algorithm. as a first step to overcome this obstacle, Censor, Gibali and Reich in [10–12] presented the *Subgradient Extragradient Method* (SEM), in which the second projection (1.7) onto C is replaced by a projection onto a specific constructible half-space which is actually one of the subgradient half-spaces.

In order to prove convergence the authors assume that F is monotone on C, Lipschitz continuous on $\mathbb{E}^n$, and the Lipschitz constant L is known, so $\tau \in (0, 1/L)$. Later Gibali [16], motivated by the work of Khobotov [33], proposed a modified version of the SEM is presented in which the mapping F is assumed to be only continuous instead of Lipschitz. The advantage of the later proposed method is that it is using an Armijo-Goldstein-type ([3]) adaptive step-rule size τ which guarantee the convergence of the algorithm. The subgradient extragradient algorithm is presented next.

The subgradient extragradient algorithm

Step 0: Select an arbitrary starting point $x^0 \in \mathbb{E}^n$ and $\tau > 0$, and set $k = 0$.

Step 1: Given the current iterate x^k, compute

$$y^k = P_C(x^k - \tau f(x^k)) \tag{1.8}$$

construct the set

$$T_k := \left\{ w \in \mathbb{E}^n \mid \left\langle \left(x^k - \tau f(x^k) \right) - y^k, w - y^k \right\rangle \leq 0 \right\} \tag{1.9}$$

and calculate the next iterate

$$x^{k+1} = P_{T_k}(x^k - \tau f(y^k)). \tag{1.10}$$

Step 2: If $x^k = y^k$ then stop. Otherwise, set $k \leftarrow (k+1)$ and return to **Step 1.**

In the rest of this paper the extragradient method is used as an iterative procedure to find a saddle-point of a zero-sum linear-quadratic differential games.

1.1.2. *The works of Glizer and Kelis.*

An abstract form of zero-sum linear-quadratic differential game contains a dynamic system (1.11) and cost functional (1.12):

$$\frac{dz(t)}{dt} = Az(t) + Bu(t) + Cv(t), \quad z(0) = z_0, \quad t \in [0, t_f], \tag{1.11}$$

$$J(u, v) = z(t_f)^T F z(t_f) + \int_0^{t_f} [z^T(t) D z(t) + u^T(t) G_u u(t) - v^T(t) G_v v(t)] dt \tag{1.12}$$

where, t_f is a given final time moment; superscript T denotes transposition; $z(t) \in \mathbb{E}^n$ is a state vector; $u(t) \in \mathbb{E}^r$, $(r \leq n)$, $v(t) \in \mathbb{E}^s$ are the players' controls; A, B and C are given constant matrices of corresponding dimensions; z_0 is a given n-vector; F, D, are given positive semi-definite symmetric matrix; G_v is a given positive definite symmetric matrix; G_u is a given symmetric matrix.

1.1.3. *The regularization approach.*
In (1.12) the weight matrix G_u can have several forms, two of them, in which the game is singular, are:

(1) $G_u = 0$, which means that all the coordinates of the minimizer's control are singular. In the study of Shinar, Glizer and Turetsky in [45] the functional $J(u, v)$ has the form:

$$J^{\varepsilon}(u,v) = z(t_f)^T F z(t_f) + \int_0^{t_f} [z^T(t) D z(t) + \varepsilon^2 u^T(t) u(t) - v^T(t) G_v v(t)] dt. \tag{1.13}$$

(2) G_u has the following diagonal form

$$G_u = \text{diag}\left(g_{u_1}, ..., g_{u_q}, \underbrace{0, ..., 0}_{r-q} \right), \qquad 0 \leq q < r, \tag{1.14}$$

which means that only part of coordinates of the minimizer's control is singular, while the rest are regular. In this case Glizer and Kelis in [22] consider the following more complicated functional:
(1.15)

$$J^{\varepsilon}(u,v) = z^T(t_f) F z(t_f) + \int_0^{t_f} \left(z^T(t) D z(t) + u^T(t) \left(G_u + \mathcal{E} \right) u(t) - v^T(t) G_v v(t) \right) dt,$$

where

$$G_u + \mathcal{E} = \text{diag}\left(g_{u_1}, ..., g_{u_q}, \underbrace{\varepsilon^2, ..., \varepsilon^2}_{r-q} \right) \tag{1.16}$$

It should be noted that in both cases $\varepsilon > 0$ is a small parameter.

In both cases the game (1.11)-(1.12) is singular and hence the approach which is used is the *regularization approach*, this means that the singular game is replaced with a "regular" form. Namely, the dynamic system (constraint) (1.11) is the same but the cost functional $J(u, v)$ in (1.12) is augmented with the control cost of the minimizer multiplied by a small positive coefficient. These "regular" cheap/partial cheap control games were analyzed by using the singular perturbation techniques which includes investigation of appropriate Riccati matrix differential equation (for finite time case) or Riccati matrix algebraic equation (for infinite time case). For more details see the works of Shinar, Glizer and Turetsky, [45], Glizer and Kelis, [22] and Glizer and Kelis [21, 23], respectively.

It is important to note that although the regularization approach which is used in the paper of [23] is similar to the approach used in the works [21, 22, 45], the main results of [23], as well as their derivation, differ considerably from the ones of these works. Namely, in [22, 45] the finite horizon games were considered in wide classes of the players' admissible feedback controls. Each control of this class guarantees the existence and uniqueness of the absolutely continuous solution to the equation of the game's dynamics (subject to a given initial condition) against any square integrable open-loop control of the opponent. Moreover, the time realization of the feedback control along this solution is square integrable. In these classes of the players' admissible feedback controls, subject to proper conditions, the complete solutions of the games (optimal control of the maximizing player, optimal control sequence of the minimizing player and the game value) were obtained.

2. Problem formulation

Consider the differential equation (1.11) controlled by two decision makers (players) with t_f a given final time moment; $z(t) \in \mathbb{E}^n$ is a state vector; $u(t) \in \mathbb{E}^r$, $(r \leq n)$, $v(t) \in \mathbb{E}^s$ are the players' controls; A, B and C are given constant matrices of corresponding dimensions; $z_0 \in \mathbb{E}^n$ is a given vector. The cost functional, to be minimized by u (the minimizer) and maximized by v (the maximizer), is defined in (1.12) with F, D, G_v and G_u are as described below.

In what follows we assume the following assumptions.

AI. The matrix B has full rank r;
AII. The matrices F , D and G_u are positive semidefinite;
AIII. The matrix G_v is positive definite;
AVI. The players' controls $u(t)$ and $v(t)$ are square integrable functions in $[0, t_f]$.

An essential result to our analysis in the sequel is the *Singular-Value Decomposition (SVD) Theorem* [50] which is phrased and proved for the convenience of the readers next.

Theorem 2.1. *Given a $m \times n$ matrix A, there exist positive constants (`singular values` of A, actually the square roots of the eigenvalues of A^TA) $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r > 0$ for some integer r such that*

$$U^TAV = \Sigma \tag{2.1}$$

where U and V are orthogonal matrices (U is $m \times m$ and V is $n \times n$) and Σ is the $m \times n$ matrix

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \sigma_1 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & & & \\ 0 & 0 & \dots & \sigma_r & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & & \end{pmatrix}. \tag{2.2}$$

Proof. Since the matrix A^TA is real symmetric then by the spectral theorem for symmetric matrices it has a complete set of orthonormal eigenvectors: $A^TAx_j = \lambda_j x_j$, and

$$x_i^T A^T A x_j = \lambda_j x_i^T x_j = \lambda_j \delta_{ij} \tag{2.3}$$

where δ_{ij} is the Kronecker delta.

For positive λ_j's (say $j = 1, ..., r$), we define $\sigma_j = \sqrt{\lambda_j}$ and $q_j = \frac{Ax_j}{\sigma_j}$; Then $q_i^T q_j = \delta_{ij}$.

Next, we extend the q_i's to a basis for $\mathbb{E}^m$ and putting x's in V and q's in U, then

$$(U^T AV)_{ij} = q_i^T Ax_j = \begin{cases} 0 & \text{if } j > r; \\ \sigma_j q_i^T q_j = \sigma_j \delta_{ij}, & \text{if } j \le r. \end{cases} \tag{2.4}$$

Hence, we obtain $U^T AV = \Sigma$ and the proof is complete. □

Using the SVD theorem there exist orthogonal matrixes U and V of appropriate dimensions such that $G_u = U^T \widehat{G}_u V$ where the $r \times r$-matrix G_u has the following form

$$G_u = \text{diag}\big(g_{u_1}, ..., g_{u_q}, \underbrace{0, ..., 0}_{r-q}\big), \qquad 0 \le q < r, \tag{2.5}$$

with $g_{u_1} \ge ... \ge g_{u_q} > 0$ and rank$G_u = q < r$. Then the cost functional translates to (1.12).

Remark 2.2. In the sequel of this paper, we deal with the differential game (1.11)-(1.12). The cost functional $J(u, v)$ of this game is minimized by the control $u(t)$ and maximized by the control $v(t)$. Since the weight matrix of the minimizer's control cost in the cost functional $J(u, v)$ is singular, the solution of (1.11)-(1.12) (if any) can be obtained neither by the Isaacs's MinMax principle nor by the Bellman-Isaacs equation method. Moreover, this game does not have, in general, an optimal control of the minimizer among regular functions. We call such a game Singular Differential Game (SDG). Singular differential games of the above mentioned form were studied in the works of [21], [22], and [23].

3. Regularization of the SDG

To study SDG we use a regularization approach. Namely, we replace it with a regular differential game, which is close in some sense to the SDG. This new game has the same dynamics (1.11) as the SDG has. However, in contrast with the SDG, the cost functional in the new game has the "regular" form, i.e., it contains a quadratic control cost of the minimizer with a "regular" (positive definite) weight matrix. Its form is as in (1.15) and (1.16).

Since the parameter $\varepsilon > 0$ is small, the problem (1.11), (1.15) is a partial cheap control differential game, i.e., a differential game with a cost of some control coordinates of at least one of the players much smaller than costs of the other control coordinates and a state cost in the cost functional. We call this game the Partial Cheap Control Game (PCCG). Due to the smallness of the cost of respective control coordinates, the boundary-value problem, as well as the Bellman-Isaacs equation, associated with partial cheap control game by controls optimality conditions, are singularly perturbed. It should be noted that PCCG (1.11), (1.15) was investigated in [22]. Also, various results on the topic of singularly perturbed differential games can be found, for instance, in the works of [17, 21, 23, 42, 45] and references therein.

4. Pursuit-evasion game

Consider a particular case of the singular problem (1.11)-(1.13). Namely, $n = 2$, $r = 1$, $s = 1$, $q = 0$. The matrices of coefficients in (1.11)-(1.13) are

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \; F = \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} \tag{4.1}$$

$$B^T = (0,1), \quad C^T = (0,1), \quad G_u = 0, \quad G_v = g \tag{4.2}$$

where the scalar g, $f > 0$.

The initial position z_0 is

$$z_0^T(0) = (0,1). \tag{4.3}$$

The system (1.11) subject to the data (4.2), (4.2) has the following form:

$$\begin{cases} \frac{dz_1(t)}{dt} &= z_2(t) \\ \frac{dz_2(t)}{dt} &= u(t) + v(t) \end{cases}. \tag{4.4}$$

The solution of (4.4) with initial position (4.3) has the following integral form:

$$\begin{aligned} z(t) &= \left(z_1(t)^T \mid z_2(t)^T\right) \\ &= M\,(t)\,M\,(0)^{-1}\,z_0(0) + \int_0^t M\,(t)\,M\,(s)^{-1}\,f\,(s)\,ds, \end{aligned} \tag{4.5}$$

where

$$M\,(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \tag{4.6}$$

is a fundamental matrix solution of the corresponding homogeneous system

$$\begin{cases} \frac{dz_1(t)}{dt} &= z_2(t) \\ \frac{dz_2(t)}{dt} &= 0 \end{cases} \tag{4.7}$$

and

$$f\,(s) = \begin{pmatrix} 0 \\ u(s) + v(s) \end{pmatrix}. \tag{4.8}$$

Thus, the analytical solution (after some technical calculations in (4.5) with (4.6), (4.8)) can be written as

$$z_1(t) \;=\; t + \int_0^t (t-s)\cdot(u(s)+v(s))ds \tag{4.9}$$

$$z_2(t) \;=\; 1 + \int_0^t (u(s)+v(s))ds. \tag{4.10}$$

The system (4.4), with (4.3) is a linearized kinematic model of a planar engagement between two vehicles - an interceptor (pursuer) and a target (evader) where both vehicles are directly controlled by their lateral accelerations $u(t) = -a_p(t)$ and $v(t) = a_e(t)$, respectively. The coordinates of the state vector $z(t)$ ($z_1(t)$ and $z_2(t)$) are the relative lateral separation and the relative lateral velocity of the vehicles. More details of such an engagement can be found, for instance, in [27] and [43].

The behavior of each player in this engagement is evaluated by the cost functional (1.13) of the form

$$J(u,v) = fz_2^2(t_f) + \int_0^{t_f} \left(-gv^2(t)\right) dt. \tag{4.11}$$

The cost functional (4.11) has to be minimized by the pursuer and maximized by the evader.

In what follows, the game, consisting of the dynamics (1.11), with the data (4.1), (4.2) and initial condition (4.3) and the cost functional (4.11) is called the (Singular) Pursuit-evasion Differential Game.

Remark 4.1. It should be noted that a more general form of such an Pursuit-evasion Differential Game was considered in [45] and was admitted an explicit analytical solution using the regularization method i.e., solving a certain Riccati differential equation.

The regularized Pursuit-evasion Differential Game has the same equation of dynamics (1.11) and the following functional

$$J^{\varepsilon}(u,v) = fz_2^2(t_f) + \int_0^{t_f} \left(\varepsilon^2 u^2\,(t) - gv^2(t)\right) dt. \tag{4.12}$$

Our goal is to minimize (with respect to u) and maximize (with respect to v) the cost functional (4.12), over a sets of admissible controls $Q \subseteq L_2[0,t_f]$ and $R \subseteq L_2[0,t_f]$ (closed, convex and bounded sets of the Hilbert space $L_2[0,t_f]$). The problem is then

$$\min_{u\in Q}\max_{v\in R} J^{\varepsilon}(u,v). \tag{4.13}$$

It is well known that J^{ε} of (4.12) is continuous, convex in u, concave in v (convex-concave) and differentiable. Hence, we will study the J^{ε} min-max problem (4.13) by investigating its equivalent saddle-point reformulation. We wish to find a saddle-point $(u^*,v^*) \in Q\times R$ (see Example 1.2) such that

$$J^{\varepsilon}(u^*,v) \le J^{\varepsilon}(u^*,v^*) \le J^{\varepsilon}(u,v^*) \tag{4.14}$$

for all $v \in R$ and $u \in Q$.

Saddle-point problems is one of the fundamental problems in Convex Programming and Game theory, for an intensive and complete investigation on saddle-point function we refer the reader to Rockafellar [41]. One class of iterative methods for approximating saddle-points is known as gradient methods. One of the earliest and simples gradient method is the Arrow-Hurwicz-Uzawa algorithm [4]. Another related method in this area is the Korpelevich's extragradient method [35], see also [2], see Subsection 1.1. Many other extension and related algorithms can also be used, for example the works of Censor, Gibali and Reich [10–12] and Iusem and Nasri [31]. In addition, in the recent book of Zaslavski [54], Chapters 12 and 13 are focus on solving variational inequalities by the extragradient method with perturbations, which are mainly extensions to inexact version of the method.

So, our goal is to test the performances of the Arrow-Hurwicz-Uzawa algorithm and Korpelevich's extragradient method for approximating the saddle-point problem (4.14). Since these methods require the evaluation of gradients of J^{ε} (subgradients in general), we first wish to explain briefly how this is done by recalling the definition of functional derivative (variation derivative) in one variable, see e.g., [15] for further details.

Definition 4.2. Given a functional of one variable $F : M \to \mathbb{E}$, where M is a manifold representing continues functions y. The `functional differential` of $F(y)$, denoted by δF, defined in terms of the functional derivative, denoted by $\frac{\delta F}{\delta y}$ as follows:

$$\delta F(y) = \int \frac{\delta F}{\delta y}(x)\, \delta y(x) dx = \lim_{h\to 0} \frac{F(y + h\delta y) - F(y)}{h}, \tag{4.15}$$

where δy is the variation of y.

We show the calculation of the functional derivatives of $J^{\varepsilon}(u, v)$ with respect to u and v based on Definition 4.2.

Proposition 4.3. *Given the following functional:*

$$J^{\varepsilon}(u, v) = f\left(1 + \int_0^{t_f} (u(t) + v(t))dt\right)^2 + \int_0^{t_f} \left(\varepsilon^2 u^2(t) - gv^2(t)\right) dt$$

its derivatives with respect to u and v, are

$$\frac{\delta J^{\varepsilon}(u, v)}{\delta u(t)} = 2f\left(1 + \int_0^{t_f} (u(t) + v(t))dt\right) + 2\varepsilon^2 u(t) \tag{4.16}$$

and

$$\frac{\delta J^{\varepsilon}(u, v)}{\delta v(t)} = 2f\left(1 + \int_0^{t_f} (u(t) + v(t))dt\right) - 2gv(t). \tag{4.17}$$

Proof. We start by calculating the functional differential with respect to u, namely, $\delta J^{\varepsilon}(u, v)$, using Definition 4.2

$$\begin{aligned}
\delta J^{\varepsilon}(u, v) &= \lim_{h\to 0} \frac{J^{\varepsilon}(u + h\delta u, v) - \delta J^{\varepsilon}(u, v)}{h} \\
&= \lim_{h\to 0} \frac{\left[f\left(1+\int_0^{t_f}(u+h\delta u+v)dt\right)^2+\int_0^{t_f}\left(\varepsilon^2(u+h\delta u)^2-gv^2\right)dt\right]-\left[f\left(1+\int_0^{t_f}(u+v)dt\right)^2+\int_0^{t_f}\left(\varepsilon^2u^2-gv^2\right)dt\right]}{h} \\
&= \lim_{h\to 0} \frac{f\left[\left(1+\int_0^{t_f}(u+h\delta u+v)dt\right)^2-\left(1+\int_0^{t_f}(u+v)dt\right)^2\right]+\int_0^{t_f}\left(\varepsilon^2\left(u^2+2uh\delta u+(h\delta u)^2\right)-gv^2\right)dt-\int_0^{t_f}\left(\varepsilon^2u^2-gv^2\right)dt}{h} \\
&= \lim_{h\to 0} \frac{f\left[\left(\left(1+\int_0^{t_f}(u+h\delta u+v)dt\right)+\left(1+\int_0^{t_f}(u+v)dt\right)\right)\left(\left(1+\int_0^{t_f}(u+h\delta u+v)dt\right)-\left(1+\int_0^{t_f}(u+v)dt\right)\right)\right]}{h} \\
&\quad + \lim_{h\to 0} \frac{h\int_0^{t_f}2\varepsilon^2u\delta udt+h^2\int_0^{t_f}(\delta u)^2dt}{h} \\
&= \lim_{h\to 0} \frac{f\left(2+2\int_0^{t_f}(u+v)dt+h\int_0^{t_f}\delta udt\right)\cdot h\int_0^{t_f}\delta udt+h\int_0^{t_f}2\varepsilon^2u\delta udt+h^2\int_0^{t_f}(\delta u)^2dt}{h} \\
&= \lim_{h\to 0} \left[f\left(2 + 2\int_0^{t_f}(u + v)\,dt + h\int_0^{t_f}\delta udt\right) \cdot \int_0^{t_f}\delta udt + \int_0^{t_f}2\varepsilon^2u\delta udt + h\int_0^{t_f}(\delta u)^2\,dt\right]
\end{aligned}$$

$= \left[2f\left(1+\int_0^{t_f}(u+v)\,dt\right)\cdot\int_0^{t_f}\delta u dt+\int_0^{t_f}2\varepsilon^2 u\delta u dt\right].$

From here one can read that the functional derivative $\frac{\delta J^\varepsilon(u,v)}{\delta u}$ has the form

$$\frac{\delta J^\varepsilon(u,v)}{\delta u}=2f\left(1+\int_0^{t_f}(u+v)\,dt\right)+2\varepsilon^2 u$$

as in (4.16).

In a similar way, using Definition 4.2 one derive the functional derivative $\frac{\delta J^\varepsilon(u,v)}{\delta v}$ as in (4.17)and the proof is complete. □

Remark 4.4. Equivalent forms of functionals derivatives in the sense of Gâteaux and Frèchet, we refer the reader, for instance, to [15].

Remark 4.5. It should be noted that investigation of a more general forms of convex-concave functionals than (4.12), in zero-sum differential games, which does not differentiable it need to use the subdifferentials (subgradients) in the sense of [41]. For these we refer the reader to [25] and references therein.

For illustrations we choose two gradient methods, the Arrow-Hurwicz-Uzawa algorithm [4] (see also [38]) Korpelevich's extragradient method [35] (consult also [2]).

The Arrow-Hurwicz-Uzawa iterative step is as follows. Given the current iterate (u_k,v_k), calculate the next iterate via

$$\begin{aligned} u_{k+1} &= \mathcal{P}_Q\left(u_k-\alpha\frac{\delta J^\varepsilon(u_k,v_k)}{\delta u\,(t)}\right)\\ &= \mathcal{P}_Q\left(u_k-2\alpha f\left(1+\int_0^{t_f}(u_k(t)+v_k(t))dt\right)+2\alpha\varepsilon^2 u_k\right), \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} v_{k+1} &= \mathcal{P}_R\left(v_k+\alpha\frac{\delta J^\varepsilon(u_k,v_k)}{\delta v\,(t)}\right)\\ &= \mathcal{P}_R\left(v_k+2\alpha f\left(1+\int_0^{t_f}(u_k(t)+v_k(t))dt\right)-2\alpha g v_k\right) \end{aligned} \tag{4.19}$$

where $\mathcal{P}_Q$ and $\mathcal{P}_R$ denote the orthogonal projection operators onto the sets Q and R, respectively. The vectors $u_0\in Q$ and $v_0\in R$ are initial iterates, and the scalar $\alpha>0$ is a constant step-size.

On the other hand, Korpelevich's extragradient computes an additional gradient per each iterations and this is the reason for its name, its iterative step is as follows. Given the current iterate (u_k,v_k), calculate the next iterate via

$$\overline{u_k} = \mathcal{P}_Q\left(u_k+2\alpha f\left(1+\int_0^{t_f}(u_k(t)+v_k(t))dt\right)+2\alpha\varepsilon^2 u_k\right); \tag{4.20}$$

$$\overline{v_k} = \mathcal{P}_R\left(v_k+2\alpha f\left(1+\int_0^{t_f}(u_k(t)+v_k(t))dt\right)-2\alpha g v_k\right); \tag{4.21}$$

$$u_{k+1} = \mathcal{P}_Q\left(u_k+2\alpha f\left(1+\int_0^{t_f}(\overline{u_k}(t)+\overline{v_k}(t))dt\right)+2\alpha\varepsilon^2\overline{u_k}\right); \tag{4.22}$$

$$v_{k+1} = \mathcal{P}_R\left(v_k+2\alpha f\left(1+\int_0^{t_f}(\overline{u_k}(t)+\overline{v_k}(t))dt\right)-2\alpha g\overline{v_k}\right); \tag{4.23}$$

A standard assumption for the convergence of the above methods is the so-called *Boundedness of the derivatives*, which means that the functional derivatives of $J^\varepsilon(u_k, v_k)$ and $J^\varepsilon(u_k, v_k)$ with respect to u and v respectively, (see above) are uniformly bounded, i.e., there is a constant $M > 0$ such that

$$\left\|\frac{\delta J^\varepsilon(u_k, v_k)}{u(t)}\right\|_{Q\times R} \leq M, \quad \left\|\frac{\delta J^\varepsilon(u_k, v_k)}{v(t)}\right\|_{Q\times R} \leq M, \tag{4.24}$$

for all $k \geq 0$.

5. Numerical Illustrations and discussion

In this section the cost functional (4.12) is considered for $f = 0.5$, $t_f = 4$. From the functional derivatives (4.16) and (4.17) once can check that a critical point of (4.12) is $(u, v) = (-0.25, 0)$ which is a saddle-point of (4.12). In Figure 1 we present a numerical experiment with the following choice of parameters: $g = 0.3$, 3000 iteration $\varepsilon = 0.005$, $\alpha = 0.001$ and $(u_0, v_0) = (1, 1)$. In Figure 2, we tested the performance of the algorithms with a smaller $\varepsilon = 0.001$. In both figures, the first rows are the graphs for the pursuer control u and the evader control v as well as of $z(t) = (z_1(t), z_2(t))$ (z_1 is the relative separation and z_2 is the relative velocity) are presented. The graphs of z_1 and z_2 obey the analytical solution (4.9) and (4.10). From these and other choices of parameters, it appears that both methods converges to the saddle-point $(u, v) = (-0.25, 0)$ but probably due to the extra evaluation of the gradients per each iteration in Korpelevich's method, the cpu time and number of computations is higher with respect to the Arrow-Hurwicz-Uzawa algorithm. In addition, the Arrow-Hurwicz-Uzawa produces u which is closer to -0.25 than Korpelevich method and consequently, z_2 approaches to zero faster in the final moment.

It should be noted and seen in Figures 1 and 2 that for $\varepsilon \to 0^+$, the value $z_2(4)$ approaching zero, this means that relative lateral velocity of the vehicles z_2 tends to zero at the end of the game. Moreover, for $\varepsilon \to 0^+$ the relative lateral separation z_1 tends to the constant function 2 at the end of the game. Hence, in this game the pursuer doesn't intercept the evader.

6. Conclusions

We considered finite-horizon singular zero-sum linear-quadratic differential game where the weight matrix of the minimizer's control cost has general singular form than in [22]. As a particular case of such a game we considered a so-called pursuit-evasion game where our purpose was to find a min-max point using gradient methods, in particular the Arrow-Hurwicz-Uzawa and the Korpelevich's algorithms. We established by calculation of appropriate functional derivatives that a min-max point of the presented pursuit-evasion game is $(-0.25, 0)$. It should be noted that the achieved saddle point $(-0.25, 0)$ by the above two gradient methods coincides with symbolic computations in `Maple` for small positive epsilon. Using MATLAB R2017a on an Intel Core i5-4200U 2.3GHz running 64-bit Window, we examined numerical illustrations showing the convergence of the gradient methods to the same saddle-point. The preliminaries results show that a more general setting of functionals and correspondingly the differential game, can be considered, for example, functionals

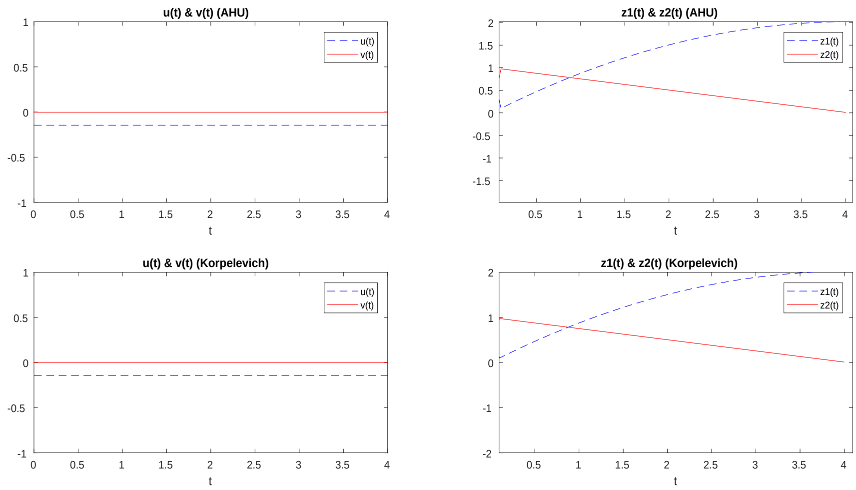

FIGURE 1. Illustration of Arrow-Hurwicz-Uzawa and Korpelevich algorithms with the parameters $\varepsilon = 0.005$, $\alpha = 0.001$, $g = 0.3$, 3000 maximum number of iteration, and starting points are $(u_0(t), v_0(t)) \equiv (1, 1)$ (constant function 1).

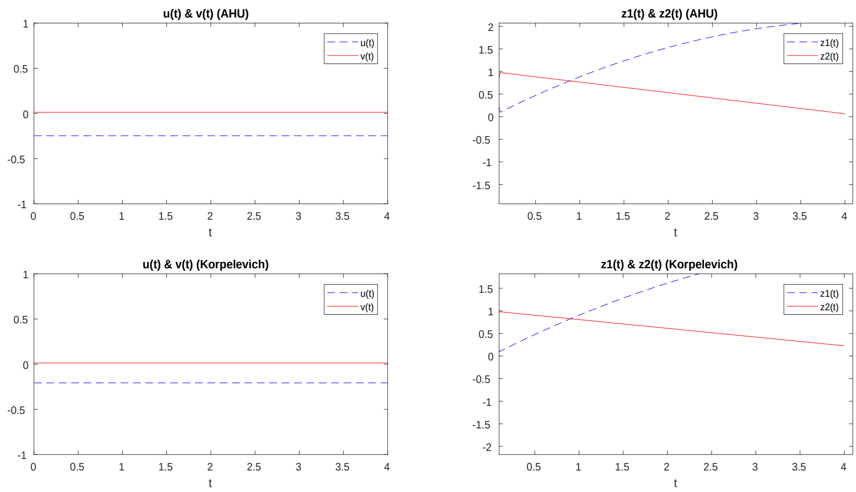

FIGURE 2. Illustration of Arrow-Hurwicz-Uzawa and Korpelevich algorithms with the parameters $\varepsilon = 0.001$, $\alpha = 0.001$, $g = 0.3$, 3000 maximum number of iteration, and starting points are $(u_0(t), v_0(t)) \equiv (1, 1)$ (constant function 1).

which are not necessary differentiable and then subgradient methods can be applied. Concerning more general pursuit-evasion games, our objective is to study the scenario in which the relative lateral separation $z_1^2(t_f)$ appears in (1.13) together or instead of the relative velocity $z_2^2(t_f)$, and moreover, the matrix D is not necessary equals zero. Of course, other iterative methods should be concerned for such games. It is all part of our future goals for research.

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A. Gibali
Department of Mathematics, ORT Braude College, 2161002 Karmiel, Israel

The Center for Mathematics and Scientific Computation, University of Haifa, Mt. Carmel, Haifa 3498838, Israel

E-mail address: `avivg@braude.ac.il`

O. Kelis
Department of Mathematics, ORT Braude College, 2161002 Karmiel, Israel,
&
Department of Mathematics, Faculty of Natural Sciences, University of Haifa, Mount Carmel, Haifa, 31905, Israel.

E-mail address: `olegkelis@braude.ac.il`