Applied Analysis and Optimization
Volume 2, Number 1, 2018, 59–73

Yokohama Publishers
ISSN 2189-1664 Online Journal

TOPOLOGICAL METHODS IN THE STUDY OF POSITIVE SOLUTIONS FOR SYSTEMS OF EQUATIONS IN ORDERED BANACH SPACES

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Abstract. In this paper, we are interested in producing sufficient conditions for the existence of coexistence states to systems of equations in ordered Banach spaces. For it, we use topological methods, more precisely, the fixed point index.

1. Introduction

In this paper we are interested in the existence of nonnegative and nontrivial solution (x, y) of the following system

$$x = G(x, y)$$

$$y = H(x, y)$$

in $E \times E$, where E is an appropriate ordered Banach space with cone P.

The so called "coexistence states" are of special importance: these are solutions (x, y) with both components nonnegative and nontrivial. Semitrivial solutions i.e., solutions (x, y) with exactly one component nonnegative and nontrivial, are also of interest.

Note that a direct application of the corresponding Amann's results in [1] in the Banach space $(E \times E, P \times P)$ for the map $F = (G, H)$ implies the existence of a solution $(x, y) \in P \times P \setminus \{(0, 0)\}$, this means that $(x, y) \neq (0, 0)$ but some component of the fixed point (x, y) may be trivial. To solve this problem, A. Cañada and A. Zertiti have given some new result (see Theorem 2.1 in [3]) to assure that each component of (x, y) belongs to $P \setminus \{0\}$. In the present paper, we shall also give some new abstract results (with different conditions) for the existence of these important solutions. Furthermore, if we suppose that F verifies the hypothesis

$$G(0, y) = H(x, 0) = 0, \quad \forall (x, y) \in E \times E,$$

we assure the existence of "semitrivial solutions". Hence, we deduce the existence of four fixed points in $P \times P : (0, 0), (x_0, 0), (0, y_0), (x_1, y_1)$ such that

$$x_j, y_j \in P \setminus \{0\} \tag{1.1}$$

for $j = 0, 1$.

2010 *Mathematics Subject Classification.* 37C25, 35P30, 47B60.

Key words and phrases. Positive solutions, fixed point index, systems of equations, nonlinear fractional differential equations, nonlinear integral equations.

Finally, in order to prove the importance of our results we give two important applications to some systems of nonlinear fractional differential equations and nonlinear integral equations.

Note that the abstract results which we obtain here may be applied to some other situations such as nonlinear boundary value problems for elliptic systems, and other kinds of systems of nonlinear integral equations.

2. Main results

Let $(E, \|.\|_E)$ be a real Banach space and P be a nonempty closed convex set in E.

P is called a cone if it satisfies the following two conditions:

(i) $x \in P, \lambda \geq 0 \Longrightarrow \lambda x \in P$

(ii) $x \in P, -x \in P \Longrightarrow x = \theta$, where θ denotes the zero element in E.

The cone P defines a linear ordering in E by

$$x \leq y \quad \text{iff} \quad y - x \in P.$$

For every open subset U of P (from now on, the topological notions of subsets of P refer to the relative topology of P as a topological subspace of E) and every compact map $F : \bar{U} \to P$ (F is continuous and $F(\bar{U})$ is relatively compact), which has no fixed points on ∂U, there exists an integer, $i_p(F, U)$, called the fixed point index of F on U with respect to P, satisfying the usual properties of the Leray-Schauder degree.

It is trivial that $P \times P$ is a cone in the Banach space $(E \times E, \|.\|_{E\times E})$ where, for each $(x, y) \in E \times E$

$$\|(x, y)\|_{E\times E} = \max\{\|x\|_E, \|y\|_E\}.$$

The cone $P \times P$ defines a linear ordering in $E \times E$ by

$$(x_1, y_1) \leq (x_2, y_2) \quad \text{iff} \quad x_2 - x_1 \in P \quad \text{and} \quad y_2 - y_1 \in P.$$

For any two real numbers $0 < \alpha < \beta$, we denote by $R_{\alpha,\beta}$ the set

$$R_{\alpha,\beta} = \{(x, y) \in P \times P : \|x\|_E < \alpha, \|y\|_E < \beta\},$$

and if $r > 0$, we denote

$$P_r = \{x \in P : \|x\|_E < r\}, \quad S_r = \{x \in P : \|x\|_E = r\}.$$

Define the compact operator $F = (G, H) : \overline{R}_{\beta,\beta} \to P \times P$, where $G : \overline{R}_{\beta,\beta} \to P$ and $H : \overline{R}_{\beta,\beta} \to P$ verifying the following hypotheses:

(F1) : There exists a continuous map $g : P \times P \to P$ which is linear with respect to the first variable such that $G(x, y) = g(x, y) + r(x, y)$. Here r is an $o(\|x\|_E)$ for $x \in P$ near zero uniformly on bounded y sets.

(F2) : There exists a continuous map $h : P \times P \to P$ which is linear with respect to the second variable such that $H(x, y) = h(x, y) + r'(x, y)$. Here r' is an $o(\|y\|_E)$ for $y \in P$ near zero uniformly on bounded x sets.

To prove the main results in this paper, we will employ the following lemmas :

Lemma 2.1. *The map g transforms every bounded set $A \times B \subset \bar{P}_\beta \times \bar{P}_\beta$ into a compact set.*

Proof. Since the operator $g(.,y)$, $(y \in \bar{P}_\beta)$ is linear it suffices to consider the case where $A = S_1$.

Assuming the contrary. Then there exists a sequence of elements $(h_n, k_n) \in A \times B$ satisfying $\|h_n\| = 1$ and

$$\|g(h_n, k_n) - g(h_m, k_m)\| \geq \varepsilon_0 \quad m \neq n,$$

where ε_0 is some positive number. From hypothesis (F1) we can choose a $\delta > 0$ such that

$$\|G(x,y) - g(x,y)\| \leq \frac{\varepsilon_0}{3}\|x\|, \quad \forall (x,y) \in \bar{P}_\beta \times B \quad \text{and} \quad \|x\| < \delta.$$

Whence, for $m \neq n$ we have

$$\begin{aligned} \|G(\delta h_n, k_n) - G(\delta h_m, k_m)\| &\geq \delta\|g(h_n, k_n) - g(h_m, k_m)\| \\ &\quad - \|G(\delta h_n, k_n) - g(\delta h_n, k_n)\| \\ &\quad - \|G(\delta h_m, k_m) - g(\delta h_m, k_m)\| \\ &\geq \frac{\varepsilon_0 \delta}{3}. \end{aligned}$$

From which it follows that $G(S_1 \times B)$ is not compact. This contradiction completes the proof of the lemma. □

In a similar way, one can see that h is compact on bounded subsets of $\bar{P}_\beta \times \bar{P}_\beta$.

In the following, define the set $A = H(\bar{P}_\beta \times \bar{P}_\beta) \cap \bar{P}_\beta$.

Lemma 2.2. *Under the previous hypotheses, we suppose that, for every $y \in \bar{P}_\beta$, 1 is not an eigenvalue of $g(.,y)$ to a positive eigenvector. Then a number $\alpha_1 > 0$ can be found such that*

$$\|x - g(x,y)\| > \alpha_1, \quad \forall x \in S_1, \forall y \in A.$$

Hence, there exists a positive number $\alpha_1 > 0$ such that

$$\|x - g(x,y)\| > \alpha_1\|x\|, \quad \forall x \in P, \forall y \in A.$$

Proof. We shall carry out the proof from the contrary. In fact, we construct a sequence of elements $x_n \in S_1, y_n \in A$ such that $\|x_n - g(x_n, y_n)\| \to 0$. Without loss of generality, it could be assumed that $g(x_n, y_n)$ would converge to some element $z \in P$ and the elements y_n also would converge to $y \in \bar{P}_\beta$. Then the elements x_n also would converge to $z \in S_1$ and the equality $z = g(z,y)$ would hold. This contradiction completes the proof of the lemma. □

After these preparations we are ready for the statement of our main results:

Theorem 2.3. *Let $F : \overline{R}_{\beta,\beta} \to P \times P$ be a compact map verifying the previous hypotheses (F1)-(F2) and*

(H1)

$$G(x,y) \neq \lambda x \quad \forall (x,y) \in S_\beta \times \bar{P}_\beta \quad \forall \lambda \geq 1$$

and

$$H(x,y) \neq \lambda y \quad \forall (x,y) \in \bar{P}_\beta \times S_\beta \quad \forall \lambda \geq 1$$

(H2) *For every* $(x, y) \in \bar{P_\beta} \times \bar{P_\beta}$, 1 *is neither an eigenvalue of* $g(., y)$ *nor of* $h(x, .)$ *to a positive eigenvector and both* $g(., y)$ *and* $h(x, .)$ *possess a positive eigenvector to an eigenvalue greater than one.*

Then F *has at least one fixed point* (x_1, y_1) *in* $P \setminus \{0\} \times P \setminus \{0\}$.

Proof. We shall use the following notation

$$U = R_{\beta,\beta}.$$

The proof is based on the following steps:

a)

$$i_{P\times P}(F, U) = 1.$$

Indeed, define the homotopy $h : [0, 1] \times \bar{U} \to P \times P$ by $h(\lambda, x, y) = \lambda F(x, y)$. It is clear that h is compact and from (H1) we have

$$h(\lambda, x, y) \neq (x, y), \quad \forall (\lambda, x, y) \in [0, 1] \times \partial U.$$

Hence, by homotopy invariance property

$$i_{P\times P}(F, U) = i_{P\times P}(h(1, .), U) = i_{P\times P}(h(0, .), U) = 1.$$

b) In view of Lemma 2.2 there exists a positive constant α_1 such that

$$\|x - g(x, y)\| \geq \alpha_1 \|x\| \quad \forall x \in P \quad \forall y \in A. \tag{2.1}$$

Choose $\rho_1 \in (0, \beta]$ such that for all $x \in \bar{P_{\rho_1}}$ and $y \in \bar{P_\beta}$

$$\|G(x, y) - g(x, y)\| \leq \alpha_1 \frac{\|x\|}{2}.$$

Then for every $\sigma \in (0, \rho_1]$, every $z_1 \in P$ satisfying $\|z_1\| < \frac{\sigma\alpha_1}{2}$ and every $\lambda \in [0, 1]$ the map $(1 - \lambda)(g + z_1, H) + \lambda F = K_\lambda$ possesses no fixed point on $\partial R_{\sigma,\beta}$.

Indeed, by taking into account that

$$\begin{aligned} \partial R_{\sigma,\beta} = &\{(x, y) \in P \times P : \|y\|_E = \beta, \|x\|_E \leq \sigma\} \\ &\cup \{(x, y) \in P \times P : \|y\|_E \leq \beta, \|x\|_E = \sigma\}, \end{aligned}$$

we distinguish two cases :

1) $\|y\|_E = \beta, \quad \|x\|_E \leq \sigma$

If $K_\lambda(x, y) = (x, y)$ then

$$(1 - \lambda)H(x, y) + \lambda H(x, y) = H(x, y) = y$$

which contradicts (H1).

2) $\|y\|_E \leq \beta, \quad \|x\|_E = \sigma$

If $K_\lambda(x, y) = (x, y)$ we get $y = H(x, y) \in A$. But from the equalities

$$\begin{aligned} \|x - (1 - \lambda)(g(x, y) + z_1) - \lambda G(x, y)\| &\geq \|x - g(x, y)\| \\ &\quad - \|G(x, y) - g(x, y)\| - \|z_1\| \\ &\geq \sigma(\alpha_1 - \frac{\alpha_1}{2} - \frac{\|z_1\|}{\sigma}) > 0. \end{aligned}$$

we obtain a contradiction.

Then by the homotopy invariance property

$$i_{P\times P}(F, R_{\sigma,\beta}) = i_{P\times P}((g + z_1, H), R_{\sigma,\beta}).$$

Next we prove that

$$i_{P\times P}((g+z_1,H),R_{\sigma,\beta})=0.$$

In fact, let $y\in\bar{P_\beta}$ and denote by $h\in S_1$ an eigenvector of $g(.,y)$ to an eigenvalue $\lambda>1$: Then we claim that, for every $\nu>0$, the equation $x-g(x,y)=\nu h$ has no positive solution, indeed, suppose that there exists a solution $x>0$ for some $\nu>0$. Then there exists a nonnegative number τ_0 such that $x\geq\tau_0 h$ and $x\not\geq\tau h$ for $\tau>\tau_0$. Hence we obtain the inequality

$$x=g(x,y)+\nu h\geq g(\tau_0 h,y)+\nu h\geq(\tau_0+\nu)h,$$

which contradicts the maximality of τ_0.

Now by setting $z=\nu h$ with $0<\nu<\frac{\sigma\alpha_1}{2}$, the solution property implies

$$i_{P\times P}(F,R_{\sigma,\beta})=i_{P\times P}((g+\nu h,H),R_{\sigma,\beta})=0.$$

c) Similarly, we find a positive constants α_2 and $\rho_2\in(0,\beta]$ satisfying

$$\|y-h(x,y)\|\geq\alpha_2\|y\|\quad\forall y\in P,\quad\forall x\in B=G(\bar{P_\beta})\times\bar{P_\beta}))\cap\bar{P_\beta}$$

and for all $y\in\bar{P_{\rho_2}}$ and $x\in\bar{P_\beta}$

$$\|H(x,y)-h(x,y)\|\leq\alpha_2\frac{\|y\|}{2}.$$

Then for every $\sigma\in(0,\rho_2]$, every $z_2\in P$ satisfying $\|z_2\|<\frac{\sigma\alpha_2}{2}$ and every $\lambda\in[0,1]$ the map $(1-\lambda)(G,h+z_2)+\lambda F=K'_\lambda$ possesses no fixed point on $\partial R_{\beta,\sigma}$. Then, from what has already been proved

$$i_{P\times P}(F,R_{\beta,\sigma})=i_{P\times P}((G,h+z_2),R_{\beta,\sigma})=0.$$

d) Next, we prove that there exists $r>0$ such that if $(x,y)\in\bar{P_\beta}\times\bar{P_\beta}$ and $\lambda>0$ satisfy $(x,y)=F(x,y)+\lambda(h_0,k_0)$ then $\|(x,y)\|>r$, where (h_0,k_0) is a fixed element in $P\times P\setminus\{(0,0)\}$. From which it will follows that (see Lemma 12.1 in [1]) $i_{P\times P}(F,R_{\sigma,\sigma})=0$, for all $\sigma\in(0,r]$. Assuming the contrary, then there exist sequences $(x_n,y_n)\in\bar{P_\beta}\times\bar{P_\beta},\lambda_n\in\mathbb{R}^+$ such that $(x_n,y_n)=F(x_n,y_n)+\lambda_n(h_0,k_0)$ and $\|(x_n,y_n)\|<\frac{1}{n}$. Then $x_n=G(x_n,y_n)+\lambda_n h_0$. (Here we suppose that $h_0\in P\setminus\{0\}$).

Writing the last equality in the form

$$\lambda_n\frac{h_0}{\|x_n\|}=\frac{x_n}{\|x_n\|}-\frac{G(x_n,y_n)-g(x_n,y_n)}{\|x_n\|}-\frac{g(x_n,y_n)}{\|x_n\|},\tag{2.2}$$

and using Lemma (2.1) we may as well assume that $\frac{g(x_n,y_n)}{\|x_n\|}$ converge to some $z\in P$. It then follows from equality (2.2) that

$$\|h_0\|\overline{\lim}\frac{\lambda_n}{\|x_n\|}\leq 1+\|z\|.$$

Therefore, without loss of generality, it can be assumed that the sequence $\frac{\lambda_n}{\|x_n\|}$ converges to some number $\alpha\geq 0$. By virtue of (2.2) we can assume that $\frac{x_n}{\|x_n\|}$ converges to some element $u_0\in P$ and $\|u_0\|=1$. Passing to the limit in (2.2) we obtain $u_0=\alpha h_0+g(u_0,0)$. It then follows from Hypothesis (H2) of the theorem that $\alpha>0$. Therefore and from what has already been proved above, we obtain a contradiction.

e) For fixed $\rho = \min\{\rho_1, \rho_2, r\}$, we shall use the following notation

$$U = R_{\beta,\beta}, \quad U_1 = R_{\rho,\beta}, \quad U_2 = R_{\beta,\rho} \quad U_3 = R_{\rho,\rho}$$

$$U_4 = U \setminus \bar{U_1} \cup \bar{U_2}, \quad U_5 = U_1 \setminus \bar{U_3}, \quad U_6 = U_2 \setminus \bar{U_3}.$$

Therefore

$$U_4 = \{(x, y) \in P \times P : \rho < \|x\|_E < \beta, \quad \rho < \|y\|_E < \beta\}.$$

Now, observe that if $\lambda = 1, F = K_1 = K_1'$ has no fixed point on $\partial U_1 \cup \partial U_2$. Then F has no fixed point on $\partial U_1 \cup \partial U_2 \cup \partial U$.

U_3 and U_5 are disjoint open subsets of U_1 such that F has no fixed points on $\bar{U_1} \setminus (U_3 \cup U_5)$, in fact $\bar{U_1} \setminus (U_3 \cup U_5) \subset \partial U_1 \cup \partial U_2$. Therefore by the additivity property

$$i_{P\times P}(F, U_5) = i_{P\times P}(F, U_1) - i_{P\times P}(F, U_3) = 0.$$

Similarly, we have

$$i_{P\times P}(F, U_6) = i_{P\times P}(F, U_2) - i_{P\times P}(F, U_3) = 0.$$

Finally, $(U_3 \cup U_5 \cup U_6)$ and U_4 are disjoint open subsets of U such that F has no fixed points on $\bar{U} \setminus (U_3 \cup U_5 \cup U_6 \cup U_4)$, in fact $\bar{U} \setminus (U_3 \cup U_5 \cup U_6 \cup U_4) \subset (\partial U \cup \partial U_1 \cup \partial U_2)$. Therefore and by the additivity property

$$\begin{aligned} i_{P\times P}(F, U_4) &= i_{P\times P}(F, U) - i_{P\times P}(F, U_3) - i_{P\times P}(F, U_5) - i_{P\times P}(F, U_6) \\ &= 1, \end{aligned}$$

which implies the existence of a fixed point (x_1, y_1) of F satisfying (1.1). □

Suppose, in addition, that the maps G and H satisfy the following hypothesis

$$G(0, y) = H(x, 0) = 0, \quad \forall (x, y) \in \bar{P_\beta} \times \bar{P_\beta}. \tag{2.3}$$

Then, we can prove the existence of two fixed point (semi-trivial solutions) $(x_0, 0), (0, y_0)$, of F satisfying (1.1).

Indeed, define the map $G_1 : \bar{P_\beta} \to P$ by $G_1(x) = G(x, 0)$. Clearly G_1 is a compact map such that $G'_{1+}(0)h = g(h, 0), \quad \forall h \in P$. Then from (H2) and lemma 13.1 in [1] there exists $\sigma_0 \in (0, \beta)$ such that for every $\sigma \in (0, \sigma_0), i_P(G_1, P_\sigma) = 0$. On the other hand from hypothesis (H1), we have $G_1(x) \neq \lambda x, \quad \forall \lambda \geq 1, \quad \forall x \in S_\beta$, then $i_P(G_1, P_\beta) = 1$ (see Lemma 12.1 in [1]). Therefore, by the additivity property we have $i_P(G_1, P_\beta \setminus \bar{P_\sigma}) = 1$. Consequently, G_1 has at least one fixed point x_0 with $\sigma < \|x_0\|_E < \beta$. Now $(x_0, 0)$ is a fixed point of F.

In a similar manner we can prove the existence of $(0, y_0)$.

Remark 1. If P has nonempty interior and $g(., y)$ (resp $h(x, 0)$) is strongly positive for every $y \in \bar{P_\beta}$ (resp $x \in \bar{P_\beta}$) then it is well known that (see [6], [5]) the spectral radius of $g(., y)$ (resp $h(x, 0)$) is an eigenvalue to a positive eigenvector, and in fact the only eigenvalue with this property. Then we have this corollary:

Corollary 2.4. *Suppose that P has nonempty interior and let $F : \bar{R_{\beta,\beta}} \to P \times P$ a compact map verifying the previous hypotheses (F1)-(F2) and* (2.3). *Moreover suppose that for every $(x,y) \in \bar{P}_\beta \times \bar{P}_\beta$ the maps $g(.,y)$ and $h(x,.)$ are strongly positive. Then if*

(H1)

$$G(x,y) \ngeq x \quad \forall (x,y) \in S_\beta \times \bar{P}_\beta \quad \textit{and} \quad H(x,y) \ngeq y \quad \forall (x,y) \in \bar{P}_\beta \times S_\beta$$

(H2) *$r(g(.,y)) > 1$ and $r(h(x,.)) > 1$ for every $(x,y) \in \bar{P}_\beta \times \bar{P}_\beta$.*

F has at least four fixed points $(0,0), (x_0,0), (0,y_0), (x_1,y_1)$ in $P \times P$ verifying (1.1).

Remark 2. The main difference between Theorem 2.3 above and Theorem (13.2) given by Amann in [1] is that a direct application of this Amann's results in the Banach space $(E \times E, P \times P)$ for the map $F = (G,H)$ implies the existence of a solution $(x,y) \in P \times P \setminus \{(0,0)\}$. This means that $(x,y) \neq (0,0)$ but some component of the fixed point (x,y) may be trivial. However, our Theorem has the advantage that it assures that each component of (x,y) belongs to $P \setminus \{0\}$.

3. Application to Systems of Fractional Differential Equations

In this section we shall study the existence of nonnegative and nontrivial solutions of boundary value problems for systems of fractional differential equations of the type

$$\begin{aligned} D^\alpha_{0+}x(t) + f(t,x(t),y(t)) = 0 \quad 0 < t < 1, \\ D^\alpha_{0+}y(t) + g(t,x(t),y(t)) = 0 \quad 0 < t < 1, \\ x(0) = x'(0) = x'(1) = y(0) = y'(0) = y'(1) = 0. \end{aligned} \tag{3.1}$$

where $f, g : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous functions, $2 < \alpha < 3$ is a real number and D^α_{0+} is a standard Riemann-Liouville fractional derivative. Problems of the form (3.1) arise in many applications in physics, mechanics, chemistry and engineering, where usually the existence of positive solutions is of interest.

Let $(x,y) \in C[0,1] \times C[0,1]$, then it is well known that (see [7]) the boundary value problem (3.1) is equivalent to the following system of integral equations

$$\begin{aligned} x(t) &= \int_0^1 G(t,s) f(s,x(s),y(s))ds, \quad t \in [0,1] \\ y(t) &= \int_0^1 G(t,s) g(s,x(s),y(s))ds, \quad t \in [0,1], \end{aligned}$$

where

$$G(t,s) = \begin{cases} \frac{(1-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1 \\ \frac{(1-s)^{\alpha-2}t^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1. \end{cases}$$

Here Γ denotes the Gamma function.

The following Lemma (see [7], Lemma 2.8) will be used later

Lemma 3.1. *$G(t,s) \ge t^{\alpha-1}G(1,s) \ge 0$ for $0 \le s,t \le 1$.*

For our considerations, we shall consider the Banach space $E = C[0,1]$ equipped with the standard norm

$$\|x\| = \max_{0\le t\le 1} |x(t)|.$$

We define the normal cone P by $P = \{x \in C[0,1] : x(t) \ge 0\}$. Defining $F : E\times E \to E \times E$ by

$$F(x,y)(t) = \Big(\int_0^1 G(t,s)f(s,x(s),y(s))ds, \int_0^1 G(t,s)g(s,x(s),y(s))ds\Big).$$
$$= \Big(H(x,y)(t), K(x,y)(t)\Big).$$

Standard arguments show that $F(P \times P) \subset P \times P$ and that F is compact on bounded subsets of $P\times P$. So we have the conditions to apply to problem (3.1) our abstract results. In fact we have

Theorem 3.2. *Suppose that*

(H'1) *there exists positive constants a_1, a_2, b_1, b_2 and a positive numbers $\nu_1, \nu_2, \mu_1, \mu_2 < 1$ such that*

$$f(t,x,y) \le a_1(x^{\nu_1} + y^{\mu_1}) + b_1,$$
$$g(t,x,y) \le a_2(x^{\nu_2} + y^{\mu_2}) + b_2,$$
$$\forall (t,x,y) \in [0,1]\times[0,+\infty)\times[0,+\infty),$$

(H'2) *there exists a continuous function $a : [0,1]\times[0+\infty) \longrightarrow \mathbb{R}$ such that*

$$\lim_{x\to 0^+} \frac{f(t,x,y)}{x} = a(t,y),$$

$$\textit{uniformly in } (t,y) \in [0,1]\times[0,\gamma], \quad \textit{for every} \quad \gamma > 0$$

where

$$a(t,y) \ge a > \frac{\Gamma(2\alpha-1)(2\alpha-1)}{\Gamma(\alpha-1)(\alpha-1)}, \quad \forall (t,y) \in [0,1]\times[0,+\infty) \tag{3.2}$$

(H'3) *there exists a continuous function $b : [0,1]\times[0,+\infty) \longrightarrow \mathbb{R}$ such that*

$$\lim_{y\to 0^+} \frac{g(t,x,y)}{y} = b(t,x),$$

$$\textit{uniformly in } (t,x) \in [0,1]\times[0,\gamma], \quad \textit{for every} \quad \gamma > 0$$

where

$$b(t,x) \ge b > \frac{\Gamma(2\alpha-1)(2\alpha-1)}{\Gamma(\alpha-1)(\alpha-1)} \quad \forall (t,x) \in [0,1]\times[0,+\infty) \tag{3.3}$$

Then (3.1) *has at least one solution (x_1, y_1) such that $x_1 \in P\setminus\{0\}$ and $x_2 \in P\setminus\{0\}$.*

Proof. We are going to prove that all conditions of Theorem 2.3 are satisfied. For it, we must observe that (E,P) is an ordered Banach space with $\overset{\circ}{P} \ne \emptyset$. Moreover, (1) In order to prove that the condition (H1) of Theorem 2.3 is satisfied, take

$$\beta > M \max\{a_1(\beta^{\nu_1} + \beta^{\mu_1}) + b_1, a_2(\beta^{\nu_2} + \beta^{\mu_2}) + b_2\}$$

where $M = \max_{[0,1]\times[0,1]} G(t,s)$. Take $(x,y) \in \bar{P_\beta} \times S_\beta$ and $\lambda \geq 1$, verifying $K(x,y) = \lambda y$. Whence

$$y(t) = \frac{1}{\lambda}\int_0^1 G(t,s)g(s,x(s),y(s))ds \quad \forall t \in [0,1].$$

Since $\|y\|_E = \beta$, there is $t_0 \in \mathbb{R}$ such that $y(t_0) = \beta$. From which it follows that

$$\beta = y(t_0) \leq \int_0^1 G(t_0,s)g(s,x(s),y(s))ds \leq M(a_2(\beta^{\nu_2} + \beta^{\mu_2}) + b_2) < \beta,$$

which is a contradiction. One may proceed in an analogous way if $\|y\|_E \leq \beta$ and $\|x\|_E = \beta$. Therefore (H1) of Theorem 2.3 is satisfied.

(2) By using (H'2) and (H'3) we prove the existence of two applications $h, k : P \times P \to P$ verifying respectively the hypotheses (F1) and (F2) of Theorem 2.3.

In fact, we are going to see that for all $x \in P$ and $y \in \bar{P_\beta}$,

$$h(x,y)(t) = L(a,y)x(t) = \int_0^1 G(t,s)a(s,y(s))x(s)ds, \quad \forall t \in [0,1]$$

and

$$\lim_{\substack{x \longrightarrow 0 \\ x\in P}} \frac{H(x,y) - L(a,y)x}{\|x\|} = 0, \quad \text{uniformly in} \quad y \in \bar{P_\beta}.$$

For it we must prove that

$$\forall \varepsilon \in \mathbb{R}^+, \exists r(\varepsilon) \in \mathbb{R}^+ : \|x\| \leq r(\varepsilon)(x \in P), \quad y \in \bar{P_\beta} \Rightarrow \frac{\|H(x,y) - L(a,y)x\|}{\|x\|} \leq \varepsilon.$$

Let $\varepsilon > 0$, then from (H'2) there is $r(\varepsilon) \in \mathbb{R}^+$ such that

$$\mid f(s,x,y) - a(s,y)x \mid \leq \varepsilon \mid x \mid, \quad \forall (s,y) \in [0,1] \times [0,\beta], \forall x \in \mathbb{R} : 0 \leq x \leq r(\varepsilon).$$

Then if $x \in P$ satisfies $\|x\| \leq r(\varepsilon)$ and $y \in \bar{P_\beta}$, we have

$$\begin{aligned}
\mid H(x,y)(t) - L(a,y)x(t) \mid &\leq \int_0^1 G(t,s) \mid f(s,x(s),y(s)) - a(s,y(s))x(s) \mid ds \\
&\leq \int_0^1 G(t,s) \mid \varepsilon x(s) \mid ds \\
&\leq \varepsilon M \|x\|, \quad \forall t \in [0,1].
\end{aligned}$$

Consequently,

$$\|H(x,y) - L(a,y)x\| \leq \varepsilon M \|x\|, \quad \forall x \in P : \|x\| \leq r(\varepsilon) \quad \forall y \in \bar{P_\beta}.$$

Similarly we can prove that

$$\lim_{\substack{y \longrightarrow 0 \\ y\in P}} \frac{K(x,y) - L(b,x)y}{\|y\|} = 0, \quad \text{uniformly in} \quad x \in \bar{P_\beta}.$$

where

$$L(b,x)y(t) = k(x,y)(t) = \int_0^1 G(t,s)b(s,x(s))y(s)ds, \quad \forall t \in [0,1].$$

(3) In order to prove that (for some $y \in \bar{P_\beta}$) 1 is not an eigenvalue of $h(.,y)$ to a positive eigenvector and that $h(.,y)$ possess a positive eigenvector to an eigenvalue

greater than one, we shall prove that $h(.,y)$ satisfies the conditions of Theorems 2.5 and 2.17 of Krasnosel'skii in [5]. To this end we begin by proving that the linear operator $\int_0^1 G(t,s)a(s,y(s))x(s)ds$ is u_0-bounded below (in the sens of Krasnosel'skii [5]) where $u_0(t) = t^{\alpha-1}$.

In fact, let $[t_1,t_2] \subset [0,1], (t_1 \neq t_2)$, then in view of Lemma 3.1 the inequality

$$\begin{aligned}\int_{t_1}^{t_2} G(t,s)ds \geq & \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} t^{\alpha-1}G(1,s)ds \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big(\frac{(1-t_1)^{\alpha-1}}{\alpha-1} - \frac{(1-t_2)^{\alpha-1}}{\alpha-1} - \frac{(1-t_1)^{\alpha}}{\alpha} + \frac{(1-t_2)^{\alpha}}{\alpha}\Big) \\ &= \frac{t^{\alpha-1}}{\Gamma(\alpha)}\delta \quad (\delta > 0)\end{aligned}$$

will be satisfied. Moreover, a direct calculation shows that $\int_0^1 G(t,s)ds = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big(\frac{1}{\alpha-1} - \frac{t}{\alpha}\Big)$. From which it follows that

$$\int_{t_1}^{t_2} G(t,s)ds \geq \delta \int_0^1 G(t,s)ds.$$

The last inequality is the condition (7.4) of Lemma 7.1 in [5]. Then the operator $\int_0^1 G(t,s)x(s)ds$ is v_0-positive where

$$v_0(t) = \int_0^1 G(t,s)ds = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big(\frac{1}{\alpha-1} - \frac{t}{\alpha}\Big).$$

But from the inequality

$$\frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big(\frac{1}{\alpha-1} - \frac{1}{\alpha}\Big) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big(\frac{1}{\alpha-1} - \frac{t}{\alpha}\Big) \leq \frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big(\frac{1}{\alpha-1}\Big),$$

one can see that the operator $\int_0^1 G(t,s)x(s)ds$ is u_0-positive. It then follows from (3.2) that the operator $\int_0^1 G(t,s)a(s,y(s))x(s)ds$ is u_0-bounded below.

In the following, by using Theorem 2.5 (in [5]) we shall prove that $h(.,y)$ has an eigenvector to an eigenvalue $\lambda > 1$. To this end, we need to prove (for fixed $y \in \bar{P_\beta}$) the existence of a positive number $\lambda_0 > 1$ satisfying

$$L(a,y)u_0(t) = \int_0^1 G(t,s)a(s,y(s))u_0(s)ds \geq \lambda_0 u_0(t), \quad t \in [0,1].$$

In fact, direct calculation shows that

$$\int_0^1 G(t,s)s^{\alpha-1}ds = \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)}(1-s)^{\alpha-2}s^{\alpha-1}ds - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{\alpha-1}ds.$$

Then, by the substitution $s = \sigma t$ and the fundamental properties of the Beta functions we find that

$$\begin{aligned}\int_0^1 G(t,s)s^{\alpha-1}ds &= = \frac{t^{\alpha-1}}{\Gamma(\alpha)}\frac{\Gamma(\alpha-1)\Gamma(\alpha)}{\Gamma(2\alpha-1)} - \int_0^1 t^{2\alpha-1}\frac{(1-s)^{\alpha-1}s^{\alpha-1}}{\Gamma(\alpha)}ds \\ &= t^{\alpha-1}\frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\Big(1 - t^{\alpha}\frac{\alpha}{2\alpha-1}\Big) \\ &\geq t^{\alpha-1}\frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\Big(\frac{\alpha-1}{2\alpha-1}\Big).\end{aligned}$$

By virtue of (3.2) we get

$$\int_0^1 G(t,s)a(s,y(s))u_0(s)ds \geq at^{\alpha-1}\frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\Big(\frac{\alpha-1}{2\alpha-1}\Big) \geq \lambda_0 u_0(t),$$

for all $t \in [0,1], y \in \bar{P_\beta}$. where $\lambda_0 = a\frac{\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\Big(\frac{\alpha-1}{2\alpha-1}\Big) > 1$.

Consequently, Theorems 2.17 and 2.5 given by Krasnosel'skii in [5] assure that h satisfies hypothesis (H2) of the Theorem 2.3. One may proceed in an analogous way for k and we can prove that hypothesis (H2) of the theorem is satisfied. Therefore, from $(1)-(3)$ and by using Theorem 2.3, problem (3.1) has at least one solution (x_1, y_1) such that $x_1 \in P \setminus \{0\}$ and $x_2 \in P \setminus \{0\}$. □

Remark 3. Suppose in addition that $f(t,0,y) > 0$ and $g(t,x,0) > 0 \quad \forall (t,x,y) \in [0,1]\times[0,\infty)\times[0,\infty)$, then hypotheses of the Theorem do not imply the existence of semi-trivial solutions of problem (3.1).

Remark 4. Note that for a metric approach of the above problem, see [9].

4. Application to systems of nonlinear integral equations

In this section we shall study the existence of positive solutions of system of nonlinear integral equations of the form

$$\begin{aligned} x(t) &= \int_0^{\tau_1(t)} f(t,s,x(t-s-l),y(t-s-l))\, ds \\ y(t) &= \int_0^{\tau_2(t)} g(t,s,x(t-s-l),y(t-s-l))\, ds \end{aligned} \tag{4.1}$$

under the following assumptions on functions f and g : $f, g : \mathbb{R}\times\mathbb{R}\times[0,+\infty[\times[0,+\infty[\longrightarrow \mathbb{R}$ are continuous functions with :

(F1) : $f(t,s,0,y) = g(t,s,x,0) = 0$ for all $(t,s,x,y) \in \mathbb{R}\times\mathbb{R}\times[0,+\infty[\times[0,+\infty[$,

(F2) : $f(t,s,x,y) \geq 0, g(t,s,x,y) \geq 0, \quad \forall (t,s,x,y) \in \mathbb{R}\times\mathbb{R}\times[0,+\infty[\times[0,+\infty[$ and there exists a positive number $w, (w > 0)$ such that $f(t+w,s,x,y) = f(t,s,x,y)$ and $g(t+w,s,x,y) = g(t,s,x,y), \forall (t,s,x,y) \in \mathbb{R}\times\mathbb{R}\times[0,+\infty[\times[0,+\infty[$,

(F3) : l is a nonnegative constant and $\tau_1, \tau_2 : \mathbb{R} \longrightarrow \mathbb{R}^+$ are continuous and λ-periodic functions $(\lambda > 0)$ such that $\frac{\omega}{\lambda} = \frac{p}{q}, p, q \in \mathbb{N}$.

System (4.1) includes the system proposed by Cooke and Kaplan [4] as a model to explain the evolution in time of two interacting species when seasonal factors are taken into account. Since in this model, $f(t,s,x,y)$ and $g(t,s,x,y)$ mean, respectively, the number of new births per unit time of the species x and y. Assumption $f(t,s,0,y) = g(t,s,x,0) = 0$ is completely coherent because of the number of individuals of the species x (or y) is zero at some time, then the number of new births of this species must be zero. In particular, this implies that (0,0) is always a solution of system (4.1).

Taking into account the origin of (3.1) we are interested in the existence of nontrivial, nonnegative, continuous and $q\omega$- periodic solutions. Especially, we are interested in the existence of coexistence states. Also the existence of semitrivial solutions of (4.1) may be of interest, i.e. solutions with exactly one nontrivial component: this means that one species may survive in the absence of the other one.

Denote by P the cone of nonnegative functions in the real Banach space E, of all real and continuous $q\omega-$ periodic functions defined on $\mathbb{R}$, where if $x \in E$

$$\|x\| = \max_{0\le t\le q\omega} |x(t)|.$$

Define the operator $F = (G,H) : P \times P \to P \times P$, by $F(x,y)(t) =$

$$\Big(\int_0^{\tau_1(t)} f(t,s,x(t-s-l),y(t-s-l))\,ds, \int_0^{\tau_2(t)} g(t,s,x(t-s-l),y(t-s-l))\,ds\Big).$$

It is easily to see that F is a compact on bounded subsets of $P \times P$ (see [2]).

Theorem 4.1. *Suppose that :*

(H'1) f *and* g *are bounded functions.*

(H'2) *there exists a continuous function* $a : \mathbb{R} \times \mathbb{R} \times [0,\beta] \longrightarrow \mathbb{R}$ *such that*

$$\lim_{x\to 0^+} \frac{f(t,s,x,y)}{x} = a(t,s,y), \quad \textit{uniformly in } (t,s,y) \in \mathbb{R}\times\mathbb{R}\times[0,\gamma],$$

for every $\gamma > 0$.

(H'3) *there exists a continuous function* $b : \mathbb{R} \times \mathbb{R} \times [0,\beta] \longrightarrow \mathbb{R}$ *such that*

$$\lim_{y\to 0^+} \frac{g(t,s,x,y)}{y} = b(t,s,x), \quad \textit{uniformly in } (t,s,x) \in \mathbb{R}\times\mathbb{R}\times[0,\gamma],$$

for every $\gamma > 0$.

(H'4) $\overset{\circ}{A}_t = \overset{\circ}{B}_t = \emptyset \quad \forall t \in \mathbb{R}$, *where* $A_t = \{(s,y) \in \mathbb{R}\times\mathbb{R}^+ : a(t,t-s,y) = 0\}$ *and* $B_t = \{(s,x) \in \mathbb{R}\times\mathbb{R}^+ : b(t,t-s,x) = 0\}$.

Then if

$$r(L(\tau_1,a,y)) > 1, \quad \textit{and} \quad r(L(\tau_2,b,x)) > 1, \quad \forall(x,y) \in \bar{P}_\beta \times \bar{P}_\beta. \tag{4.2}$$

F *has at least four fixed points in* $P \times P$: $(0,0), (x_0,0), (0,y_0), (x_1,y_1)$ *verifying* (1.1), *where* $r(L(\tau_1,a,y))$ *means the spectral radius of the linear operator* $L(\tau_1,a,y) : E \longrightarrow E$ *defined by*

$$L(\tau_1,a,y)x(t) = \int_0^{\tau_1(t)} a(t,s,y(t-s-l))x(t-s-l)\,ds, \quad \forall(x,y) \in E\times E,$$

(analogously for $r(L(\tau_2, b, x))$ and $L(\tau_2, b, x)$).

Proof. We are going to prove that all conditions of Theorem 2.3 and Remark 1 are satisfied. For it, we must observe that (E, P) is an ordered Banach space with $\overset{\circ}{P} \neq \emptyset$.

In order to prove that the condition (H1) of Theorem 2.3 is satisfied, take β such that

$$\beta > M \max\{\tau_1^* = \max_{0 \leq t \leq \lambda} \tau_1(t), \tau_2^* = \max_{0 \leq t \leq \lambda} \tau_2(t)\}$$

where M is a constant satisfying

$$M \geq \sup\{f(t, s, x, y), g(t, s, x, y), (t, s, x, y) \in \mathbb{R} \times \mathbb{R} \times [0, +\infty[\times[0, +\infty[\}.$$

Take $(x, y) \in P_\beta \times S_\beta$ and $\lambda \geq 1$, verifying $H(x, y) = \lambda y$, whence

$$y(t) = \frac{1}{\lambda} \int_0^{\tau_2(t)} g(t, s, x(t - s - l), y(t - s - l))\, ds \quad \forall t \in \mathbb{R}.$$

Since $\|y\|_E = \beta$, there is $t_0 \in \mathbb{R}$ such that $y(t_0) = \beta$. From which it follows that

$$\beta = y(t_0) \leq \int_0^{\tau_2(t_0)} g(t_0, s, x(t_0 - s - l), y(t_0 - s - l))\, ds \leq M\tau_2^* < \beta,$$

which is a contradiction. One may proceed in an analogous way if $\|y\|_E \leq \beta$ and $\|x\|_E = \beta$. Therefore (H1) of Theorem 2.3 is satisfied.

As in the proof of Theorem 3.2 one can see that for all $x \in P$ and $y \in \bar{P}_\beta$

$$\lim_{\substack{x \longrightarrow 0 \\ x \in P}} \frac{G(x, y) - L(\tau_1, a, y)x}{\|x\|} = 0, \quad \text{uniformly in} \quad y \in \bar{P}_\beta.$$

Similarly

$$\lim_{\substack{y \longrightarrow 0 \\ y \in P}} \frac{H(x, y) - L(\tau_2, b, x)y}{\|y\|} = 0, \quad \text{uniformly in} \quad x \in \bar{P}_\beta.$$

Now, it is easily seen (see[2, Theorem 2.1]) that $L(\tau_1, a, y)$ and $L(\tau_2, b, x)$ are strongly positive. It then follows from (4.2) that hypothesis (H2) of Theorem 2.3 is satisfied. This completes the proof of the theorem. □

Now we present an example of Theorem 4.1

Example 4.2. Let $f_1 : [0, +\infty) \to \mathbb{R}^+$ defined by

$$f_1(x) = \begin{cases} x(1 - x), & 0 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$$

$g_2 = f_1$, $g_1(y) = 1 + \sin^2 y \quad \forall y \in [0, \infty)$ $f_2(x) = 1 + \cos^2 x \quad \forall x \in [0, \infty)$.

And take $d, d' : \mathbb{R} \to \mathbb{R}$ a continuous, positive and ω-periodic functions ($\omega > 0$) and $l = 0$.

Let the system of nonlinear integral equations

$$x(t) = \int_0^{\tau_1(t)} d(t-s) f_1(x(s)) g_1(y(s))\ ds$$
$$y(t) = \int_0^{\tau_2(t)} d'(t-s) f_2(x(s)) g_2(y(s))\ ds$$

hypotheses (H'1) -(H'4) of Theorem 4.1 are satisfied with $a(t,s,y) = d(t-s)(1 + \sin^2 y)$, and $b(t,s,x) = d'(t-s)(1 + \cos^2 x)$.

Consequently if

$$r(L(\tau_1, a, y)) > 1, \quad r(L(\tau_2, b, x)) > 1, \quad \forall (x,y) \in \bar{P}_\beta \times \bar{P}_\beta, \tag{4.3}$$

(where β is defined as in Theorem 4.1) the above system has at least four fixed points $(0,0), (x_0, 0), (0, y_0), (x_1, y_1)$ in $P \times P$ verifying (1.1). Note that in the particular case where $d(t) \equiv d \in \mathbb{R}^+$ and $d'(t) \equiv d' \in \mathbb{R}^+$conditions (4.3) are satisfied if we take

$$\frac{1}{d} < \min_{t\in\mathbb{R}} \tau_1(t) \quad \text{and} \quad \frac{1}{d'} < \min_{t\in\mathbb{R}} \tau_2(t).$$

Here we use that fact that (see [8, 2])

$$\min_{t\in\mathbb{R}} \int_0^{\tau_1(t)} a(t,s,y(t-s)ds \le r(L(\tau_1, a, y)),$$

and

$$\min_{t\in\mathbb{R}} \int_0^{\tau_1(t)} b(t,s,x(t-s))ds \le r(L(\tau_1, b, x)).$$

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Manuscript received November 15 2017
revised March 17 2018

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