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ON SOME OPTIMIZATION PROBLEMS FOR A CLASS OF FRACTIONAL ORDER FEEDBACK CONTROL SYSTEMS

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Abstract. We study some optimization problems for a class of feedback control systems governed by fractional order semilinear differential inclusions in a Banach space. As example, optimization problems for a time-fractional diffusion type system which include as a particular case the same problems for a controlled process of fractional heat transfer are presented.

1. Introduction

In the present paper we study some optimization problems for a class of feedback control systems governed by fractional order semilinear differential inclusions in a Banach space. It is well known that differential inclusions in infinite dimensional Banach spaces represent a convenient and effective model for the investigation of systems with distributed parameters (see, e.g., [1, 8] and other sources). At the same time the theory of differential equations of fractional order attracts the attention of many researchers due to interesting applications in physics, enginery, biology, economics and other branches of natural sciences (see, e.g., monographs [9, 11, 13] and references therein). In the recent papers [6, 7] the authors proved the existence results for the Cauchy problem for a semilinear fractional order differential inclusion in a Banach space, described the topological structure of the solution set and investigated its continuous dependence on parameters and initial data.

We consider a feedback control system in a Banach space E of the form:

$$^{C}D^{q}x(t) \in Ax(t) + F(t, x(t), u(t)), \quad 0 \leq t \leq T; \tag{1.1}$$

$$u(t) \in U(t, x(t)), \quad 0 \leq t \leq T; \tag{1.2}$$

Here $^{C}D^{q}$ denotes the Caputo fractional derivative of the order $0 < q < 1$, $x\colon [0,T] \to E$ is a trajectory of the system; $A\colon D(A) \subseteq E \to E$ is a closed (not necessarily bounded) linear operator; $u\colon [0,T] \to E_1$ is a control function; E_1 is a Banach space of controls; $F\colon [0,T] \times E \times E_1 \multimap E$ is a multivalued nonlinearity; $U\colon [0,T] \times E \multimap E_1$ is a feedback multimap.

For the above system, we study the optimization of a given lower semicontinuous functional on a set of trajectories of system (1.1)-(1.2) satisfying a given initial

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condition

$$x(0) = x_0 \in E. \tag{1.3}$$

Notice that the problem under consideration includes, as particular cases,

1) $j_0(x(T)) \longrightarrow \min$ (terminal control problem);
2) $j_0(x(T)) + \int_0^T f_0(t, x(t))dt \longrightarrow \min$ (particular Bolza problem).

We deal also with the time-optimal problem which consists in finding of such a trajectory of the system which, starting from a given initial set $\mathcal{M}_0 \subset E$ attains a certain target set $\mathcal{M} \subset E$ in the shortest time.

As example, we consider optimization problems for a time-fractional diffusion type system which include as a particular case the same problems for a controlled process of fractional heat transfer.

2. Preliminaries

Let X, Y be Banach spaces;

$$P(Y) = \{D \subseteq Y, D \neq \varnothing\},$$

$$K(Y) = \{D \in P(Y) : D \text{ is compact }\},$$

$$Kv(Y) = \{D \in K(Y) : D \text{ is convex}\}.$$

The Hausdorff metric on the collection $K(Y)$ is defined as a function $H : K(Y) \times K(Y) \to R_+ :$

$$H(B_1, B_2) = inf\{\varepsilon{>}0 : B_1 \subset U_\varepsilon(B_2), B_2 \subset U_\varepsilon(B_1)\},$$

where $U_\varepsilon(B)$ denotes the ε-neighborhood of a set B.

We will need the following notions (see, e.g., [2, 3, 8]).

Definition 2.1. A multivalued map (multimap) $F\colon X \to P(Y)$ is called upper semicontinuous if the set

$$F_+^{-1}(V) = \{x \in X : F(x) \subset V\}$$

is open in X for every open set $V \subset Y$. A multimap $F : X \to P(Y)$ is called lower semicontinuous if the set $F^{-1}(W)$ is closed in X for every closed set $W \subset Y$. If a multimap F is upper and lower semicontinuous it is called continuous.

Let us mention the following property (see, e.g., [2], Proposition 17.30; [8], Theorem 1.1.7).

Lemma 2.2. *Let $F\colon X \to K(Y)$ be a upper semicontinuous multimap. If $A \subset X$ be a compact set then its image $F(A)$ is a compact subset of Y.*

Let E be a separable Banach space; an interval $[a, b]$ is equipped with the Lebesgue measure.

Recall that a multifunction $G : [a, b] \to K(E)$ is called *measurable* provided it satisfies one of two equivalent conditions:) the set $G^{-1}(V)$ is measurable for every open set $V \subset E$;) there exists a sequence $\{g_n\}_{n=1}^{\infty}$ of measurable functions $g_n : [a,b] \to E$ such that

$$G(t) = \overline{\{g_n(t)_{n=1}^{\infty}\}}$$

for almost all $t \in [a,b]$ (see, e.g., [2, 3, 4]).

Let E' be a separable Banach space.

Definition 2.3. A multimap $\mathcal{F}\colon [a,b] \times E' \to K(E)$ is called superpositionally measurable provided the multifunction $\mathcal{H}\colon [a,b] \to K(E)$, $\mathcal{H}(t) = \mathcal{F}(t, \mathcal{Q}(t))$ is measurable for every measurable multifunction $\mathcal{Q}\colon [a,b] \to K(E')$.

It is known (see, e.g., [8], Proposition 1.3.1 and Theorem 1.3.4) that a multimap $\mathcal{F}$ is superpositionally measurable provided it is upper semicontinuous or satisfies the following *Carathéodory conditions*:

(i) for every $x \in E'$ the multifunction $\mathcal{F}(\cdot, x)\colon [a,b] \to K(E)$ is measurable;
(ii) for a.e. $t \in [a,b]$ the multimap $\mathcal{F}(t, \cdot)\colon E' \to K(E)$ is continuous.

From the other side, it is clear that if a multimap $\mathcal{F}$ is superpositionally measurable then it satisfies the above condition (i).

In the sequel, we will need the following assertion which is due to Castaing (see, e.g., [2], Theorem 20.7; [8], Corollary 1.3.3).

Lemma 2.4. *Let a multimap $\mathcal{F}\colon [a,b] \times E' \to K(E)$ satisfies the following conditions:*

(i') *for each $x \in E'$ the multifunction $\mathcal{F}(\cdot, x)\colon [a,b] \to K(E)$ admits admits a measurable selection, i.e., there exists a measurable function $f : [a,b] \to E$ such that $f(t) \in \mathcal{F}(t,x)$ for a.e. $t \in [a,b]$;*
(ii') *for a.e. $t \in [a,b]$ the multimap $\mathcal{F}(t,\cdot)\colon E' \to K(E)$ is upper semicontinuous.*

Then for every for every measurable multifunction $\mathcal{Q}\colon [a,b] \to K(E')$, the multifunction $\mathcal{H}\colon [a,b] \to K(E)$, $\mathcal{H}(t) = \mathcal{F}(t, \mathcal{Q}(t))$ admits a measurable selection.

We will use also some notions from the fractional analysis (see, e.g., [9, 11, 13]).

Definition 2.5. The Riemann–Liouville fractional derivative of the order $q \in (0,1)$ of a continuous function $g : [0,T] \to E$ is the function $D^q g$ of the following form:

$$D^q g(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_0^t (t-s)^{-q} g(s)\, ds$$

provided the right-hand side of this equality is well defined.

Here Γ is the Euler gamma-function

$$\Gamma(r) = \int_0^{\infty} s^{r-1} e^{-s} ds.$$

Definition 2.6. The Caputo fractional derivative of the order $q \in (0,1)$ of a continuous function $g : [0,T] \to E$ is the function ${}^C D^q g$ defined in the following way:

$${}^C D^q g(t) = \Big(D^q (g(\cdot) - g(0))\Big)(t)$$

provided the right-hand side of this equality is well defined.

Consider the Cauchy problem for a fractional order semilinear differential inclusion in E :

$$ {}^{C}D^{q}x(t) \in Ax(t) + \mathfrak{F}(t, x(t)), \quad 0 \leqslant t \leqslant T, \tag{2.1} $$

$$ x(0) = x_0 \in E, \tag{2.2} $$

under the following assumptions.

(A) $A\colon D(A) \subseteq E \to E$ is a linear closed not necessarily bounded operator generating a bounded C_0-semigroup $\{e^{At}\}_{t\geq 0}$ of linear operators in E.

For a multimap $\mathfrak{F} : [0,T] \times E \to Kv(E)$ we will suppose that:

($\mathfrak{F}1$) for each $x \in E$ the multifunction $\mathfrak{F}(\cdot, x) : [0,T] \to Kv(E)$ admits a measurable selection, i.e., there exists a measurable function $f : [0,T] \to E$ such that $f(t) \in \mathfrak{F}(t,x)$ for a.e. $t \in [0,T]$;

($\mathfrak{F}2$) for almost every $t \in [0,T]$ the multimap $\mathfrak{F}(t,\cdot) : E \to Kv(E)$ is upper semicontinuous;

($\mathfrak{F}3$) there exists a function $\alpha(\cdot) \in L^{\infty}_{+}([0,T])$ such that

$$ \|\mathfrak{F}(t,x)\| := sup\{\|y\| : y \in \mathfrak{F}(t,x)\} \leq \alpha(t)(1 + \|x\|) \text{ a.e. } t \in [0,T] $$

for all $x \in E$;

($\mathfrak{F}4$) there exists a function $k(\cdot) \in L^{\infty}_{+}([0,T])$ such that for every nonempty bounded set $D \subset E$ the following estimate holds true for a.e. $t \in [0,T]$:

$$ \chi(\mathfrak{F}(t,D)) \leq k(t)\chi(D), $$

where χ is the Hausdorff measure of noncompactness in E:

$$ \chi(D) = inf\{\varepsilon{>}0 : D \text{ has a finite } \varepsilon - \text{net}\}. $$

It is clear that condition ($\mathfrak{F}1$) is fulfilled if for every $x \in E$ the multifunction $\mathfrak{F}(\cdot, x)$ is measurable.

Definition 2.7. (cf. [6]) A mild solution of problem (2.1)-(2.2) is a function $x \in C([0,T];E)$ which can be represented as:

$$ x(t) = \mathcal{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f(s)ds, \qquad t \in [0,T], $$

where $f(\cdot) \in L^{\infty}_{+}([0,T];E)$, $f(t) \in \mathfrak{F}(t, x(t))$ for a.e. $t \in [0,T]$ and

$$ \mathcal{G}(t) = \int_0^{\infty} \xi_q(\theta)e^{A(t^q\theta)}d\theta, \qquad \mathcal{T}(t) = q\int_0^{\infty} \theta\xi_q(\theta)e^{A(t^q\theta)}d\theta, $$

$$ \xi_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\psi_q(\theta^{-1/q}), $$

$$ \psi_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \theta \in \mathbb{R}^{+}. $$

The next existence and continuous dependence result follows from the results of the papers [6, 7].

Theorem 2.8. *Under conditions* (A) *and* $(\mathfrak{F}1)-(\mathfrak{F}4)$ *the set* Σ_{x_0} *of all mild solutions of the Cauchy problem* (2.1)-(2.2) *is a nonempty compact subset of* $C([0,T]);E)$ *and, moreover, the multimap*

$$\Sigma\colon E \multimap K(C([0,T]);E)), \quad \Sigma(x) = \Sigma_x$$

is upper semicontinuous.

We will need also the following assertion (see [8], Proposition 2.2.2).

Theorem 2.9. *Let a multimap* $\Phi\colon E \times E \to K(E)$ *be such that:*

i) for every $x \in E$ *and bounded set* $\Omega \subset E$ *the set* $\Phi(x, \Omega)$ *is relatively compact in* E*;*

ii) for every $y \in E$ *the multimap* $\Phi(\cdot, y)\colon E \to K(E)$ *is* κ*-Lipschitz* ($\kappa \geq 0$) *with respect to the Hausdorff metric* H *in* $K(E)$*, i.e.,*

$$H\left(\Phi(x_1, y), \Phi(x_2, y)\right) \leq \kappa \|x_1 - x_2\|$$

for every $x_1, x_2 \in E$.

Then the multimap $\Psi : E \to K(E), \Psi(x) = \Phi(x, x)$ *is* (κ, χ)*-bounded, i.e.,*

$$\chi(\psi(\Omega)) \leq \kappa\chi(\Omega)$$

for every bounded set $\Omega \subset E$.

3. Optimization problems

We will consider system (1.1)-(1.2) under the following assumptions.

Let the linear operator A satisfy condition (A) of the previous section.

Further, let E_1 be a separable space of controls. The multimap $F\colon [0,T] \times E \times E_1 \to K(E)$ obeys the conditions:

$(F1)$ the multifunction $F(\cdot, x, u)\colon [0,T] \to K(E)$ admits a measurable selection for every $(x, u) \in E \times E_1$;

$(F2)$ the multimap F satisfies the following Lipschitz condition in the second argument with respect to the Hausdorff metric H:

$$H\left(F(t, x_1, u), F(t, x_2, u)\right) \leq k(t)\|x_1 - x_2\|$$

for each $x_1, x_2 \in E$, $u \in U([0,T] \times E)$, where $k(\cdot) \in L^\infty_+([0,T])$ does not depend on u;

$(F3)$ the multimap $F(t, \cdot, \cdot)\colon E \times E_1 \to K(E)$ is upper semicontinuous for a.e. $t \in [0,T]$.

For the feedback multimap $U : [0,T] \times E \to K(E_1)$ we will assume that:

$(U1)$ the multimap U is superpositionally measurable;

$(U2)$ the multimap $U(t, \cdot) : E \to K(E_1)$ is upper semicontinuous for a.e. $t \in [0,T]$;

$(U3)$ the set

$$\mathfrak{F}(t, x) = F(t, x, U(t, x))$$

is convex for all $(t, x) \in [0,T] \times E$;

$(U4)$ the multimap $\mathfrak{F} : [0,T] \times E \to Kv(E)$ satisfies the boundedness condition $(\mathfrak{F}3)$;

$(U5)$ for every $(t,x) \in [0,T] \times E$ and a bounded set $D \subset E$ the set
$$F(t,x,U(t,D))$$
is relatively compact in E.

In accordance with Definition 2.7 we introduce the following notion.

Definition 3.1. A pair $\{x,u\}$, where $x \in C([0,T];E)$ and $u\colon [0,T] \to E_1$ is a measurable function, is called a mild solution to problem (1.1)-(1.3) if
$$x(t) = \mathcal{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f(s)ds, \qquad t \in [0,T],$$
where $f(\cdot) \in L^\infty_+([0,T];E)$, $f(t) \in F(t,x(t),u(t))$ for a.e. $t \in [0,T]$, and $u(t) \in U(t,x(t)), t \in [0,T]$.
The function x is called a mild trajectory of the system and the function u is a control function.

The following result on the optimization of a cost functional holds true.

Theorem 3.2. *Let $j : C([0,T];E) \to R$ be a lower semicontinuous functional. Then under assumptions (A), $(F1)$-$(F3)$, and $(U1)$-$(U5)$ there exists a mild solution $\{x_\star, u_\star\}$ to problem* (1.1)-(1.3) *such that*
$$j(x_\star) = \min_{x \in \Sigma_{x_0}} j(x),$$
where Σ_{x_0} is the set of all mild trajectories of problem (1.1)-(1.3).

Proof. Let us show that the multimap $\mathfrak{F}(t,x) = F(t,x,U(t,x))$ satisfies conditions of Theorem 2.8. First of all, let us mention that due to Lemma 2.2 and conditions $(F3),(U3)$ the multimap $\mathfrak{F}$ has compact convex values. Further, condition $(\mathfrak{F}1)$ follows from conditions $(F1),(F3)$, $(U1)$ and Lemma 2.4. Conditions $(F3),(U2)$ and the continuity property of the composition of multimaps (see, e.g., [8], Theorem 1.2.8) imply condition $(\mathfrak{F}2)$. Condition $(\mathfrak{F}3)$ follows directly from $(U4)$.

Now, let us verify that the multimap $\mathfrak{F}$ satisfies condition $(\mathfrak{F}4)$. Fix $t \in [0,T]$ and consider the multimap $\Phi : E \times E \to K(E)$ defined as
$$\Phi(x,y) = F(t,x,U(t,y)).$$
This multimap satisfies conditions of Theorem 2.9. In fact, the validity of condition (i) of this theorem follows immediately from $(U5)$. Now, fix $y \in E$ and take $x_1, x_2 \in E$ and arbitrary $\varphi_1 \in \Phi(x_1,y)$. Then $\varphi_1 \in F(t,x_1,u)$, for some $u \in U(t,y)$. From condition $(F2)$ it follows that there exists $\varphi_2 \in F(t,x_2,u) \subset \Phi(x_2,y)$ such that
$$\|\varphi_1 - \varphi_2\| \le k(t)\|x_1 - x_2\|$$
that implies condition (ii) of Theorem 2.9. By applying this theorem we get that the multimap $\mathfrak{F}(t,x)$ satisfies condition $(\mathfrak{F}4)$.

Now, from Theorem 2.8 it follows that there exists a nonempty compact set of functions $x \in C([0,T];E)$ of the form
$$x(t) = \mathcal{G}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{T}(t-s)f(s)ds, \qquad t \in [0,T],$$

where $f(t) \in \mathfrak{F}(t, x(t)) = F(t, x(t), U(t, x(t)))$ for a.e. $t \in [0, T]$ is a measurable selection. By using condition $(U1)$ and applying the Filippov implicit function lemma (see, e.g., [8], Theorem 1.3.3) we conclude that every such function is a mild trajectory of the system (1.1)-(1.3). Let $x_\star$ be a mild trajectory minimizing the functional j and $u_\star$ be a corresponding control function. The pair $\{x_\star, u_\star\}$ is the required optimal solution of problem (1.1)-(1.3). □

Now we will consider the time-optimality problem for system (1.1)-(1.2) satisfying conditions (A), $(F1)-(F3)$, and $(U1)-(U5)$. Let $\mathcal{M}_0 \subset E$ be a given compact initial set, $\mathcal{M} \subset E$ a given closed target set.

Theorem 3.3. *Let there exist a mild trajectory of system* (1.1)-(1.2) *starting from the set $\mathcal{M}_0$ and attaining the set $\mathcal{M}$ at a certain moment $t_1 \in (0, T]$. Then there exists a mild trajectory of the system attaining the set $\mathcal{M}$ from the set $\mathcal{M}_0$ in a shortest time.*

Proof. Denote by $\Sigma(x) \subset C([0, T]; E)$ the set of all mild trajectories of the system emanating from a point $x \in \mathcal{M}_0$. From Theorem 2.8 we know that the multimap $\Sigma\colon E \multimap C([0, T]; E)$ is compact-valued and upper semicontinuous. Hence the set $\Sigma(\mathcal{M}_0)$ of all mild trajectories emanating from $\mathcal{M}_0$ is compact (Lemma 2.2).

Consider the attainability multifunction $\Pi\colon [0, T] \to K(E)$ defined as

$$\Pi(t) = \Sigma(\mathcal{M}_0)(t) = \{x(t) \colon x \in \Sigma(\mathcal{M}_0)\}.$$

and the set

$$\mathfrak{T} = \Pi_-^{-1}(\mathcal{M}) = \{t \in [0, T] \colon \Pi(t) \cap \mathcal{M} \neq \emptyset\}.$$

This set is nonempty since, by assumption, it contains the point t_1. Further, it is easy to verify that the multimap Π is upper semicontinuous and therefore the set $\mathfrak{T} \subset [0, T]$ is closed and contains its lower bound $t_\star$ which is the minimal moment of attainability of the set $\mathcal{M}$. □

4. Optimization of a time-fractional diffusion system

We will consider optimization problems for the following time-fractional diffusion system. Let $G \subset \mathbb{R}^n$ be a domain of a finite measure with a smooth boundary ∂G. The state of a system will be characterized by a function $z\colon [0, T] \times G \to \mathbb{R}$ and its dynamics is described by the following relations

$$(4.1) \qquad {}^C D_t^q z(t, y) = \sum_{k=1}^{n} \frac{\partial^2}{\partial y_k^2} z(t, y) + h(y, z(t, y), v(t, y)), \quad 0 \le t \le T$$

$$(4.2) \qquad z(t, \cdot)|_{\partial G} = 0, \quad 0 \le t \le T.$$

It is worth noting that the equation $D_t^\alpha z = z_{xx}$, known as time-fractional diffusion equation presents a mathematical model finding wide applications, due to anomalous diffusion effects in disordered materials, where the environment is constrained and trapping and binding of particles can occur. It describes anomalous diffusion characterized by the mean square displacement of particles from the original starting site, verifying the generalized Fick's second law. Important applications include viscoelasticity and seismic-wave theory, diffusion in turbulent plasma, fractal media and porous media (see, e.g., [5] and the references therein). We consider a perturbed equation modeling the influence of a control on the dynamics of the process.

As the space of the system states we will consider a Hilbert space $E = L^2(G;\mathbb{R})$ and a Hilbert space $E_1 = L^2(G,\mathbb{R}^m)$ will be considered as the space of controls. Let a feedback multimap $U\colon [0,T] \times E \to K(E_1)$ at each moment $t \in [0,T]$ associates to a state of the system $z(t,\cdot)$ the set of admissible controls

$$v(t,\cdot) \in U(t, z(t,\cdot)). \tag{4.3}$$

We will assume that the function $h : G \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ satisfies the following conditions:

($h1$) the function $h(\cdot, z, u) : G \to \mathbb{R}$ is measurable $\forall z \in \mathbb{R}, u \in \mathbb{R}^m$;
($h2$) the function $h(y, \cdot, \cdot) : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is continuous $\forall y \in G$;
($h3$) there exists a function $a \in L^2_+(G;\mathbb{R})$ such that

$$|h(y, z, u)| \le a(y) \quad \forall z \in \mathbb{R},\, u \in U([0,T] \times E);$$

($h4$) $|h(y, z_1, u) - h(y, z_2, u)| \le k|z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{R},\, y \in G,\, u \in U([0,T] \times E)$, where $k > 0$ does not depend on y and u.

Let the multimap U satisfy conditions ($U1$) and ($U2$) of the previous section and, moreover, let the following condition hold true:

($U5'$) for every $t \in [0,T]$ and $x \in E$ the set of functions

$$\{h(\cdot, x(\cdot), u(\cdot)) : u \in U(t, D)\}$$

is relatively compact in E for each bounded set $D \subset E$.

The map h generates the map $F : E \times E_1 \to E$ as

$$F(x, u) = h(\cdot, x(\cdot), u(\cdot)).$$

We will assume that for every $t \in [0,T]$ and $x \in E$ the set of functions

$$\{h(\cdot, x(\cdot), u(\cdot)) : u \in U(t, x)\}$$

is convex in E.

The continuity of the map F follows from the Krasnoselskii theorem on the continuity of the superposition operator (see, e.g., [10]) and hence condition ($F3$) is fulfilled for the map F.

Notice that from the continuity of the map $F(x,\cdot)$ it follows that condition ($U5'$) will be fulfilled provided the multimap U is completely upper semicontinuous in the second argument, i.e., for each $t \in [0,T]$ the image $U(t, D)$ of every bounded set $D \subset E$ is relatively compact.

Let us verify now condition ($F2$). Take $x_1, x_2 \in E$, $u \in U([0,T] \times E)$, then

$$\begin{aligned}\|F(x_1, u) - F(x_2, u)\|_E &= \|h(\cdot, x_1(\cdot), u(\cdot)) - h(\cdot, x_2(\cdot), u(\cdot))\|_E \\ &= \Big(\int_G |h(y, x_1(y), u(y)) - h(y, x_2(y), u(y))|^2 dy\Big)^{1/2} \\ &\le k\Big(\int_G |x_1(y) - x_2(y)|^2 dy\Big)^{1/2} = k\|x_1 - x_2\|_E.\end{aligned}$$

In the space $L^2(G;\mathbb{R})$ consider the Laplace operator

$$\Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial y_k^2}$$

with the domain $W^{2,2}(G;\mathbb{R}) \cap W_0^{1,2}(G;\mathbb{R})$. It is known (see, e.g. [12]) that Δ generates a C_0-semigroup of contractions and so condition (A) is fulfilled.

Now we see that system (4.1)-(4.3) satisfies all conditions described in the previous section. It means that we can formulate the following optimization results.

Theorem 4.1. *Let $j\colon C([0,T];L^2(G;\mathbb{R})) \to \mathbb{R}$ be a given lower semicontinuous functional. Then there exists a mild solution $\{z_\star, v_\star\}$ of problem* (4.1)-(4.3) *satisfying a given initial value condition*

$$z(t,\cdot) = x_0 \in L^2(G;\mathbb{R}) \tag{4.4}$$

with $z_\star \in C([0,T];L^2(G;\mathbb{R}))$ and a measurable function $t \in [0,T] \to v_\star(t,\cdot) \in L^2(G;\mathbb{R}^m)$ such that

$$j(x_\star) = \min_{x \in \Sigma_{x_0}} j(x),$$

where $\Sigma_{x_0} \subset C([0,T];L^2(G;\mathbb{R}))$ is the set of all mild trajectories of problem (4.1)-(4.4)*.*

Theorem 4.2. *Let there exist a mild trajectory of system* (4.1)-(4.3) *starting from a compact set $\mathcal{M}_0 \subset L^2(G;\mathbb{R})$ and attaining a closed set $\mathcal{M} \subset L^2(G;\mathbb{R})$ at a certain moment $t_1 \in (0,T]$. Then there exists a mild trajectory of the system attaining the set $\mathcal{M}$ from the set $\mathcal{M}_0$ in a shortest time.*

Example 4.3. We will consider optimization problems in a time-fractional process of heat transfer. Let $G \subset \mathbb{R}^3$ be a domain of a finite measure with a smooth boundary ∂G. A function

$$z(t,y),\ z \in C([0,T];L^2(G;\mathbb{R}))$$

characterizes the temperature in a point $y \in G$ at a moment $t \in [0,T]$.

Let in the domain G there exist m heat sources whose properties depend on the temperature and whose densities are described by functions $\varphi_i(y,z), i = 1,\dots,m, \varphi_i : G\times\mathbb{R} \to \mathbb{R}$. The intensity of sources is regulated by the controls $u_i : [0,T] \to \mathbb{R}(i = 1,...,m)$ which are measurable functions satisfying the following feedback condition:

$$u(t) = (u_1(t),\dots,u_m(t)) \in W(z(t,\cdot)), \quad t \in [0,T], \tag{4.5}$$

where W a upper semicontinuous multimap from $L^2(G;\mathbb{R})$ to $\mathbb{R}^m$ with convex closed values such that

$$\|W(x)\| \le M$$

for all $x \in L^2(G;\mathbb{R})$, where $M{>}0$.

Now the controlled time-fractional process of heat transfer in the domain G is described, together with (4.5), by the following relations:

$$^{C}D_t^q z(t,y) = \sum_{k=1}^{3} \frac{\partial^2}{\partial y_k^2} z(t,y) + \sum_{i=1}^{m} u_i(t)\varphi_i(y,z(t,y)), \quad 0 \le t \le T; \tag{4.6}$$

$$z(t,\cdot)|_{\partial G} = 0, \quad 0 \le t \le T; \tag{4.7}$$

$$z(0,\cdot) = x_0 \in L_2(G;\mathbb{R}). \tag{4.8}$$

We will assume that the functions φ_i satisfy for each $i = 1, ..., m$ the following conditions:

$\varphi 1)$ $\varphi_i(\cdot, z) : G \to \mathbb{R}$ is measurable $\forall z \in \mathbb{R}$;
$\varphi 2)$ $|\varphi_i(y,z)| \le \alpha_i(y) \quad \forall z \in \mathbb{R}$, where $\alpha_i \in L^2_+(G;\mathbb{R})$;
$\varphi 3)$ $|\varphi_i(y,z_1) - \varphi_i(y,z_2)| \le k_i|z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{R}, y \in G$, where k_i does not depend on y.

Then it is easy to see that the function $h : G \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$

$$h(y,z,u) = \sum_{i=1}^{m} u_i \varphi_i(y,z)$$

satisfies conditions $(h1)$-$(h4)$.

Making the natural embedding of the space $\mathbb{R}^m$ into $L^2(G;\mathbb{R}^m)$, we will induce by the multimap W a upper semicontinuous multimap U from $L^2(G;\mathbb{R})$ to $L^2(G,\mathbb{R}^m)$ for which all necessary conditions will be fulfilled. In particular, the validity of property $(U5')$ follows from the fact that for every $x \in L^2(G;\mathbb{R})$ the set

$$\{h(\cdot, x(\cdot), u(\cdot)) : u \in U(L^2(G;\mathbb{R}))\}$$

is a bounded subset of a linear hull of the functions

$$\varphi_1(\cdot, x(\cdot)), \ldots, \varphi_m(\cdot, x(\cdot)) \in L^2(G;\mathbb{R}).$$

Therefore, from Theorem 4.1 we conclude that there exists a mild solution $\{z_\star(t,y), u_\star(t)\}$ of problem (4.5)-(4.8) such that the function $z_\star$ minimizers the functional

$$j(z) = \int_G |z(T,y) - z_0(y)|^2 dy,$$

expressing the mean square deviation of the temperature distribution at the final moment $t = T$ from a prescribed distribution $z_0 \in L^2(G;\mathbb{R})$.

A similar application of Theorem 4.2 yields the existence of a time-optimal solution of problem (4.5)-(4.8) attaining a given closed set $\mathcal{M} \subset L^2(G;\mathbb{R})$ of temperature distributions.

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