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ON ELLIPTIC EQUATIONS WITH COMBINED NONLINEARITIES

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ABSTRACT. We study certain elliptic equations with combined nonlinearities for the asymptotic behavior of the solution as the spatial variable $\mathbf{x}$ approaches the infinity. We have found that the smooth solution that decays sufficiently fast at the infinity must be identically zero.

1. Introduction

The elliptic equation with combined nonlinearities

$$\Delta u = G(\mathbf{x}, u) + F(\mathbf{x}, u), \mathbf{x} \in \Omega \tag{1.1}$$

where Ω is a smooth domain in $\mathbf{R}^n$, has application in the theory for studying the activator-inhibitor systems modeling biological pattern formation [7, 8]. Problems of this type as well as the associated evolution equations have been proposed in the study of cellular automata and interacting particle systems with self-organized criticality [3]. It also appears in the study of long range Van der Waals interactions in thin films spreading on solid surfaces [6] and the study of the flow over an impermeable plate [2, 12]. This equation also appears in the study of the heat conduction in materials with corroded boundary [15] as well as in the study of the curvature of multiply warped products [4].

The mathematical study of the type of equation (1.1) has been an active field of study. Please see [1, 5, 12, 13, 14, 16, 17] and the references in these papers.

In this paper, we would like to examine the asymptotic behavior of the smooth solution as $\mathbf{x}$ approaches the infinity in the case of $\Omega = \mathbf{R}^n$ for the elliptic equation with combined nonlinearities.

$$-\Delta u + g|u|^{q-1}u + h|u|^{p-1}u = 0 \tag{1.2}$$

where $0 < q < p$, g and h are continuous differentiable functions in $\mathbf{R}^n$. We will show that the solution which decays sufficiently fast at the infinity must be identically zero. The method follows [10, 11] in using the Morawetz multiplier [9]. Similar result also holds for the case of P-Laplacian equations with combined nonlinearities

$$-\nabla \cdot (|\nabla u|^{p-2}\nabla u) + g|u|^{a-1}u + h|u|^{b-1}u = 0 \tag{1.3}$$

where $p > 1, 0 < a < b$, $u = u(\mathbf{x})$, $\mathbf{x} \in \mathbf{R^n}$, and biharmonic equations with combined nonlinearities,

$$\Delta^2 u + g|u|^{q-1}u + h|u|^{p-1}u = 0, \tag{1.4}$$

where $0 < q < p$, g and h are in $C^3(\mathbf{R}^n)$.

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As usual, $\mathbf{x} = (x_1, x_2, \ldots, x_n), \nabla u$ denotes the gradient of u, $\nabla \cdot u$ denotes the divergence of u, and $r = |\mathbf{x}|$. We use the notation $u_r = \frac{\partial u}{\partial r} = ((\frac{\mathbf{x}}{r}) \cdot \nabla u)$ and $\partial_j = \partial/\partial x_j$. $F_r(\mathbf{x}, s)$ denotes $\partial F(\mathbf{x}, s)/\partial r = ((\mathbf{x}/r) \cdot \nabla_x F(\mathbf{x}, s))$.

$C^k(\mathbf{R}^n)$ is the space of functions whose partial derivatives of order up to and including k are continuously differentiable. Finally, we define

$$\begin{aligned} H(\zeta, q, p) = {} & ((n-1)/(2r))\zeta(g|u|^{q-1}u + h|u|^{p-1}u)u \\ & - [\zeta_r + ((n-1)/r)\zeta][(1/(q+1))g|u|^{q+1} + (1/(p+1))h|u|^{p+1}] \\ & - \zeta[(1/(q+1))g_r|u|^{q+1} + (1/(p+1))h_r|u|^{p+1}] \end{aligned}$$

2. Elliptic equations with combined nonlinearities

Multiplying Equation (1.2) by $\zeta(u_r + ((n-1)u/(2r)))$, where $\zeta \in C^2(\mathbf{R}^n)$ and $\zeta(\mathbf{x}) = \zeta(|\mathbf{x}|) = \zeta(r)$,
we get

$$0 = (-\Delta u + g|u|^{q-1}u + h|u|^{p-1}u)\zeta(u_r + ((n-1)u/(2r))) = \nabla \cdot Y + Z, \tag{2.1}$$

where

$$\begin{aligned} Y = {} & (\nabla u)[-\zeta(u_r + ((n-1)/(2r))u)] + (\nabla\zeta)((n-1)/(4r))u^2 \\ & + (\mathbf{x}/r)\zeta[(1/2)|u|^2 - ((n-1)/(4r^2))u^2 + (1/(q+1))g|u|^{q+1} \\ & + (1/(p+1))h|u|^{p+1}] \end{aligned}$$

and

$$\begin{aligned} Z = {} & (1/2)\zeta_r|\nabla u|^2 + (\nabla u \cdot \nabla\zeta)u_r - \zeta_r|u_r|^2 \\ & + ((1/r)\zeta - \zeta_r)(|\nabla u|^2 - |u_r|^2) \\ & + [(n-1)/(2r)][(1/r)\zeta_r - (1/2)(\Delta\zeta) + ((n-3)/(2r^2))\zeta]u^2 \\ & + H(\zeta, q, p) \end{aligned}$$

Theorem 2.1. *Let $n > 3$. Assume that u is a C^2 solution which satisfies*

(A) $\lim_{R\to\infty} (\sup_{|\mathbf{x}|\le R} (|x^{\boldsymbol{\alpha}}||D^{\boldsymbol{\beta}}u(x)|)) = 0$, *for all multi-indices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that*

$$|\boldsymbol{\alpha}| \le n-1 \quad \textit{and} \quad |\boldsymbol{\beta}| \le 1,$$

(B) $\lim_{R\to\infty} R^{n-1} \sup_{|\mathbf{x}|=R} |(1/(q+1))g|u|^{q+1} + (1/(p+1))h|u|^{p+1}| = 0,$ *and*

(C) $H(1, q, p) \ge 0$.
Then $u \equiv 0$.

Proof. Let $\zeta = 1$. Integrating both sides of (2.1) in $\mathbf{R}^n$ and using the conditions (A) and (B), we get

$$\int_{\mathbf{R}^n} [(1/r)(|\nabla u|^2 - |u_r|^2) + ((n-1)(n-3)/(4r^3))u^2 + H(1, q, p)]d\mathbf{x} = 0.$$

Thus

$$\begin{aligned}0 &\le \int_{\mathbf{R}^n} [(1/r)(|\nabla u|^2 - |u_r|^2 + ((n-1)(n-3)/(4r^3))u^2]d\mathbf{x} \\ &= \int_{\mathbf{R}^n} -H(1,q,p)d\mathbf{x} \le 0,\end{aligned}$$

since u satisfies the assumption (C).

Therefore $\int_{\mathbf{R}^n}[(1/r)(|\nabla u|^2 - |u_r|^2) + ((n-1)(n-3)/(4r^2))u^2]d\mathbf{x} = 0$. Since $n > 3, u \equiv 0$. □

Remark 2.2. Assume that g and h are constants.

The condition (A) in the hypothesis will be satisfied if

$$\lim_{R\to\infty} R^{n-1} \sup_{|\mathbf{x}|=R} |u| = 0.$$

Also any of the following conditions would satisfy the condition (C).

(a) $g > 0$ and $h > 0$ with $1 < q < p$,
(b) $g < 0$ and $h > 0$ with $0 < q < 1 < p$,
(c) $g < 0$ and $h < 0$ with $0 < q < p < 1$.

Remark 2.3. In case either g or h is not a constant, then the following condition would satisfy (C).

$$(n-1)(1-q)g + 2rg_r \le 0 \quad \text{and} \quad (n-1)(1-p)h + 2rh_r \le 0.$$

Remark 2.4. In the case of $n \ge 2$, by taking appropriate function for ζ, we can also get conditions for $u \equiv 0$. The details will be in a forthcoming article.

3. P-Laplacian equations with combined nonlinearities

Multipying equation (1.3) by $\zeta(u_r + ((n-1)u/(2r)))$, where $\zeta \in C^1(\mathbf{R}^n)$ and $\zeta(\mathbf{x}) = \zeta(|\mathbf{x}|) = \zeta(r)$, we get

$$\begin{aligned}0 &= (-\nabla\cdot(|\nabla u|^{p-2}\nabla u) + g|u|^{a-1}u + h|u|^{b-1}u)\zeta(u_r + ((n-1)u/(2r))) \\ &= \nabla\cdot Y + Z,\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}Y = &-\zeta|\nabla u|^{p-2}(\nabla u)(u_r + ((n-1)u/(2r))) \\ &+ (\zeta/p)(\mathbf{x}/r)|\nabla u|^p + \zeta(\mathbf{x}/r)[(1/(a+1))g|u|^{a+1} \\ &+ (1/(b+1))h|u|^{b+1}]\end{aligned}$$

and

$$\begin{aligned}Z = &-((\zeta/r) - \zeta')|\nabla u|^{p-2}[(u_r)^2 + ((n-1)/(2r))u_r u] \\ &- [(\zeta'/p) - (\zeta/(2pr))((n+1)p - 2(n-1))]|\nabla u|^p \\ &+ H(\zeta, a, b).\end{aligned}$$

Theorem 3.1. *Let $n \ge 2$. Assume that u is a C^2 solution of (1.3) such that*

(A) $\lim_{R\to\infty} \sup_{|\mathbf{x}|\le R} (|\mathbf{x}^{\boldsymbol{\alpha}}||D^{\boldsymbol{\beta}}u(\mathbf{x})|) = 0$, *for all multi-indices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such that $|\boldsymbol{\alpha}| \le n$ and $|\boldsymbol{\beta}| \le 1$, and*

(B) $\lim_{R\to\infty} R^{n-1} \sup_{|\mathbf{x}|=R} |(1/(a+1))g|u|^{a+1} + (1/(b+1))h|u|^{b+1}| = 0.$

(C) *If* $1 < p < 2n/(n+1)$ *and* $H(r,a,b) \le 0$, *then* $u \equiv 0$.

(D) *If* $p > 2n/(n+1)$ *and* $H(r,a,b) \ge 0$, *then* $u \equiv 0$.

Proof. Let $\zeta = r$. Integrating both sides of (3.1) in $\mathbf{R}^n$ and using the assumptions (A) and (B), we get, after some calculation,

$$\int_{\mathbf{R}^n} [(2n-(n+1)p)/(2p)]|\nabla u|^p d\mathbf{x} = \int_{\mathbf{R}^n} H(r,a,b)d\mathbf{x}.$$

The conclusion of this theorem follows directly from the assumptions (C) and (D). □

Remark 3.2. Assume g and h are constants. Then any of the following conditions would imply $u \equiv 0$:

(a) $1 < p < 2n/(n+1), g > 0, h > 0, 0 < a < b < (n+1)(n-1)$.

(b) $1 < p < 2n/(n+1), g > 0, h < 0, 0 < a < (n+1)/(n-1) < b$.

(c) $1 < p < 2n/(n+1), g < 0, h < 0, (n+1)/(n-1) < a < b$.

(d) $p > 2n/(n+1), g > 0, h > 0, (n+1)/(n-1) < a < b$.

(e) $p > 2n/(n+1), g < 0, h > 0, 0 < a < (n+1)/(n-1) < b$.

(f) $p > 2n/(n+1), g < 0, h < 0, 0 < a < b < (n+1)/(n-1)$.

4. Biharmonic equations with combined nonlinearities

Multiplying both sides of (1.4) by $\zeta(u_r + ((n-1)u/(2r)))$, where $\zeta(\mathbf{x}) = \zeta(|\mathbf{x}|) = \zeta(r)$ is in $C^4(\mathbf{R}^n)$, we get

$$0 = \nabla \cdot Y + Z \tag{4.1}$$

where Y depends on g, h, ζ and u as well as their partial derivatives up to and including the third order

$$Z = (3\zeta'/2)(\Delta u)^2 + A(u_r)^2 + B(|\nabla u|^2 - |u_r|^2) + Cu^2 + (\zeta - r\zeta')P + H(\zeta,q,p)$$

where

$$A = -7\zeta'''/2 - (n-1)(n-3)(\zeta' - (\zeta/r))/(2r^2),$$
$$B = -3\zeta'''/2 + (n-5)\zeta''/r - (n^2+2n-19)(\zeta' - \zeta/r)/(2r^2),$$
$$C = ((n-1)/2)[\zeta''''/(2r) + (n-3)\zeta'''/(r^2) + (n-3)(n-7)\zeta''/(2r^3) - 3(n-3)(n-5)(\zeta' - \zeta/r)/(2r^4)].$$

and

$$P = (2/r)\Big[\sum_{i,j}(S_{ij}u)^2 - \sum_i \Big(\sum_j (x_j/r)S_{ij}u\Big)^2\Big] \ge 0,$$

where

$$S_{ij}u = (x_i/r^3)\sum_k [x_k(x_k\partial_j - x_j\partial_k)u_r] + \partial_i \sum_k [(x_k/r^2)(x_k\partial_i - x_i\partial_k)u].$$

Theorem 4.1. *Assume* $n \ge 5$. *Assume that* u *is a* C^4 *solution of (1.4) such that*

(A) $\lim_{R\to\infty}(\sup_{|\mathbf{x}|\le R}(|\mathbf{x}^{\boldsymbol{\alpha}}||D^{\boldsymbol{\beta}}u(\mathbf{x})|))=0$ *for all multi-indices* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *such that* $|\boldsymbol{\alpha}|\le n-1$ *and* $|\boldsymbol{\beta}|\le 3$,

(B) $\lim_{R\to\infty}(R^{n-1}\sup_{|\mathbf{x}|=R}|(1/(q+1))g|u|^{q+1}+(1/(p+1))h|u|^{p+1}|)=0$, *and*

(C) $H(1,q,p)\ge 0$.

Then $u\equiv 0$.

Proof. Let $\zeta=1$. Integrating both sides of (4.1) in $\mathbf{R}^n$ and using the assumptions (A) and (B), we get

$$0=\int_{\mathbf{R}^n}[(n-1)(n-3)(u_r)^2/(2r^3)+(n^2+2n-19)(|\nabla u|^2-|u_r|^2)/(2r^3)$$
$$+3(n-3)(n-5)u^2/(2r^5)+P+H(1,q,p)]d\mathbf{x}.$$

Therefore

$$\begin{aligned}0&\le\int_{\mathbf{R}^n}[(n-1)(n-3)(u_r)^2/(2r^3)+(n^2+2n-19)(|\nabla u|^2-|u_r|^2)/(2r^3)\\&\quad+3(n-3)(n-5)u^2/(2r^5)+P]d\mathbf{x}\\&=\int_{\mathbf{R}^n}-H(1,q,p)d\mathbf{x}\\&\le 0\end{aligned}$$

from the assumption (C).

Thus $\int_{R^n}[(n-1)(n-3)(u_r)^2/(2r^3)+(n^2+2n-19)(|\nabla u|^2-|u_r|^2)/(2r^3)+3(n-3)(n-5)u^2/(2r^5)+P]d\mathbf{x}=0$.

Since $n\ge 5$, $u\equiv 0$. □

Remark 4.2. Assume g and h are constants. Then any of the following conditions would satisfy the assumption (C).

(a) $g>0$ and $h>0$ with $1<q<p$,
(b) $g<0$ and $h>0$ with $0<q<1<p$,
(c) $g<0$ and $h<0$ with $0<q<p<1$,

Remark 4.3. In the case of $n\ge 2$, by taking appropriate function for ζ, we can also get conditions for $u\equiv 0$. The details will be in a forthcoming article.

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