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POINCARÉ LEMMA ON SOME SUBRIEMANNIAN MANIFOLDS

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ABSTRACT. The article widely recalls recent developements on characterizations of conservative vector fields on two very important examples in subRiemannian manifolds: the Heisenberg group and the quaternion Heisenberg group. We also generalize these results to nonisotropic quaternion Heisenberg groups. The potential functions related to conservative vector fields are able to be solved explicitly in integral forms.

1. Introduction

Conservative vector fields that appear in vector calculus are ones in which integrating along two paths connecting the same two points are equal. From mathematical or physical point of view, it is worthwhile to consider conservative vector fields on subRiemannian manifolds.

Let $\mathbf{X} = \{X_1, X_2, \ldots, X_m\}$ be m linearly independent vector fields defined on an n-dimensional manifold $\mathcal{M}_n$ with $m \leq n$. The subspace $T_{\mathbf{X}}$ spanned by $X_1, \ldots, X_m$ is called the *horizontal subspace*, and its complement is referred to as the *missing directions*. When $T_{\mathbf{X}} = T\mathcal{M}_n$, then $m = n$ and hence $\mathcal{M}_n$ is a Riemannian manifold. Let $V = (a_1, a_2, \ldots, a_n)$ be a vector-valued function defined on $\mathcal{M}_n$ where a_j, $j = 1, \ldots, n$ are smooth functions. One wishes to find necessary and sufficient conditions on a_j's so that V is conservative, *i.e.,* there exists a function f, called the *potential function*, that satisfies the following system

$$X_1 f = a_1, \quad X_2 f = a_2, \quad \cdots \quad X_n f = a_n. \tag{1.1}$$

For example, let $V = (a, b)$ be a vector-valued function defined on $\mathbb{R}^2$ where a and b are two smooth functions. Assume that $X_1 = \frac{\partial}{\partial x}$ and $X_2 = \frac{\partial}{\partial y}$. Then V is conservative if and only if $\frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}$. In fact, denote $\omega = adx + bdy$ and

$$f(x,y) = \int_{r(t)} \omega = \int_0^1 \omega(r'(t))dt = \int_0^1 a(tx,ty)x + b(tx,ty)ydt, \tag{1.2}$$

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where $r(t) = t(x,y), t \in [0,1]$, is a straight line joining the origin and the point (x,y). Then by straightforward computations,

$$\frac{\partial f}{\partial x}(x,y) = a(x,y) + \int_0^1 ty\left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dt,$$
$$\frac{\partial f}{\partial y}(x,y) = b(x,y) + \int_0^1 tx\left(\frac{\partial a}{\partial y} - \frac{\partial b}{\partial x}\right) dt.$$

The result follows immediately. The potential function f in (1.2) can be interpreted as the work done by the force $\omega = adx + bdy$ from the origin to the point (x,y) connected by the straight line $r(t)$.

Now let us turn to the case when $T_{\mathbf{X}} \neq T\mathcal{M}_n$. Since the complement of $T_{\mathbf{X}}$, by definition, is the missing directions, extra vector fields are needed so as to generate $T\mathcal{M}_n$. Assume $\mathbf{X}$ satisfies the *bracket generating property*: the horizontal vector fields $\mathbf{X}$ and their Lie brackets span $T\mathcal{M}_n$. Then by Chow's Theorem [6], we know that given any two points $A, B \in \mathcal{M}_n$, there is a piecewise C^1 *horizontal curve* $\gamma : [0,1] \to \mathcal{M}_n$ such that

$$\gamma(0) = A, \quad \gamma(1) = B,$$

and

$$\dot\gamma(s) = \sum_{k=1}^m a_k(s) X_k.$$

Then we may define the "length" of γ as usual:

$$\ell(\gamma) \;=\; \int_0^1 \sqrt{a_1^2(s) + a_2^2(s) + \cdots + a_m^2(s)}\, ds.$$

The shortest length $d_{cc}(A,B)$ is called the *Carnot-Carathéodory distance* between A, $B \in \mathcal{M}_n$ which is given by

$$d_{cc}(A,B) := \inf \ell(\gamma)$$

where the infimum is taken over all absolutely continuous horizontal curves joining A and B. Hence, we may define a geometry on $\mathcal{M}_n$ which is so-called *subRiemannian geometry*. One notes that in place of $r(t)$ in $\mathbb{R}^2$, the horizontal curve γ and the Carnot-Carathéodory distance will play an essential role in deriving our results in a subRiemannian setting. Characterizations of conservative vector fields on Heisenberg groups and quaternion Heisenberg groups are considered by Chang, *et al.* ([2, 3, 4, 5]) recently. We are going to recall results in turn, briefly sketch their proofs if necessary, and add remarks to them.

The Heisenberg group $\mathcal{H}^n$ may be considered as $\mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law [1]

$$(\tilde x_1, \tilde y_1, \ldots, \tilde x_n, \tilde y_n, \tilde z) \;=\; (x_1, y_1, \ldots, x_n, y_n, z) \cdot (x_1', y_1', \ldots, x_n', y_n', z')$$
$$= (x_1 + x_1', y_1 + y_1', \ldots, x_n + x_n', y_n + y_n', z + z' + 2\sum_{j=1}^n \alpha_j (x_j y_j' - y_j x_j')).$$

The *Heisenberg vector fields* on $\mathcal{H}^n$ are given by

(1.3) $$X_j = \partial_{x_j} - 2\alpha_j y_j \partial_z, \quad Y_j = \partial_{y_j} + 2\alpha_j x_j \partial_z, \quad j = 1, \ldots, n,$$

where the coefficients α_j are assumed not all zero so that ∂_z can be generated by X_j and Y_j by their Lie bracket $[X_j, Y_j] = X_jY_j - Y_jX_j = 4\alpha_j\partial_z$.

The quaternion Heisenberg group qH^{n-1} is a $4n-1$ real-dimensional nilpotent Lie group with the group law

$$(p, w) \cdot (q, v)$$
$$= (p + q, w + v + (\sum_{j,k=1}^{4(n-1)} a^1_{jk}x'_jx_k)\mathbf{i} + (\sum_{j,k=1}^{4(n-1)} a^2_{jk}x'_jx_k)\mathbf{j} + (\sum_{j,k=1}^{4(n-1)} a^3_{jk}x'_jx_k)\mathbf{k}),$$

where $(p, w) = (p_1, \ldots, p_{n-1}, w)$ and $(q, v) = (q_1, \ldots, q_{n-1}, v)$ are in $\mathbb{R}^{4n-4}\times\mathbb{R}^3, p_j = (x_{4j-3}, x_{4j-2}, x_{4j-1}, x_{4j}), q_j = (x'_{4j-3}, x'_{4j-2}, x'_{4j-1}, x'_{4j})$, and all a^l_{jk} are real which satisfying

$$a^l_{jk} = -a^l_{kj}. \tag{1.4}$$

Consider $n = 2$ for simplicity, the vector fields on qH^1 are given by

$$X_j = \frac{\partial}{\partial x_j} + \sum_{k=1}^{4}\sum_{l=1}^{3} a^l_{jk}x_k\frac{\partial}{\partial y_l}, \quad j = 1, \ldots, 4. \tag{1.5}$$

Missing directions are generated by their Lie brackets given by

$$[X_n, X_m] = X_nX_m - X_mX_n = -2\sum_{l=1}^{3} a^l_{nm}\frac{\partial}{\partial y_l}. \tag{1.6}$$

Remark 1.1. The quaternion Heisenberg group qH^1 can be reduced to an isotropic quaternion Heisenberg group, which is identified to the boundary of the Siegel upper half plane of high dimensional quaternion space, if we modify (1.4) to the following assumption (see *e.g.,* [5] and [8])

$$\begin{cases} a^l_{jk} = -a^l_{kj}, \\ a^1_{21} = a^1_{34} = a^2_{31} = a^2_{42} = a^3_{41} = a^3_{23} = a > 0, \\ a^l_{jk} = 0, \quad \text{otherwise}. \end{cases} \tag{1.7}$$

Since qH^1 is a larger class than the isotropic quaternion Heisenberg group, this article will focus on qH^1, the (nonisotropic) quaternion Heisenberg group.

Now we are ready to recall characterizations of conservative vector fields on $\mathcal{H}^n$ and qH^1.

2. Integrability condition

We now consider the solvability condition, called the *integrability condition*, for (1.1). That is, we wish to know characterizations for conservative vector fields. The first result was discovered by Chang, *et al.* [2] on $\mathcal{H}^1$ which we recorded as follows.

Theorem 2.1 ([2])**.** *Let $X_1 = \partial_x - 2y\partial_z, X_2 = \partial_y + 2x\partial_z$ be the Heisenberg vector fields. The system $X_1f = a, X_2f = b$ has a solution if and only if*

$$X_1^2b = (X_1X_2 + [X_1, X_2])a,$$
$$X_2^2a = (X_2X_1 + [X_2, X_1])b.$$

Followed by Theorem 2.1, the same question was considered on $\mathcal{H}^n$ by Chang, *et al.* [4]. The biggest difference between $\mathcal{H}^n$ for $n \geq 2$ and $\mathcal{H}^1$ is the concept of curl, which was used essentially in deriving the integrability condition.

Theorem 2.2. *Let $X_j, Y_j, j = 1, \ldots, n$ be the Heisenberg vector fields on $\mathcal{H}^n$ defined in* (1.3). *For smooth functions $a_1, b_1, \ldots, a_n, b_n$, the system $X_j f = a_j, Y_j f = b_j, j = 1, \ldots, n$ is solvable if and only if*

$$\begin{cases} X_l a_j = X_j a_l, \quad X_j b_l = Y_l a_j, \quad Y_l b_j = Y_j b_l, \\ [X_{j_0}, Y_{j_0}] a_j = X_j (X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}), \\ [X_{j_0}, Y_{j_0}] b_j = Y_j (X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}), \\ X_j b_j - Y_j a_j = \begin{cases} 0, & \text{if } \alpha_j = 0, \\ \frac{\alpha_j}{\alpha_{j_0}} (X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}), & \text{if } \alpha_j \neq 0, \end{cases} \end{cases} \tag{2.1}$$

where $1 \leq j \neq l \leq n$ and j_0 is a positive integer no larger than n such that $\alpha_{j_0} \neq 0$.

Proof. Since α_j are not all zero, choose an index j_0 such that $\alpha_{j_0} \neq 0$. Let $T_j = [X_j, Y_j]$ for $1 \leq j \leq n$, $T = T_{j_0}$, and $c = X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}$. Then $\{X_1, Y_1, \ldots, X_n, Y_n, T\}$ forms an orthonormal basis on $\mathcal{H}^n$ with respect to a Riemannian metric g. Observe that

$$X_j f = a_j, Y_j f = b_j, \quad j = 1, 2, \ldots, n$$

if and only if

$$X_j f = a_j, Y_j f = b_j, T f = c, \quad j = 1, 2, \ldots, n.$$

Consider vector fields

$$\operatorname{grad} f = \sum_{j=1}^n \big((X_j f) X_j + (Y_j f) Y_j\big) + (Tf) T \quad \text{and} \quad U = \sum_{j=1}^n (a_j X_j + b_j Y_j) + cT.$$

Then

$$\begin{aligned} & X_j f = a_j, Y_j f = b_j, \quad j = 1, 2, \ldots, n \\ \Longleftrightarrow & X_j f = a_j, Y_j f = b_j, Tf = c, \quad j = 1, 2, \ldots, n \\ \Longleftrightarrow & \operatorname{grad} f = U \Longleftrightarrow \operatorname{curl} U = 0 \\ \Longleftrightarrow & A(X_j, Y_j) = A(X_j, X_l) = A(X_j, Y_l) = A(Y_j, Y_l) \\ & = A(X_j, T) = A(Y_j, T) = 0, \quad 1 \leq j \neq l \leq n, \end{aligned} \tag{2.2}$$

where $\operatorname{curl} U$ is a 2-covariant antisymmetric tensor A on a pair of vector fields (X, Y) defined by

$$A(X, Y) = Y g(U, X) - X g(U, Y) + g(U, [X, Y]). \tag{2.3}$$

The proof of $\operatorname{grad} f = U \Leftrightarrow \operatorname{curl} U = 0$ can be found in [7]. Applying (2.3) on $\{X_1, Y_1, \ldots, X_n, Y_n, T\}$ we have

$$\begin{aligned} & A(X_j, Y_j) = Y_j a_j - X_j b_j + g\left(U, \frac{\alpha_j}{\alpha_{j_0}} T\right) = \begin{cases} Y_j a_j - X_j b_j, & \alpha_j = 0, \\ Y_j a_j - X_j b_j + \frac{\alpha_j}{\alpha_{j_0}} c, & \alpha_j \neq 0, \end{cases} \\ & A(X_j, Y_l) = Y_l a_j - X_j b_l, \quad A(X_j, X_l) = X_l a_j - X_j a_l, \quad A(Y_j, Y_l) = Y_l b_j - Y_j b_l, \\ & A(X_j, T) = [X_{j_0}, Y_{j_0}] a_j - X_j (X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}), \\ & A(Y_j, T) = [X_{j_0}, Y_{j_0}] b_j - Y_j (X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}). \end{aligned}$$

Thus (2.2) is equivalent to

$$X_j b_j - Y_j a_j = \begin{cases} 0, & \alpha_j = 0, \\ \frac{\alpha_j}{\alpha_{j_0}} c, & \alpha_j \neq 0, \end{cases} \quad X_l a_j = X_j a_l, \quad X_j b_l = Y_l a_j, \quad Y_l b_j = Y_j b_l,$$
$$[X_{j_0}, Y_{j_0}] a_j = X_j (X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}), \quad [X_{j_0}, Y_{j_0}] b_j = Y_j (X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}).$$

□

Remark 2.3. The earlier version of Theorem 2.2 may be referred to [4], where the authors assumed $\alpha_j = 1$ for $j = 1, 2, \ldots, n$. In contrast to Theorem 2.2, we called the corresponding theorem in [4] the integrability condition of isotropic case. Theorem 2.2 is thus the integrability condition of nonisotropic case.

The following theorem is the integrability condition on qH^1. The key point is to deal with multiple missing directions of qH^1, in contrast to one missing direction of $\mathcal{H}^n$.

Theorem 2.4. *Let X_1, X_2, X_3, X_4 be the vector fields on qH^1 which are defined as in* (1.5). *Let $\{[X_{i_1}, X_{j_1}], [X_{i_2}, X_{j_2}], [X_{i_3}, X_{j_3}]\}$ be a basis of* $\operatorname{span}\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\}$, *where X_{i_l}, X_{j_l} are chosen from $\{X_1, X_2, X_3, X_4\}$ with $i_l < j_l$. Then for any smooth functions a_1, a_2, a_3, a_4 and some scalars α_l^{ij} we have*

$$X_i a_j - X_j a_i = \sum_{l=1}^{3} \alpha_l^{ij} (X_{i_l} a_{j_l} - X_{j_l} a_{i_l}), \quad 1 \le i < j \le 4, \tag{2.4}$$

$$[X_{i_l}, X_{j_l}] a_k = X_k (X_{i_l} a_{j_l} - X_{j_l} a_{i_l}), \quad 1 \le k \le 4, 1 \le l \le 3 \tag{2.5}$$

if and only if there exists a function f such that $X_1 f = a_1, X_2 f = a_2, X_3 f = a_3$, and $X_4 f = a_4$.

Proof. For any smooth functions a_1, a_2, a_3, a_4, we have

$$\begin{cases} X_1 f = a_1, \\ X_2 f = a_2, \\ X_3 f = a_3, \\ X_4 f = a_4, \end{cases} \iff \begin{cases} X_1 f = a_1, & X_2 f = a_2, \\ X_3 f = a_3, & X_4 f = a_4, \\ [X_1, X_2] f = c_{12}, & [X_1, X_3] f = c_{13}, \quad [X_1, X_4] f = c_{14}, \\ [X_2, X_3] f = c_{23}, & [X_2, X_4] f = c_{24}, \quad [X_3, X_4] f = c_{34}, \end{cases}$$

where $c_{ij} = X_i a_j - X_j a_i, 1 \le i < j \le 4$. Each Lie bracket $T_{ij} := [X_i, X_j], 1 \le i < j \le 4$ on the right of the last statement, as shown in (1.6), is spanned by $\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\}$, and thus lies in a 3-dimensional subbundle. From which, the collection of $T_{ij}, 1 \le i < j \le 4$ are linearly dependent in the subbundle. By the bracket generating property, $T_{ij}, 1 \le i < j \le 4$ form a generating set of this subbundle, i.e.,

$$\operatorname{span}\{T_{12}, T_{13}, T_{14}, T_{23}, T_{24}, T_{34}\} = \operatorname{span}\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\}.$$

Thus, there exists a linearly independent subset $\{T_{i_1 j_1}, T_{i_2 j_2}, T_{i_3 j_3}\}$ of $\{T_{12}, T_{13}, T_{14}, T_{23}, T_{24}, T_{34}\}$. Therefore, with a Riemannian metric g defined on qH^1, $\{X_1, X_2, X_3, X_4, T_{i_1 j_1}, T_{i_2 j_2}, T_{i_3 j_3}\}$ forms an orthonormal basis for qH^1. Let

$$U = a_1 X_1 + a_2 X_2 + a_3 X_3 + a_4 X_4 + c_{i_1 j_1} T_{i_1 j_1} + c_{i_2 j_2} T_{i_2 j_2} + c_{i_3 j_3} T_{i_3 j_3}.$$

The equivalence of $X_1 f = a_1$, $X_2 f = a_2, X_3 f = a_3, X_4 f = a_4$ becomes

$$\begin{cases} X_1 f = a_1, \\ X_2 f = a_2, \\ X_3 f = a_3, \\ X_4 f = a_4, \end{cases} \Longleftrightarrow \begin{cases} X_1 f = a_1, & X_2 f = a_2, \\ X_3 f = a_3, & X_4 f = a_4, \\ T_{i_1 j_1} f = c_{i_1 j_1}, & T_{i_2 j_2} f = c_{i_2 j_2}, \quad T_{i_3 j_3} f = c_{i_3 j_3}, \end{cases}$$

$$\Longleftrightarrow \operatorname{grad} f = U \Longleftrightarrow \operatorname{curl} U = 0$$

$$\Longleftrightarrow A(X_i, X_j) = A(X_k, T_{i_l j_l}) = A(T_{i_l j_l}, T_{i_s j_s}) = 0$$

for $1 \le i, j, k \le 4$ and $1 \le l, s \le 3$, where $\operatorname{grad} f$ is defined by

$$\begin{aligned} \operatorname{grad} f = (X_1 f) X_1 + (X_2 f) X_2 + (X_3 f) X_3 + (X_4 f) X_4 \\ + (T_{i_1 j_1} f) T_{i_1 j_1} + (T_{i_2 j_2} f) T_{i_2 j_2} + (T_{i_3 j_3} f) T_{i_3 j_3}, \end{aligned}$$

and $\operatorname{curl} U$ is a 2-covariant antisymmetric tensor A on a pair of vector fields (X, Y) defined by

$$A(X, Y) = Y g(U, X) - X g(U, Y) + g(U, [X, Y]).$$

Now we calculate the contents of $A(X_i, X_j)$, $A(X_k, T_{i_l j_l})$, and $A(T_{i_l j_l}, T_{i_s j_s})$ as follows. Since $\{T_{i_1 j_1}, T_{i_2 j_2}, T_{i_3 j_3}\}$ is linearly independent, T_{ij} can be expressed as

$$T_{ij} = \sum_{l=1}^{3} \alpha_l^{ij} T_{i_l j_l},$$

which yields

$$A(X_i, X_j) \;=\; X_j a_i - X_i a_j + g(U, T_{ij}) = X_j a_i - X_i a_j + \sum_{l=1}^{3} \alpha_l^{ij} c_{i_l j_l}.$$

So $A(X_i, X_j) = 0$ implies (2.4). Next, since $[X_j, [X_n, X_m]] = 0$, we have

$$A(X_k, T_{i_l j_l}) = [X_{i_l}, X_{j_l}] a_k - X_k (X_{i_l} a_{j_l} - X_{j_l} a_{i_l}).$$

So $A(X_k, T_{i_l j_l}) = 0$ implies (2.5). Now the fact $[T_{i_l j_l}, T_{i_s j_s}] = 0$ implies

$$A(T_{i_l j_l}, T_{i_s j_s}) = [X_{i_s}, X_{j_s}](X_{i_l} a_{j_l} - X_{j_l} a_{i_l}) - [X_{i_l}, X_{j_l}](X_{i_s} a_{j_s} - X_{j_s} a_{i_s}).$$

Due to $[X_j, [X_n, X_m]] = 0$, the previous equality becomes

$$A(T_{i_l j_l}, T_{i_s j_s}) = -X_{j_l} A(X_{i_l}, T_{i_s j_s}) + X_{i_l} A(X_{j_l}, T_{i_s j_s}), \quad 1 \le l, s \le 3.$$

Applying $A(X_{i_l}, T_{i_s j_s}) = A(X_{j_l}, T_{i_s j_s}) = 0$ to obtain $A(T_{i_l j_l}, T_{i_s j_s}) = 0$. In summary, $X_1 f = a_1, X_2 f = a_2, X_3 f = a_3, X_4 f = a_4$ is solvable if and only if (2.4) and (2.5) hold. □

Theorem 2.4 is actually a revised version of the corresponding theorem in [5]. We newly discovered (2.4) as part of the integrability condition due to incomplete computation of $g(U, T_{ij})$ of the corresponding theorem in [5].

3. Poincaré lemma

If the system (1.1) is solvable, the potential function f can be solved explicitly in an integral form. The following theorem is the case in $\mathcal{H}^1$.

Theorem 3.1 ([3]). *Let $X_1 = \partial_x - 2y\partial_z, X_2 = \partial_y + 2x\partial_z$ be the Heisenberg vector fields and $\mathbf{p} = (x, y, z)$ in $\mathcal{H}^1$. Given any smooth functions a and b, and set*

$$c = X_1 b - X_2 a, \quad a_1 = a + y\frac{c}{2}, \quad b_1 = b - x\frac{c}{2}, \quad c_1 = \frac{c}{4}.$$

Consider

$$f(\mathbf{p}) = \int_0^1 \big[a_1(t\mathbf{p})x + b_1(t\mathbf{p})y + c_1(t\mathbf{p})z\big]dt.$$

Then

$$(X_1 f)(\mathbf{p}) = a(\mathbf{p}) + \int_0^1 \frac{tz}{4}(X_1^2 b - (X_1X_2 + [X_1, X_2])a)(t\mathbf{p})dt,$$
$$(X_2 f)(\mathbf{p}) = b(\mathbf{p}) - \int_0^1 \frac{tz}{4}(X_2^2 a - (X_2X_1 + [X_2, X_1])b)(t\mathbf{p})dt.$$

If the conditions

$$X_1^2 b = (X_1X_2 + [X_1, X_2])a, \quad X_2^2 a = (X_2X_1 + [X_2, X_1])b,$$

hold, then $X_1 f = a, X_2 f = b$ with

$$f(\mathbf{p}) = \int_0^1 \big[a(t\mathbf{p})x + b(t\mathbf{p})y\big]dt.$$

Based on the integrability condition for $\mathcal{H}^n$ in Theorem 2.2, the Poincaré's lemma for $\mathcal{H}^n$ is able to be deduced. The derivation of the potential function should also be considered in nonisotropic case.

Theorem 3.2. *Let $X_j, Y_j, j = 1, \ldots, n$ be the Heisenberg vector fields on $\mathcal{H}^n$ defined in* (1.3). *Given smooth functions $a_1, b_1, \ldots, a_n, b_n$ with*

$$X_j b_j - Y_j a_j = \begin{cases} 0, & \text{if } \alpha_j = 0, \\ \frac{\alpha_j}{\alpha_{j_0}}(X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}), & \text{if } \alpha_j \neq 0, \end{cases}$$

$$c^* = \frac{X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}}{4\alpha_{j_0}}, \quad a_j^* = a_j + 2\alpha_j y_j c^*, \quad b_j^* = b_j - 2\alpha_j x_j c^*, \tag{3.1}$$

for $1 \leq j \leq n$. Consider

$$f(\mathbf{p}) = \int_0^1 \sum_{j=1}^n \big(a_j^*(t\mathbf{p})x_j + b_j^*(t\mathbf{p})y_j\big) + c^*(t\mathbf{p})z dt,$$

where $\mathbf{p} = (x_1, y_1, \ldots, x_n, y_n, z)$ *in* $\mathcal{H}^n$. *Then for* $l = 1, \ldots, n$,

$$\begin{aligned} X_l f(\mathbf{p}) = a_l(\mathbf{p}) + \int_0^1 t\Big\{ & \sum_{\substack{j=1\\ j\neq l}}^n [x_j(X_l a_j - X_j a_l)(t\mathbf{p}) + y_j(X_l b_j - Y_j a_l)(t\mathbf{p})] \\ & + \frac{z}{4\alpha_{j_0}}\Big(X_l(X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}) - [X_{j_0}, Y_{j_0}]a_l\Big)(t\mathbf{p})\Big\} dt, \\ Y_l f(\mathbf{p}) = b_l(\mathbf{p}) + \int_0^1 t\Big\{ & \sum_{\substack{j=1\\ j\neq l}}^n [x_j(Y_l a_j - X_j b_l)(t\mathbf{p}) + y_j(Y_l b_j - Y_j b_l)(t\mathbf{p})] \\ & + \frac{z}{4\alpha_{j_0}}\Big(Y_l(X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}) - [X_{j_0}, Y_{j_0}]b_l\Big)(t\mathbf{p})\Big\} dt. \end{aligned}$$

If the conditions

$$\begin{aligned} & [X_{j_0}, Y_{j_0}]a_j = X_j(X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}), \quad [X_{j_0}, Y_{j_0}]b_j = Y_j(X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}), \\ & X_l a_j = X_j a_l, \quad X_j b_l = Y_l a_j, \quad Y_l b_j = Y_j b_l, \quad 1 \le j \ne l \le n \end{aligned}$$

hold, then the system $X_j f = a_j, Y_j f = b_j, j = 1, \ldots, n$ *is solvable and*

$$f(\mathbf{p}) = \int_0^1 g\big(U(\gamma(t)), \dot\gamma(t)\big) dt,$$

where γ *is a horizontal curve joining the origin and* $\mathbf{p}$, $U = \sum_{j=1}^n (a_j X_j + b_j Y_j)$, *and* $g(\cdot,\cdot)$ *is the subRiemannian metric.*

Proof. By (1.3),

$$\begin{cases} X_1 f = a_1 \\ Y_1 f = b_1 \\ \quad\vdots \\ X_n f = a_n \\ Y_n f = b_n \end{cases} \Leftrightarrow \begin{cases} \partial_{x_1} f = a_1 + 2\alpha_1 y_1 \partial_z f \\ \partial_{y_1} f = b_1 - 2\alpha_1 x_1 \partial_z f \\ \quad\vdots \\ \partial_{x_n} f = a_n + 2\alpha_n y_n \partial_z f \\ \partial_{y_n} f = b_n - 2\alpha_n x_n \partial_z f \end{cases} \Leftrightarrow \begin{cases} \partial_{x_1} f = a_1^* \\ \partial_{y_1} f = b_1^* \\ \quad\vdots \\ \partial_{x_n} f = a_n^* \\ \partial_{y_n} f = b_n^* \end{cases} \Leftrightarrow \begin{cases} \partial_{x_1} f = a_1^* \\ \partial_{y_1} f = b_1^* \\ \quad\vdots \\ \partial_{x_n} f = a_n^* \\ \partial_{y_n} f = b_n^* \\ \partial_z f = c^*, \end{cases}$$

where c^*, a_j^*, b_j^* are defined in (3.1). Consider

$$f(\mathbf{p}) = \int_{\gamma(t)} \omega = \int_0^1 \sum_{j=1}^n \big(a_j^*(t\mathbf{p})x_j + b_j^*(t\mathbf{p})y_j\big) + c^*(t\mathbf{p})z dt, \tag{3.2}$$

where $\omega = \sum_{j=1}^n \big(a_j^* dx_j + b_j^* dy_j\big) + c^* dz$ and

$$\begin{aligned} \gamma(t) = t\mathbf{p} &= (tx_1, ty_1, \ldots, tx_n, ty_n, tz) \\ &= (x_1(t), x_2(t), \ldots, x_n(t), y_n(t), z(t)), \quad t \in [0,1] \end{aligned}$$

is a horizontal curve connecting the origin and $\mathbf{p} = (x_1, y_1, \ldots, x_n, y_n, z)$ in $\mathcal{H}^n$. Applying partial derivatives $\partial_{x_l}, \partial_{y_l}$, and ∂_z on (3.2), and by (1.3),

$$\begin{pmatrix} X_1 f(\mathbf{p}) \\ Y_1 f(\mathbf{p}) \\ \vdots \\ X_n f(\mathbf{p}) \\ Y_n f(\mathbf{p}) \end{pmatrix} = B \begin{pmatrix} \partial_{x_1} f(\mathbf{p}) \\ \partial_{y_1} f(\mathbf{p}) \\ \vdots \\ \partial_{x_n} f(\mathbf{p}) \\ \partial_{y_n} f(\mathbf{p}) \\ \partial_z f(\mathbf{p}) \end{pmatrix} = B \begin{pmatrix} a_1^*(\mathbf{p}) \\ b_1^*(\mathbf{p}) \\ \vdots \\ a_n^*(\mathbf{p}) \\ b_n^*(\mathbf{p}) \\ c^*(\mathbf{p}) \end{pmatrix} + \int_0^1 (tBM\mathbf{p}^T)(t\mathbf{p})dt$$

$$= \begin{pmatrix} a_1(\mathbf{p}) \\ b_1(\mathbf{p}) \\ \vdots \\ a_n(\mathbf{p}) \\ b_n(\mathbf{p}) \end{pmatrix} + \int_0^1 (tBM\mathbf{p}^T)(t\mathbf{p})dt, \tag{3.3}$$

where

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & -2\alpha_1 y_1 \\ & 1 & \cdots & 0 & 0 & 2\alpha_1 x_1 \\ & & \ddots & & & \vdots \\ & & & 1 & 0 & -2\alpha_n y_n \\ & & & & 1 & 2\alpha_n x_n \end{pmatrix}$$

is a $2n \times (2n+1)$ upper-triangular matrix, and $M = (m_{ij})$ is a $(2n+1) \times (2n+1)$ skew-symmetric matrix with entries

$$m_{ij} := \begin{cases} \left.\begin{array}{lcl} m_{(2l-1)(2s-1)} & = & \partial_{x_l} a_s^* - \partial_{x_s} a_l^* \\ m_{(2l)(2s-1)} & = & \partial_{y_l} a_s^* - \partial_{x_s} b_l^* \\ m_{(2l-1)(2s)} & = & \partial_{x_l} b_s^* - \partial_{y_s} a_l^* \\ m_{(2l)(2s)} & = & \partial_{y_l} b_s^* - \partial_{y_s} b_l^* \end{array}\right\} 1 \le l \le s \le n \\ \left.\begin{array}{lcl} m_{(2l-1)(2n+1)} & = & \partial_{x_l} c^* - \partial_z a_l^* \\ m_{(2l)(2n+1)} & = & \partial_{y_l} c^* - \partial_z b_l^* \end{array}\right\} 1 \le l \le n. \end{cases}$$

The integrand $tBM\mathbf{p}^T$ of (3.3) is a vector with $2n$ entries

$$\begin{aligned} (tBM\mathbf{p}^T)_{2l-1} = t\Big\{ & \sum_{\substack{j=1 \\ j \ne l}}^{n} [x_j(X_l a_j - X_j a_l) + y_j(X_l b_j - Y_j a_l)] \\ & + \frac{z}{4\alpha_{j_0}} \Big(X_l(X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}) - [X_{j_0}, Y_{j_0}] a_l \Big) \Big\}, \end{aligned}$$

and

$$\begin{aligned} (tBM\mathbf{p}^T)_{2l} = t\Big\{ & \sum_{\substack{j=1 \\ j \ne l}}^{n} [x_j(Y_l a_j - X_j b_l) + y_j(Y_l b_j - Y_j b_l)] \\ & + \frac{z}{4\alpha_{j_0}} \Big(Y_l(X_{j_0} b_{j_0} - Y_{j_0} a_{j_0}) - [X_{j_0}, Y_{j_0}] b_l \Big) \Big\}, \end{aligned}$$

for $l = 1, \ldots, n$. Under the integrability condition (2.1), the entries of the integrand $tBM\mathbf{p}^T$ are all zero. Hence the system $X_j f = a_j, Y_j f = b_j, j = 1, \ldots, n$ holds and its solution f can be deduced from (3.2) as

$$\int_{\gamma(t)} \omega = \int_0^1 \sum_{j=1}^n \big(a_j^*(\gamma(t))\dot{x}_j + b_j^*(\gamma(t))\dot{y}_j\big) + c^*(\gamma(t))\dot{z}dt$$

$$= \int_0^1 \sum_{j=1}^n \big(a_j(\gamma(t))\dot{x}_j + b_j(\gamma(t))\dot{y}_j\big) + [\dot{z} - 2\sum_{j=1}^n \alpha_j(x_j\dot{y}_j - y_j\dot{x}_j)]c^*(\gamma(t))dt. \tag{3.4}$$

Note that

$$\dot{\gamma} = \sum_{j=1}^n (\dot{x}_j X_j + \dot{y}_j Y_j) + [\dot{z} - 2\sum_{j=1}^n \alpha_j(x_j\dot{y}_j - y_j\dot{x}_j)]\partial_z. \tag{3.5}$$

Since γ is horizontal, $\dot{\gamma}$ can be constructed only by X_j's and Y_j's. Hence by (3.5), $\dot{z} = 2\sum_{j=1}^n \alpha_j(x_j\dot{y}_j - y_j\dot{x}_j)$ and so (3.4) turns into

$$\int_0^1 \sum_{j=1}^n \big(a_j(\gamma(t))\dot{x}_j + b_j(\gamma(t))\dot{y}_j\big)dt = \int_0^1 g\big(U(\gamma(t)), \dot{\gamma}(t)\big)dt,$$

where $U = \sum_{j=1}^n (a_j X_j + b_j Y_j)$ and $g(\cdot,\cdot)$ is the subRiemannian metric. □

The isotropic version for Poincaré's lemma in $\mathcal{H}^n$ is particularly true. We simply recorded this particular result from our earlier paper as follows.

Corollary 3.3 ([4]). *Let $X_j, Y_j, j = 1, \dots, n$ be the Heisenberg vector fields on $\mathcal{H}^n$ defined in* (1.3) *with $\alpha_j = 1$ for $j = 1, \dots, n$. Given smooth functions $a_1, b_1, \dots, a_n, b_n$ with*

$$X_1 b_1 - Y_1 a_1 = \cdots = X_n b_n - Y_n a_n,$$

and let

$$c^* = \frac{X_j b_j - Y_j a_j}{4}, \quad a_j^* = a_j + 2y_j c^*, \quad b_j^* = b_j - 2x_j c^*, \quad 1 \le j \le n.$$

Consider

$$f(\mathbf{p}) = \int_0^1 \sum_{j=1}^n \big(a_j^*(t\mathbf{p})x_j + b_j^*(t\mathbf{p})y_j\big) + c^*(t\mathbf{p})zdt,$$

where $\mathbf{p} = (x_1, y_1, \dots, x_n, y_n, z)$ in $\mathcal{H}^n$. Then for $l = 1, \dots, n$,

$$X_l f(\mathbf{p}) = a_l(\mathbf{p}) + \int_0^1 t\Big\{\sum_{\substack{j=1\\ j\ne l}}^n [x_j(X_l a_j - X_j a_l)(t\mathbf{p}) + y_j(X_l b_j - Y_j a_l)(t\mathbf{p})] + \frac{z}{4}\Big(X_l^2 b_l - (X_l Y_l + [X_l, Y_l])a_l\Big)(t\mathbf{p})\Big\}dt,$$

$$Y_l f(\mathbf{p}) = b_l(\mathbf{p}) + \int_0^1 t\Big\{\sum_{\substack{j=1\\ j\ne l}}^n [x_j(Y_l a_j - X_j b_l)(t\mathbf{p}) + y_j(Y_l b_j - Y_j b_l)(t\mathbf{p})] + \frac{z}{4}\Big((Y_l X_l + [Y_l, X_l])b_l - Y_l^2 a_l\Big)(t\mathbf{p})\Big\}dt.$$

If the conditions

$$X_j^2 b_j = ([X_j, Y_j] + X_j Y_j)a_j, \quad Y_j^2 a_j = ([Y_j, X_j] + Y_j X_j)b_j,$$
$$X_l a_j = X_j a_l, \quad X_j b_l = Y_l a_j, \quad Y_l b_j = Y_j b_l, \quad 1 \le j \ne l \le n$$

hold, then the system $X_j f = a_j, Y_j f = b_j, j = 1, \ldots, n$ *is solvable and*

$$f(\mathbf{p}) = \int_0^1 g\big(U(\gamma(t)), \dot\gamma(t)\big)dt, \tag{3.6}$$

where γ *is a horizontal curve joining the origin and* $\mathbf{p}$, $U = \sum_{j=1}^n (a_j X_j + b_j Y_j)$, *and* $g(\cdot,\cdot)$ *is the subRiemannian metric.*

The potential function on qH^1 explored by Wu [8] was recorded as follows.

Theorem 3.4 ([8]). *Let* X_1, X_2, X_3, X_4 *be the vector fields on* qH^1 *given in* (1.5). *Let* $\{[X_{i_1}, X_{j_1}], [X_{i_2}, X_{j_2}], [X_{i_3}, X_{j_3}]\}$ *be a basis of* $\mathrm{span}\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\}$, *where* X_{i_l}, X_{j_l} *are chosen from* $\{X_1, X_2, X_3, X_4\}$ *with* $i_l < j_l$. *Consider any smooth functions* a_1, a_2, a_3, a_4,

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} a^1_{i_1j_1} & a^2_{i_1j_1} & a^3_{i_1j_1} \\ a^1_{i_2j_2} & a^2_{i_2j_2} & a^3_{i_2j_2} \\ a^1_{i_3j_3} & a^2_{i_3j_3} & a^3_{i_3j_3} \end{pmatrix}^{-1} \begin{pmatrix} X_{i_1}a_{j_1} - X_{j_1}a_{i_1} \\ X_{i_2}a_{j_2} - X_{j_2}a_{i_2} \\ X_{i_3}a_{j_3} - X_{j_3}a_{i_3} \end{pmatrix},$$

$$a_j^* = a_j + \frac{1}{2}\sum_{k=1}^4 x_k (X_j a_k - X_k a_j), \quad j = 1,2,3,4,$$

and let

$$f(\mathbf{p}) = \int_0^1 \Big[\sum_{j=1}^4 a_j^*(t\mathbf{p})x_j + \sum_{l=1}^3 c_l(t\mathbf{p})y_l\Big]\,dt,$$

where $\mathbf{p} = (x_1, x_2, x_3, x_3, y_1, y_2, y_3)$. *Then for* $i = 1,2,3,4$,

$$(X_i f)(\mathbf{p}) = a_i(\mathbf{p}) - \int_0^1 \frac{t}{2}(y_1, y_2, y_3)\begin{pmatrix} a^1_{i_1j_1} & a^2_{i_1j_1} & a^3_{i_1j_1} \\ a^1_{i_2j_2} & a^2_{i_2j_2} & a^3_{i_2j_2} \\ a^1_{i_3j_3} & a^2_{i_3j_3} & a^3_{i_3j_3} \end{pmatrix}^{-1}$$
$$\times \left[X_i \begin{pmatrix} X_{i_1}a_{j_1} - X_{j_1}a_{i_1} \\ X_{i_2}a_{j_2} - X_{j_2}a_{i_2} \\ X_{i_3}a_{j_3} - X_{j_3}a_{i_3} \end{pmatrix} - \begin{pmatrix} [X_{i_1}, X_{j_1}] \\ [X_{i_2}, X_{j_2}] \\ [X_{i_3}, X_{j_3}] \end{pmatrix} a_i \right] dt.$$

If the integrability conditions (2.4) *and* (2.5) *hold, then the system of equations* $X_1 f = a_1, X_2 f = a_2, X_3 f = a_3, X_4 f = a_4$ *is solvable and*

$$f(\mathbf{p}) = \int_0^1 g\big(U(\gamma(t)), \gamma'(t)\big)dt,$$

where $U = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4$, $\gamma(t)$ *is a horizontal curve connecting the origin and* $\mathbf{p}$, *and* $g(\cdot,\cdot)$ *is the subRiemannian metric.*

As of Corollary 3.3, we may also state the Poincaré's lemma for isotropic quaternion Heisenberg groups. However, the contents will be almost the same as Theorem 3.4. One only has to add an additional assumption (1.7) for vector fields (1.5) comparing with Theorem 3.4. We thus omit it.

Remark 3.5. The Heisenberg groups $\mathcal{H}^n$ and the quaternion Heisenberg groups qH^1 are, in fact, nilpotent Lie groups of step two since by (1.3),

$$[X_j, [X_k, Y_k]] = [Y_j, [X_k, Y_k]] = 0, \quad j, k = 1, 2, \ldots, n,$$

and since by (1.5),

$$[X_j, [X_k, X_l]] = 0, \quad j, k, l = 1, 2, 3, 4.$$

An article of the Poincaré lemma on nilpotent Lie groups of step two will appear soon.

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