Applied Analysis and Optimization
Volume 2, Number 1, 2018, 41–46

Yokohama Publishers
ISSN 2189-1664 Online Journal

A FIXED SET THEOREM FOR SET-TO-SET MAPS

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Abstract. A fixed set theorem in term of $T(A) = A$ for set-to-set Hausdorff continuous self-mappings on a family of all nonempty compact convex subset of a normed space was given by using an embedding idea by Radström [8].

1. Introduction

Let X be a compact convex subset of a normed space. For a set-valued map $T : X \to 2^X$, $\bar{x} \in X$ is said to be a fixed point of T if $T(\bar{x}) \ni \bar{x}$. Nadler established a fixed point theorem for set-valued maps in [7] which is an extension of the Banach contraction principle, Mizoguchi and Takahashi have extended Nadler's results in [6]. Also Fakhar, Soltani and Zafarani gave a maximal invariant set (fixed set) theorem for set-valued maps in [3].

On the other hand, for a set-to-set map $T : 2^X \to 2^X$ and a nonempty set $A \in 2^X$, there are four type *fixed set* notions which are generalizations of the fixed point notion:

(1) $T(A) = A$;
(2) $T(A) \subset A$;
(3) $T(A) \supset A$;
(4) $T(A) \cap A \neq \emptyset$.

We can find the following previous works for such fixed set theorems: Pradip, Binayak and Murchana showed a fixed set theorem in term of $T(A) \supset A$ in [2], which is a generalization of Nadler's result, and Robert, Klaus and Bradon showed a fixed set theorem in term of $T(B) = B$ for a monotone map T under the existence of A such that $T(A) \subset A$ in [1], and applied to study of a boundary value problem for a system of differential equations. In this paper, we give another fixed set theorem for set-to-set maps, by using an embedding idea in [8], which is a generalization of the following Schauder fixed point theorem, see [9]:

Theorem 1.1. *Let X be a nonempty convex subset of a normed space E, and let T be a continuous self-mapping on X. If $T(X)$ is compact, then there exists $\bar{x} \in X$ such that $T(\bar{x}) = \bar{x}$.*

2. Main results

Throughout this paper, let E be a normed space, let X be a nonempty compact convex subset of E, and let $\mathcal{C}_X$ be the family of all nonempty compact convex subsets of X.

2010 *Mathematics Subject Classification.* 54H25, 47H10.

Key words and phrases. Fixed set, set-valued map, Hausdorff continuous, embedding space.

The second author was partially supported by a grant of JSPS KAKENHI Grant Numbers 16K05274.

Lemma 2.1. *Define* $H : \mathcal{C}_X \times \mathcal{C}_X \to [0, +\infty)$ *by*

$$H(A,B) := \max\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\},$$

for any $A, B \in \mathcal{C}_X$. *Then* H *is a metric on* $\mathcal{C}_X$, *which is called the Hausdorff metric, and the metric space* $(\mathcal{C}_X, H)$ *is compact.*

Proof. We give a proof based on the non-convex version, see [4]. Since X is compact, that is, X is totally bounded, for any $\varepsilon > 0$, there exists a finite set $Y \subset X$ such that

$$\min_{y\in Y} d(x,y) < \varepsilon \text{ for any } x \in X.$$

For any $C \in \mathcal{C}_X$, put $S = \{y \in Y \mid d(C,y) < \varepsilon\}$, then $H(C,S) < \varepsilon$ holds, that is, $H(C, \text{co}S) < \varepsilon$ holds. Put a finite subfamily $\mathcal{T} = \{\text{co}S \mid S \in 2^Y\}$, then $\mathcal{T} \subset \mathcal{C}_X$ and

$$\min_{T\in\mathcal{T}} H(C,T) < \varepsilon \text{ for any } C \in \mathcal{C}_X.$$

This shows that $(\mathcal{C}_X, H)$ is also total bounded. Next, for any Cauchy sequence $\{A_n\} \subset \mathcal{C}_X$, define

$$A := \{x \in X \mid \exists\{x_n\} \subset X \text{ s.t. } x_n \to x, x_n \in A_n \ \forall n \in \mathbb{N}\},$$

then we can see that A is a nonempty compact convex subset of X and $\{A_n\}$ converges to A with respect to the Hausdorff metric H. Then $(\mathcal{C}_X, H)$ is complete, and consequently $(\mathcal{C}_X, H)$ is compact. □

Now we give the main theorem.

Theorem 2.2. *Let* $\mathcal{A}$ *be a subfamily of* $\mathcal{C}_X$ *satisfying*

$$A, B \in \mathcal{A}, \lambda \in (0,1) \Rightarrow (1-\lambda)A + \lambda B \in \mathcal{A}, \tag{2.1}$$

and let $T : \mathcal{A} \to \mathcal{A}$ *be continuous with respect to the Hausdorff metric* H. *If either the following (i) or (ii) holds:*

(i) $\mathcal{A}$ *is closed with respect to the Hausdorff metric* H,

(ii) $T(\mathcal{A}) := \{T(A) \mid A \in \mathcal{A}\}$ *is closed with respect to the Hausdorff metric* H,

then T *has a fixed set, that is, there exists* $\bar{A} \in \mathcal{A}$ *such that* $T(\bar{A}) = \bar{A}$.

Proof. We may assume (ii). Indeed, if (i) holds, then $\mathcal{A}$ is compact because $\mathcal{A}$ is closed and $\mathcal{C}_X$ is compact with respect to the Hausdorff metric H, therefore, the image $T(\mathcal{A})$ is also compact because T is continuous.

Let $\mathcal{C}$ be the family of all nonempty compact convex subsets of E, and define a binary relation $\equiv$ on $\mathcal{C}^2$ by, for all $(A,B), (C,D) \in \mathcal{C}^2$,

$$(A,B) \equiv (C,D) \text{ if } A + D = B + C,$$

then $\equiv$ is an equivalence relation on $\mathcal{C}^2$. The cancellation low on $\mathcal{C}$, that is,

$$A + B \subset A + C \Rightarrow B \subset C$$

is essential to show the equivalence. Define the quotient space

$$\mathcal{C}^2/{\equiv} := \{[A,B] \mid (A,B) \in \mathcal{C}^2\},$$

where

$$[A,B] := \{(C,D) \in \mathcal{C}^2 \mid (A,B) \equiv (C,D)\},$$

and define the following addition and scholar multiplication on $\mathcal{C}^2/\!\equiv$ by

$$[A,B]+[C,D]=[A+C,B+D],$$

$$\lambda[A,B]=\begin{cases}[\lambda A,\lambda B] & \text{if } \lambda\geq 0\\ [-\lambda B,-\lambda A] & \text{if } \lambda<0,\end{cases}$$

for any $[A,B],[C,D]\in\mathcal{C}^2/\!\equiv$ and $\lambda\in\mathbb{R}$, then $\mathcal{C}^2/\!\equiv$ is a vector space over $\mathbb{R}$. Also define

$$\|[A,B]\|=H(A,B)$$

for each $[A,B]\in\mathcal{C}^2/\!\equiv$, then $(\mathcal{C}^2/\!\equiv,\|\cdot\|)$ becomes a normed space. For details about these arguments, see [5, 8].

Define

$$\begin{array}{cccc}\psi: & \mathcal{A} & \to & \mathcal{C}^2/\!\equiv\\ & \Cup & & \Cup\\ & A & \longmapsto & [A,\{0\}].\end{array}$$

Note that

$$\|\psi(A)-\psi(B)\|=\|[A,\{0\}]-[B,\{0\}]\|=\|[A,B]\|=H(A,B),$$

for any $A,B\in\mathcal{C}_X$. Consequently, ψ is continuous because

$$\|\psi(A_n)-\psi(A)\|=H(A_n,A)\to 0$$

for a sequence $\{A_n\}_{n\in\mathbb{N}}\subset\mathcal{A}$ converges to $A\in\mathcal{A}$ with respect to the Hausdorff metric H. Also $\psi(\mathcal{A})$ is a convex subset of $\mathcal{C}^2/\!\equiv$. Indeed, for any $\psi(A),\psi(B)\in\psi(\mathcal{A})$ and $\lambda\in(0,1)$, from

$$(1-\lambda)\psi(A)+\lambda\psi(B)=[(1-\lambda)A+\lambda B,\{0\}]=\psi((1-\lambda)A+\lambda B)$$

and $(1-\lambda)A+\lambda B\in\mathcal{A}$, then $(1-\lambda)\psi(A)+\lambda\psi(B)\in\psi(\mathcal{A})$.

Consider a self-mapping on convex set $\psi(\mathcal{A})$ defined by

$$\begin{array}{cccc}\mathcal{T}: & \psi(\mathcal{A}) & \to & \psi(\mathcal{A})\\ & \Cup & & \Cup\\ & [A,\{0\}] & \longmapsto & [T(A),\{0\}],\end{array}$$

then $\mathcal{T}$ is continuous. Indeed, if a sequence $\{\psi(A_n)\}\subset\psi(\mathcal{A})$ converges to $\psi(A)\in\psi(\mathcal{A})$, that is $\|\psi(A_n)-\psi(A)\|\to 0$, then $H(A_n,A)\to 0$ and

$$\|\mathcal{T}(\psi(A_n))-\mathcal{T}(\psi(A))\|=\|[T(A_n),\{0\}]-[T(A),\{0\}]\|=H(T(A_n),T(A)).$$

Since T is continuous with respect to H, then $H(T(A_n),T(A))\to 0$. This shows $\mathcal{T}$ is continuous. Also $\mathcal{T}(\psi(\mathcal{A}))$ is compact because $T(\mathcal{A})$ is compact, ψ is continuous, and

$$\begin{aligned}\mathcal{T}(\psi(\mathcal{A}))&=\{\mathcal{T}(\psi(A))\mid A\in\mathcal{A}\}\\ &=\{\mathcal{T}([A,\{0\}])\mid A\in\mathcal{A}\}\\ &=\{[T(A),\{0\}]\mid A\in\mathcal{A}\}\\ &=\{\psi(T(A))\mid A\in\mathcal{A}\}\\ &=\psi(T(\mathcal{A})).\end{aligned}$$

By using Theorem 1.1, there exists $\bar{A}\in\mathcal{A}$ such that $\mathcal{T}(\psi(\bar{A}))=\psi(\bar{A})$, that is, $T(\bar{A})=\bar{A}$. □

Remark 2.3. It is clear that Theorem 2.2 is different from the previous fixed set theorems in [1, 2].

We can obtain the following corollaries by using Theorem 2.2:

Corollary 2.4. *Let T be a continuous self-mapping on $\mathcal{C}_X$ with respect to the Hausdorff metric H. Then T has a fixed set, that is, there exist $\bar{A} \in \mathcal{C}_X$ such that $T(\bar{A}) = \bar{A}$.*

Proof. Put $\mathcal{A} := \mathcal{C}_X$, then we can see that $\mathcal{A}$ is closed with respect to H satisfying (2.1). Therefore we can apply Theorem 2.2 to show the existence of fixed sets of T. □

Corollary 2.5 (Theorem 1.1)**.** *Let X be a nonempty convex subset of a normed space E, and let T be a continuous self-mapping on X. If $T(X)$ is compact, then there exists $\bar{x} \in X$ such that $T(\bar{x}) = \bar{x}$.*

Proof. Put $\mathcal{A} := \{\{x\} \mid x \in X\}$, then we can see that $\mathcal{A}$ is closed with respect to H satisfying (2.1) and $\hat{T} : \mathcal{A} \to \mathcal{A}$, defined by $\hat{T}(\{x\}) = \{T(x)\}$, is continuous with respect to the Hausdorff metric H. Therefore we can apply Theorem 2.2 to show the existence of fixed sets of T. □

Remark 2.6. Theorem 2.2 does not guarantee an existence fixed set $\bar{A}$ is a non-singleton set. However by constructing $\mathcal{A}$ which does not include singleton sets, every existence fixed set becomes non-singleton. We give the following examples to explain this remark:

Example 2.7. Let $X = [0, 2]^2$ and

$$\mathcal{A} = \{B(x_1, x_2, r) \mid B(x_1, x_2, r) \subset X, (x_1, x_2) \in \mathbb{R}^2, r \geq 0\},$$

where $B(x_1, x_2, r) = \{(y_1, y_2) \in \mathbb{R}^2 \mid (y_1 - x_1)^2 + (y_2 - x_2)^2 \leq r^2\}$. Then $\mathcal{A} \subset \mathcal{C}_X$ is closed with respect to the Hausdorff metric H. Hence each continuous self-mapping on $\mathcal{A}$ with respect to H has a fixed set from Theorem 2.2. For example, define

$$T(B(x_1, x_2, r)) = B(x_2, x_1, r^2),$$

then we can check that $T : \mathcal{A} \to \mathcal{A}$ is continuous with respect to H and then there exists a fixed set $\bar{A} \in \mathcal{A}$ such that $T(\bar{A}) = \bar{A}$. However, we can not see whether an existence fixed set $\bar{A}$ is a non-singleton set or not. Indeed, $B(1, 1, 1)$ and $B(x, x, 0)$, $0 \leq x \leq 2$, are fixed sets of T. On the other hand, let

$$\mathcal{A}' = \{B(x_1, x_2, r) \mid B(x_1, x_2, r) \subset X, (x_1, x_2) \in \mathbb{R}^2, r > 0\}$$

and assume that a self-mapping T on $\mathcal{A}'$ has a fixed set $\bar{A}$, then $\bar{A}$ should be non-singleton. However $\mathcal{A}'$ is not closed with respect to H and Theorem 2.2 can not be applied to the situation.

Example 2.8. Let $X = [0, 4] \times [0, 4] \subset \mathbb{R}^2$ and let

$$\mathcal{A} = \{[a, b] \times [c, d] \mid 0 \leq a \leq b \leq 4, 0 \leq c \leq d \leq 4, (b - a)(d - c) = 1\}.$$

Consider a self-mapping $T : \mathcal{A} \to \mathcal{A}$ defined by

$$T([a, b] \times [c, d]) = [a + t_W(a) - h, b - t_E(b) + h] \times [c + t_S(c) - h, d - t_N(d) + h]$$

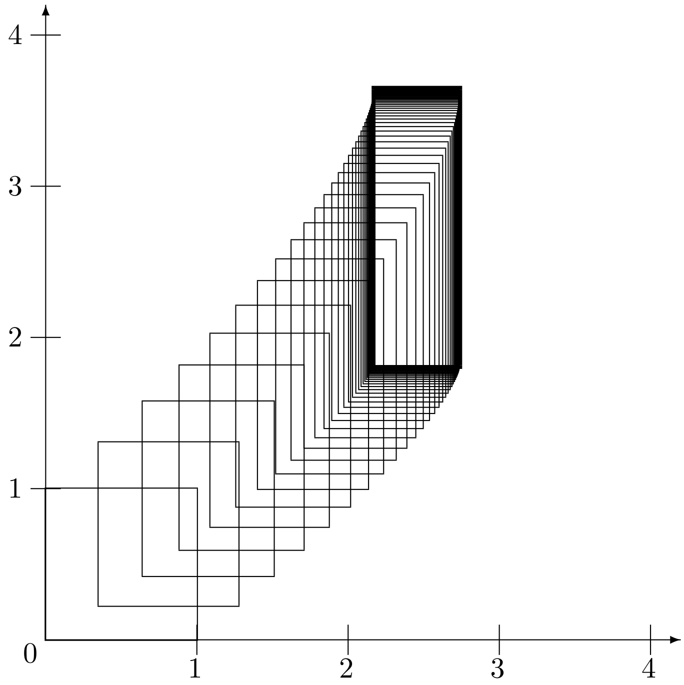

FIGURE 1. The orbit $\{T^n([0,1]\times[0,1])\}$ in Example 2.8

where $t_W(a) = 3(4-a)/16$, $t_E(b) = b/8$, $t_S(c) = 5(4-c)/32$, $t_N(d) = 3d/32$, and h is the biggest solution of the following quadratic function:

$$(b - t_E(b) - a - t_W(a) + 2h)(d - t_N(d) - c - t_S(c) + 2h) = 1.$$

Since $\mathcal{A}$ does not include any singleton, every fixed set $\bar{A}$ of T is non-singleton. We can see that the only fixed set is $[4 - k/6, k/4] \times [4 - k/5, k/3]$ where $k = (171 - 3\sqrt{249})/20$. This example shows a model of residence movement against natural threats from north, south, east, and west. The orbit $\{T^n([0,1]\times[0,1])\}$ is given in Figure 1.

Example 2.9. Let X be a nonempty compact convex subset of a normed space E and for any $\varepsilon > 0$, define

$$\mathcal{A}_\varepsilon = \{A \subset \mathcal{C}_X \mid \text{there exists } x \in X \text{ such that } B(x,\varepsilon) \subset A\},$$

where $B(x,\varepsilon) = \{y \in X \mid \|y - x\| \leq \varepsilon\}$. Then we can check that $\mathcal{A}_\varepsilon$ is closed with respect to the Hausdorff metric H and (2.1). If T is a continuous self-mapping on $\mathcal{A}_\varepsilon$, then there exists a fixed set $\bar{A} \in \mathcal{A}_\varepsilon$. Clearly, $\bar{A}$ is a non-singleton set.

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Manuscript received December 5 2017
revised February 1 2018

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