Applied Analysis and Optimization
Volume 2, Number 1, 2018, 27–39

Yokohama Publishers
ISSN 2189-1664 Online Journal

SURROGATE DUALITY FOR ROBUST QUASICONVEX VECTOR OPTIMIZATION

SATOSHI SUZUKI AND DAISHI KUROIWA

Abstract. In this paper, we study quasiconvex vector optimization with data uncertainty via robust optimization. By using scalarization, we introduce two types of surrogate duality theorems for robust quasiconvex vector optimization. We show surrogate min-max duality theorems for quasiconvex vector optimization with uncertain objective and/or constraints. For the problem with uncertain objective, we introduce its robust counterpart as a set-valued optimization problem.

1. Introduction

Recently, mathematical programming problems with data uncertainty have been investigated in order to deal with complexity of real-world optimization problems. Robust optimization is one of the approach to solve these problems robustly. In robust optimization, we replace the problem which has data uncertainty with a semi-infinite programming problem. One of the research aspects of robust optimization is a strong duality theorem, which is a result guaranteeing that the pessimistic and the optimistic counterparts of a given uncertain problem have the same optimal values and the dual optimal value is attained. Many researchers introduce robust strong duality theorems for mathematical programming problems with data uncertainty by duality theorems in infinite programming, see [1, 2, 5, 7, 10, 14, 27] and references therein.

In mathematical programming, many researchers study problems whose objective function is real-valued, see [3, 8, 11, 16–26]. Especially, in quasiconvex optimization, surrogate duality is a well known and important element. Recently, the authors introduce necessary and sufficient constraint qualifications for surrogate duality, see [22, 24, 27]. Also, vector optimization, which is concerned with a problem whose objective function is vector-valued, have been investigated by many researchers, for example, see [4, 6, 15, 28]. In vector optimization, solutions are defined by an ordering cone. By using scalarization, we replace the problem with a real-valued optimization problem. There are so many results of real-valued robust optimization, and some results of robust convex vector optimization. However, as far as we know, robust quasiconvex vector optimization has not been investigated yet. Additionally, some types of robust counterpart for vector optimization have been introduced, for example see [5, 14]. It is not so evident which is the suitable robust counterpart

2010 *Mathematics Subject Classification.* 90C26, 90C29, 90C34, 26B25 .

Key words and phrases. Multiobjective programming, robust vector optimization, surrogate duality, constraint qualification.

This work was supported by JSPS KAKENHI Grant Numbers 15K17588, 16K05274. This work was supported by the Research Institute for Mathematical Sciences, a Joint Usage/Research Center located in Kyoto University.

of uncertain problem whenever the uncertainty aspects the vector-valued objective function.

Hence, in this paper, we study quasiconvex vector optimization with data uncertainty via robust optimization. To the purpose, we introduce surrogate duality for robust vector optimization by using scalarization. We investigate surrogate min-max duality for quasiconvex vector optimization with uncertain objective and/or constraints. For the problem with uncertain objective, we introduce its robust counterpart as a set-valued optimization problem.

The remainder of the paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we investigate surrogate min-max duality for quasiconvex vector optimization with uncertain constraints. In Section 4, we investigate surrogate min-max duality for quasiconvex vector optimization with uncertain objective and constraints. In Section 5, we discuss about our results.

2. Preliminaries

Let X be a locally convex Hausdorff topological vector space, X^* the continuous dual space of X, and $\langle x^*, x\rangle$ the value of a functional $x^* \in X^*$ at $x \in X$. Given a set $A^* \subset X^*$, we denote the w^*-closure, the boundary, the interior, the convex hull, and the conical hull generated by A^*, by $\mathrm{cl}\, A^*$, $\mathrm{bd}\, A^*$, $\mathrm{int}\, A^*$, $\mathrm{conv} A^*$, and $\mathrm{cone}\, A^*$, respectively. By convention, we define $\mathrm{cone}\, \emptyset = \{0\}$. The indicator function δ_A of $A \subset X$ is defined by

$$\delta_A(x) := \begin{cases} 0, & x \in A, \\ \infty, & \textit{otherwise}. \end{cases}$$

Let f be a function from X to $\overline{\mathbb{R}} := [-\infty, +\infty]$. We denote the domain of f by $\mathrm{dom} f := \{x \in X \mid f(x) < \infty\}$. The epigraph of f is $\mathrm{epi} f := \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\}$, and f is said to be convex if $\mathrm{epi} f$ is convex. The Fenchel conjugate of f, $f^* : X^* \to \overline{\mathbb{R}}$, is defined as $f^*(u) := \sup_{x \in \mathrm{dom} f}\{\langle u, x\rangle - f(x)\}$. Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$L(f, \diamond, \beta) := \{x \in X \mid f(x) \diamond \beta\}$$

for any $\beta \in \mathbb{R}$. A function f is said to be quasiconvex if for each $\beta \in \mathbb{R}$, $L(f, \leq, \beta)$ is convex. Any convex function is quasiconvex, but the opposite is not generally true.

Let Y be a locally convex Hausdorff topological vector space, partially ordered by a nonempty, closed and convex cone $K \subset Y$, that is, for y, $z \in Y$, the notation $y \leq_K z$ will mean $z - y \in K$. A cone K is said to be solid if $\mathrm{int}\, K$ is nonempty. Let Y^* be the continuous dual space of Y, and g a function from X to Y. The positive polar cone of K is $K^+ := \{\lambda \in Y^* \mid \forall y \in K, \langle \lambda, y\rangle \geq 0\}$. A function g is said to be K-convex if for all x_1, $x_2 \in X$, and $\alpha \in [0, 1]$, $(1-\alpha)g(x_1) + \alpha g(x_2) \in g((1-\alpha)x_1 + \alpha x_2) + K$. It is well known that g is K-convex if and only if $\lambda \circ g$ is convex for all $\lambda \in K^+$. A function g is said to be K-quasiconvex if for all $y \in Y$, x_1, $x_2 \in X$, and $\alpha \in [0, 1]$ with $y \in (g(x_1)+K) \cap (g(x_2)+K)$, $y \in g((1-\alpha)x_1 + \alpha x_2) + K$. Also, g is said to be proper quasi K-concave if for each x, $y \in X$ and $\alpha \in (0, 1)$, $g((1-\alpha)x + \alpha y) \in (g(x) + K) \cup (g(y) + K)$, in detail, see [4, 28].

The following set containment characterizations are well known in convex optimization. There are some similar results in convex and quasiconvex optimization, for example see, [3, 8, 9, 19, 22].

Theorem 2.1 ([3]). *Let X be a locally convex Hausdorff topological vector space, I an arbitrary set, g_i a proper lower semicontinuous convex function from X to $\overline{\mathbb{R}}$ for each $i \in I$, A a closed convex subset of X, $\{x \in A \mid \forall i \in I, g_i(x) \leq 0\} \neq \emptyset$, $x^* \in X^*$, and $\alpha \in \mathbb{R}$. Then, the following statements are equivalent:*

(i) $\{x \in A \mid \forall i \in I, g_i(x) \leq 0\} \subset \{x \in X \mid \langle x^*, x \rangle \leq \alpha\}$,

(ii) $(x^*, \alpha) \in \mathrm{cl}\,\mathrm{cone}\,\mathrm{conv}\left(\bigcup_{i \in I} \mathrm{epi} g_i^* + \mathrm{epi}\delta_A^*\right)$.

Theorem 2.2 ([8]). *Let X and Y be locally convex Hausdorff topological vector spaces, $K \subset Y$ a nonempty, closed and convex cone, A a closed convex subset of X, Y partially ordered by K, and g a continuous and K-convex function. Assume that $S = \{x \in A \mid g(x) \in -K\}$ is nonempty. Then,*

$$\mathrm{epi}\delta_S^* = \mathrm{cl}\left(\bigcup_{\lambda \in K^+} \mathrm{epi}(\lambda \circ g)^* + \mathrm{epi}\delta_A^*\right).$$

3. Quasiconvex vector optimization with uncertain constraints

In this section, let X, Y and Z be locally convex Hausdorff topological vector spaces, $K \subset Y$ and $C \subset Z$ nonempty, solid, pointed, closed and convex cones, A a closed convex subset of X, Y partially ordered by K, Z partially ordered by C, f a continuous C-quasiconvex function from X to Z, $\mathcal{V}$ a set, g a function from $X \times \mathcal{V}$ to Y such that $g(\cdot, v)$ is continuous and K-convex for each $v \in \mathcal{V}$, $F_{(v,\lambda)} = \{x \in A \mid \lambda \circ g(x, v) \leq 0\}$ for each $(v, \lambda) \in \mathcal{V} \times K^+$, and $F = \{x \in A \mid \forall v \in \mathcal{V}, g(x, v) \in -K\} \neq \emptyset$.

We investigate the following vector optimization problem with data uncertainty (UP):

$$\begin{array}{ll} \text{Minimize} & f(x), \\ \text{subject to} & x \in A,\ g(x, v) \in -K. \end{array}$$

In this problem, $v \in \mathcal{V}$ indicates data uncertainty. Because of the complexity of real-world optimization problems, measurement errors, and the other uncertainty, it is difficult to determine constraint (or objective) functions clearly. In (UP), we cannot determine v clearly, however, we know that v is an element of the uncertainty set $\mathcal{V}$. In order to solve such a problem robustly, robust optimization have been investigated. In robust optimization, we consider the following robust counterpart (RC):

$$\begin{array}{ll} \text{Minimize} & f(x), \\ \text{subject to} & x \in A,\ \forall v \in \mathcal{V},\ g(x, v) \in -K. \end{array}$$

In (RC), the constraint set is the intersection of the constraint sets of (UP). Hence, a feasible solution of (RC) is also a feasible solution of (UP) for each v. It is clear that $val(RC) \geq val(UP)$, where $val(RC)$ ($val(UP)$) is the minimum value of (RC) ((UP), respectively). Because of these properties, robust optimization is called 'worst-case approach'. Since (RC) is a semi-infinite programming problem, we investigate (RC) by the similar way in previous results via semi-infinite programming. We define a solution of (RC) as follows.

Definition 3.1. An element $x \in F$ is said to be a weakly optimal solution of (RC) if $f(F) \cap (f(x) - \mathrm{int}C)$ is empty.

Let $e \in \mathrm{int}C$ and $z_0 \in Z$. We denote by φ_{e,z_0} the scalarizing function from Z to $\overline{\mathbb{R}}$, that is,

$$\varphi_{e,z_0}(z) = \inf\{t \in \mathbb{R} \mid z \in z_0 + te - C\}.$$

It is well known that the following statements hold, in detail, see [6, 15].

Theorem 3.2 ([6, 15])**.** *Let $e \in \mathrm{int}C$ and $z_0 \in Z$. Then, the following statements hold:*

(i) *φ_{e,z_0} is a real-valued continuous function,*
(ii) *$L(\varphi_{e,z_0}, \leq, t) = z_0 + te - C$,*
(iii) *$L(\varphi_{e,z_0}, <, t) = z_0 + te - \mathrm{int}C$,*
(iv) *$L(\varphi_{e,z_0}, =, t) = z_0 + te - \mathrm{bd}C$,*
(v) *if f is C-quasiconvex, then $\varphi_{e,z_0} \circ f$ is quasiconvex,*
(vi) *if f is proper quasi C-concave, then $\varphi_{e,z_0} \circ f$ is quasiconcave,*
(vii) *x_0 is a weakly optimal solution of (RC) if and only if for each $e \in \mathrm{int}C$, there exists $z_0 \in Z$ such that $\varphi_{e,z_0} \circ f(x_0) = \min_{x \in F} \varphi_{e,z_0} \circ f(x)$.*

We need the following proposition.

Proposition 3.3. *Let $e \in \mathrm{int}C$ and $z_0 \in Z$ Then, for each $\mu \in \mathbb{R}$,*

$$\varphi_{e,z_0}(\mu e) = \mu + \varphi_{e,z_0}(0).$$

Proof. By the definition of φ_{e,z_0}, for each $\varepsilon > 0$, there exists $t_\varepsilon < \varphi_{e,z_0}(0) + \varepsilon$ such that $0 \in z_0 + t_\varepsilon e - C$. Hence,

$$\mu e \in \mu e + z_0 + t_\varepsilon e - C = z_0 + (\mu + t_\varepsilon)e - C.$$

This shows that

$$\varphi_{e,z_0}(\mu e) \leq \mu + t_\varepsilon < \mu + \varphi_{e,z_0}(0) + \varepsilon.$$

Therefore, $\varphi_{e,z_0}(\mu e) \leq \mu + \varphi_{e,z_0}(0)$.

Assume that $\varphi_{e,z_0}(\mu e) < \mu + \varphi_{e,z_0}(0)$. Then, there exists $t \in \mathbb{R}$ such that $\varphi_{e,z_0}(\mu e) < t < \mu + \varphi_{e,z_0}(0)$ and $\mu e \in z_0 + te - C$. Hence, $0 \in z_0 + (t - \mu)e - C$. This shows that $\varphi_{e,z_0}(0) \leq t - \mu$. This is a contradiction. $\square$

We show the following characterizations for robust vector optimization.

Theorem 3.4. *The following conditions hold:*

(i) $\mathrm{epi}\delta^*_{F_{(v,\lambda)}} = \mathrm{cl}\,\{\mathrm{cone}\,\mathrm{epi}\,(\lambda \circ g(\cdot, v))^* + \mathrm{epi}\delta^*_A\}$,
(ii) $\mathrm{epi}\delta^*_F = \mathrm{cl}\,\mathrm{conv} \bigcup_{v \in \mathcal{V}, \lambda \in K^+} \mathrm{cl}\,\{\mathrm{cone}\,\mathrm{epi}\,(\lambda \circ g(\cdot, v))^* + \mathrm{epi}\delta^*_A\}$.

Proof. By Theorem 2.2,

$$\begin{aligned}\mathrm{epi}\delta^*_{F_{(v,\lambda)}} &= \mathrm{cl}\left\{\bigcup_{t\geq 0}\mathrm{epi}\,(t\lambda\circ g(\cdot,v))^* + \mathrm{epi}\delta^*_A\right\}\\ &= \mathrm{cl}\left\{\mathrm{cone}\,\mathrm{epi}\,(\lambda\circ g(\cdot,v))^* + \mathrm{epi}\delta^*_A\right\}.\end{aligned}$$

This shows that the condition (i) holds.

(ii) Since K is a closed convex cone,

$$\begin{aligned}F &= \{x\in A \mid \forall v\in\mathcal{V}, \forall\lambda\in K^+, \lambda\circ g(x,v)\leq 0\}\\ &= \bigcap_{v\in\mathcal{V},\lambda\in K^+} F_{(v,\lambda)}\\ &= \bigcap_{v\in\mathcal{V},\lambda\in K^+} L(\delta_{F_{(v,\lambda)}},\leq,0).\end{aligned}$$

Since $\mathrm{epi}\delta^*_{F_{(v,\lambda)}}$ is a convex cone, by Theorem 2.1 and the condition (i) of this theorem,

$$\begin{aligned}\mathrm{epi}\delta^*_F &= \mathrm{cl}\,\mathrm{cone}\,\mathrm{conv}\left(\bigcup_{v\in\mathcal{V},\lambda\in K^+}\mathrm{epi}\delta^*_{F_{(v,\lambda)}} + \mathrm{epi}\delta^*_X\right)\\ &= \mathrm{cl}\,\mathrm{cone}\,\mathrm{conv}\bigcup_{v\in\mathcal{V},\lambda\in K^+}\mathrm{epi}\delta^*_{F_{(v,\lambda)}}\\ &= \mathrm{cl}\,\mathrm{conv}\bigcup_{v\in\mathcal{V},\lambda\in K^+}\mathrm{epi}\delta^*_{F_{(v,\lambda)}}\\ &= \mathrm{cl}\,\mathrm{conv}\bigcup_{v\in\mathcal{V},\lambda\in K^+}\mathrm{cl}\left\{\mathrm{cone}\,\mathrm{epi}\,(\lambda\circ g(\cdot,v))^* + \mathrm{epi}\delta^*_A\right\}.\end{aligned}$$

□

Remark 3.5. It is clear that

$$\begin{aligned}&\mathrm{cl}\,\mathrm{conv}\bigcup_{v\in\mathcal{V},\lambda\in K^+}\mathrm{cl}\left\{\mathrm{cone}\,\mathrm{epi}\,(\lambda\circ g(\cdot,v))^* + \mathrm{epi}\delta^*_A\right\}\\ =\ &\mathrm{cl}\,\mathrm{conv}\bigcup_{v\in\mathcal{V},\lambda\in K^+}\left\{\mathrm{cone}\,\mathrm{epi}\,(\lambda\circ g(\cdot,v))^* + \mathrm{epi}\delta^*_A\right\}.\end{aligned}$$

However, the operator 'cl' in the union is necessary and important to prove surrogate duality and its constraint qualification.

Remark 3.6. By Theorem 3.4, we show that the following set containment characterization. Let $x^*\in X^*$, and $\alpha\in\mathbb{R}$. Then, the following statements are equivalent:

(i) $F\subset\{x\in X\mid\langle x^*,x\rangle\leq\alpha\}$,

(ii) $(x^*,\alpha)\in\mathrm{cl}\,\mathrm{conv}\bigcup_{v\in\mathcal{V},\lambda\in K^+}\mathrm{cl}\{\mathrm{cone}\,\mathrm{epi}\,(\lambda\circ g(\cdot,v))^* + \mathrm{epi}\delta^*_A\}$.

By Theorem 3.4, the following robust characteristic cone

$$\bigcup_{v\in\mathcal{V},\lambda\in K^+} \mathrm{cl}\left\{\mathrm{cone}\,\mathrm{epi}\,(\lambda\circ g(\cdot,v))^* + \mathrm{epi}\delta_A^*\right\}$$

is closed and convex if and only if

$$\mathrm{epi}\delta_F^* \subset \bigcup_{v\in\mathcal{V},\lambda\in K^+} \mathrm{cl}\left\{\mathrm{cone}\,\mathrm{epi}\,(\lambda\circ g(\cdot,v))^* + \mathrm{epi}\delta_A^*\right\}.$$

In the following theorem, we investigate surrogate min-max duality for quasiconvex vector optimization with uncertain constraints.

Theorem 3.7. *The following statements are equivalent:*

(i) *the following robust characteristic cone is closed and convex:*

$$\bigcup_{v\in\mathcal{V},\lambda\in K^+} \mathrm{cl}\left\{\mathrm{cone}\,\mathrm{epi}\,(\lambda\circ g(\cdot,v))^* + \mathrm{epi}\delta_A^*\right\},$$

(ii) *for each $x_0 \in F$ and continuous C-quasiconvex function f, x_0 is a weakly optimal solution of (RC) if and only if for each $e \in \mathrm{int}C$, there exists $z_0 \in Z$ such that*

$$\varphi_{e,z_0}\circ f(x_0) = \max_{v\in\mathcal{V},\lambda\in K^+} \inf_{x\in A}\{\varphi_{e,z_0}\circ f(x) \mid \lambda\circ g(x,v)\le 0\}.$$

Proof. Assume that (i) holds. Let $x_0 \in F$ and f a continuous C-quasiconvex function. By Theorem 3.2 (vii), x_0 is a weakly optimal solution of (RC) if and only if for each $e \in \mathrm{int}C$, there exists $z_0 \in Z$ such that $\varphi_{e,z_0}\circ f(x_0) = \min_{x\in F}\varphi_{e,z_0}\circ f(x)$. We show that

$$\inf_{x\in F}\varphi_{e,z_0}\circ f(x) = \max_{v\in\mathcal{V},\lambda\in K^+} \inf_{x\in A}\{\varphi_{e,z_0}\circ f(x) \mid \lambda\circ g(x,v)\le 0\}. \tag{3.1}$$

It is clear that the weak duality holds, that is,

$$\inf_{x\in F}\varphi_{e,z_0}\circ f(x) \ge \sup_{v\in\mathcal{V},\lambda\in K^+} \inf_{x\in A}\{\varphi_{e,z_0}\circ f(x) \mid \lambda\circ g(x,v)\le 0\}.$$

Let $m = \inf_{x\in F}\varphi_{e,z_0}\circ f(x)$. If $L(\varphi_{e,z_0}\circ f, <, m) = \emptyset$, then for each $v \in \mathcal{V}$ and $\lambda = 0$, the equation (3.1) holds. If $L(\varphi_{e,z_0}\circ f, <, m) \ne \emptyset$, there exists $(x^*, \alpha) \in X^* \times \mathbb{R}$ such that for all $x \in F$ and $y \in L(\varphi_{e,z_0}\circ f, <, m)$,

$$\langle x^*, x\rangle \le \alpha < \langle x^*, y\rangle,$$

since $L(\varphi_{e,z_0}\circ f, <, m) \cap F = \emptyset$ and $L(\varphi_{e,z_0}\circ f, <, m)$ is a nonempty, open and convex set. Because of the condition (i) and Theorem 3.4,

$$\begin{aligned}(x^*,\alpha) &\in \mathrm{epi}\,\delta_F^* \\ &\subset \bigcup_{v\in\mathcal{V},\lambda\in K^+} \mathrm{cl}\left\{\mathrm{cone}\,\mathrm{epi}\,(\lambda\circ g(\cdot,v))^* + \mathrm{epi}\delta_A^*\right\} \\ &= \bigcup_{v\in\mathcal{V},\lambda\in K^+} \mathrm{epi}\,\delta_{F_{(v,\lambda)}}^*.\end{aligned}$$

Therefore, there exists $(\bar{v}, \bar{\lambda}) \in \mathcal{V} \times K^+$ such that $(x^*, \alpha) \in \operatorname{epi} \delta^*_{F_{(\bar{v},\bar{\lambda})}}$. For each $x \in F_{(\bar{v},\bar{\lambda})}$,

$$\langle x^*, x\rangle \leq \alpha \Longrightarrow x \notin L(\varphi_{e,z_0} \circ f, <, m) \iff \varphi_{e,z_0} \circ f(x) \geq m,$$

by the above separation inequality. This shows that $\inf_{x\in A}\{\varphi_{e,z_0} \circ f(x) \mid \bar{\lambda} \circ g(x, \bar{v}) \leq 0\} \geq m$. Hence, the equation (3.1) holds. This shows that (i) implies (ii).

Next, we prove that (ii) implies (i). We only show that

$$\operatorname{epi}\delta^*_F \subset \bigcup_{v\in\mathcal{V},\lambda\in K^+} \operatorname{cl}\left\{\operatorname{cone}\operatorname{epi}(\lambda \circ g(\cdot, v))^* + \operatorname{epi}\delta^*_A\right\}.$$

Let $(x^*, \alpha) \in \operatorname{epi} \delta^*_F$. We define a function f from X to Z as follows:

$$f(x) = \begin{cases} -\langle x^*, x\rangle\, e, & \langle x^*, x\rangle > \delta^*_F(x^*), \\ -\delta^*_F(x^*)e, & \langle x^*, x\rangle \leq \delta^*_F(x^*), \end{cases}$$

where $e \in \operatorname{int}C$. Then, f is continuous C-quasiconvex and for each $x \in F$, x is a weakly optimal solution of (RC). Let $x_0 \in F$. By the statement (ii), there exist $z_0 \in Z$, $v \in \mathcal{V}$, and $\lambda \in K^+$ such that

$$\varphi_{e,z_0} \circ f(x_0) = \inf_{x\in A}\{\varphi_{e,z_0} \circ f(x) \mid \lambda \circ g(x, v) \leq 0\}.$$

By Proposition 3.3,

$$\varphi_{e,z_0} \circ f(x) = \begin{cases} -\langle x^*, x\rangle + \varphi_{e,z_0}(0), & \langle x^*, x\rangle > \delta^*_F(x^*), \\ -\delta^*_F(x^*) + \varphi_{e,z_0}(0), & \langle x^*, x\rangle \leq \delta^*_F(x^*). \end{cases}$$

Hence, for each $x \in F_{(v,\lambda)}$,

$$\begin{aligned} \varphi_{e,z_0} \circ f(x) \geq \varphi_{e,z_0} \circ f(x_0) \quad &\iff \quad \varphi_{e,z_0} \circ f(x) \geq -\delta^*_F(x^*) + \varphi_{e,z_0}(0) \\ &\iff \quad \langle x^*, x\rangle \leq \delta^*_F(x^*). \end{aligned}$$

This implies that $\delta^*_{F_{(v,\lambda)}}(x^*) \leq \delta^*_F(x^*) \leq \alpha$. By Theorem 3.4.

$$(x^*, \alpha) \in \operatorname{epi} \delta^*_{F_{(v,\lambda)}} = \operatorname{cl}\left\{\operatorname{cone}\operatorname{epi}(\lambda \circ g(\cdot, v))^* + \operatorname{epi}\delta^*_A\right\}.$$

This shows that (i) holds. □

4. Quasiconvex vector optimization with uncertain objective and constraints

In this section, let X and Y be locally convex Hausdorff topological vector spaces, $K \subset Y$ a nonempty, solid, pointed, closed and convex cone, A a closed convex subset of X, Y partially ordered by K, $\mathcal{U}$ a compact set, f a continuous function from $X \times \mathcal{U}$ to $\mathbb{R}^n$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n \mid \forall i \in \{1, \dots, n\}, x_i > 0\}$, $\mathcal{V}$ a set, g a function from $X \times \mathcal{V}$ to Y such that $g(\cdot, v)$ is continuous K-convex for each $v \in \mathcal{V}$, $F_{(v,\lambda)} = \{x \in A \mid \lambda \circ g(x, v) \leq 0\}$ for each $(v, \lambda) \in \mathcal{V} \times K^+$, and $F = \{x \in A \mid \forall v \in \mathcal{V}, g(x, v) \in -K\} \neq \emptyset$.

We study vector optimization with uncertain objective and constraints. Consider the following uncertain problem:

$$\begin{aligned} &\text{Minimize} && f(x, u), \\ &\text{subject to} && x \in A,\ g(x, v) \in -K. \end{aligned}$$

In this paper, we introduce the following robust counterpart (RC):

$$\begin{array}{ll} \text{Minimize} & f(x,\mathcal{U}), \\ \text{subject to} & x \in A, \forall v \in \mathcal{V}, g(x,v) \in -K, \end{array}$$

where $f(x,\mathcal{U}) = \{f(x,u) \mid u \in \mathcal{U}\}$. Hence, we regard (RC) as a set-valued optimization problem with vector-valued constraints. For each $i \in \{1,\ldots,n\}$, there exists $f_i : X \times \mathcal{U} \to \mathbb{R}$, $i \in \{1,\ldots,n\}$ such that

$$f(x,u) = (f_1(x,u),\ldots,f_n(x,u)).$$

Since f is continuous and $\mathcal{U}$ is compact, $f_i(x,\cdot)$ attains its maximum on $\mathcal{U}$ for each i and x. Let $\hat{f}$ be a function from X to $\mathbb{R}^n$ as follows:

$$\hat{f}(x) = (\max_{u\in\mathcal{U}} f_1(x,u),\ldots,\max_{u\in\mathcal{U}} f_n(x,u)).$$

In [14], Kuroiwa and Lee introduce the function, and study a similar robust counterpart. Now we define a solution of (RC).

Definition 4.1. An element $x_0 \in F$ is said to be a solution of (RC) if $f(x,\mathcal{U}) \not\subset (\hat{f}(x_0) - \mathrm{int}C)$ for each $x \in F$.

We compare a solution of (RC) with a solution of set optimization in Section 5.

Theorem 4.2. *Let $x_0 \in F$, $e \in \mathbb{R}^n_{++}$ and $z_0 \in \mathbb{R}^n$. Then,*

$$\varphi_{e,z_0}(\hat{f}(x_0)) = \sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x_0,u).$$

Proof. It is clear that $f(x_0,\mathcal{U}) \subset \hat{f}(x_0) - \mathbb{R}^n_+$. Hence,

$$\varphi_{e,z_0}(\hat{f}(x_0)) \geq \sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x_0,u).$$

By Theorem 3.2, there exists $\bar{z} \in \mathrm{bd}\mathbb{R}^n_+$ such that $\hat{f}(x_0) = z_0 + \varphi_{e,z_0}(\hat{f}(x_0))e - \bar{z}$. Since $\bar{z} \in \mathrm{bd}\mathbb{R}^n_+$, there exists $i \in \{1,\ldots,n\}$ such that $\bar{z}_i = 0$. Let $\bar{y} = \hat{f}(x_0) - f(x_0,u_i)$ where $u_i \in \mathcal{U}$ such that $f_i(x,u_i) = \max_{u\in\mathcal{U}} f_i(x,u)$. Then, $\bar{y} \in \mathbb{R}^n_+$ and $\bar{y}_i = 0$. This implies that $\bar{z} + \bar{y} \in \mathrm{bd}\mathbb{R}^n_+$. Hence,

$$\begin{array}{rcl} f(x_0,u_i) & = & \hat{f}(x_0) - \bar{y} \\ & = & z_0 + \varphi_{e,z_0}(\hat{f}(x_0))e - \bar{z} - \bar{y} \\ & \in & z_0 + \varphi_{e,z_0}(\hat{f}(x_0))e - \mathrm{bd}\mathbb{R}^n_+. \end{array}$$

This shows that $\varphi_{e,z_0}(f(x,u_i)) = \varphi_{e,z_0}(\hat{f}(x_0))$. This completes the proof. □

In the following theorem, we show a characterization of the solution of (RC).

Theorem 4.3. *Let $x_0 \in F$. The following statements are equivalent:*

(i) *x_0 is a solution of (RC),*
(ii) *for each $e \in \mathbb{R}^n_{++}$, there exists $z_0 \in \mathbb{R}^n$ such that*

$$\sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x_0,u) = \min_{x\in F} \sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x,u).$$

Proof. We show that (i) implies (ii). Let $e \in \mathbb{R}^n_{++}$ and $z_0 = \hat{f}(x_0)$. Then, by Theorem 4.2, $\sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x_0, u) = \varphi_{e,z_0}(\hat{f}(x_0)) = 0$. By the statement (i), for each $x \in F$, there exists $u \in \mathcal{U}$ such that $f(x, u) \notin \hat{f}(x_0) - \mathbb{R}^n_{++}$. Hence $\varphi_{e,z_0} \circ f(x, u) \geq \varphi_{e,z_0}(\hat{f}(x_0))$. This shows that (ii) holds.

Conversely, assume that (ii) holds and there exists $\bar{x} \in F$ such that $f(\bar{x}, \mathcal{U}) \subset \hat{f}(x_0) - \mathbb{R}^n_{++}$. Let $e \in \mathbb{R}^n_{++}$. By the assumption and Theorem 4.2, there exists $z_0 \in \mathbb{R}^n$ such that $\varphi_{e,z_0}(\hat{f}(x_0)) = \sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x_0, u) = \min_{x\in F} \sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x, u)$. By monotonicity of the scalarizing function, $\varphi_{e,z_0}(\hat{f}(x_0)) > \varphi_{e,z_0} \circ f(\bar{x}, u)$ for each $u \in \mathcal{U}$. Since $\mathcal{U}$ is compact and f is continuous, $\varphi_{e,z_0}(\hat{f}(x_0)) > \max_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(\bar{x}, u)$. This is a contradiction. □

Let $\mathcal{F}$ be the following set of functions;

$$\mathcal{F} = \left\{ f : X \times \mathcal{U} \to \mathbb{R}^n \;\middle|\; \begin{array}{l} f : \text{continuous}, \\ \mathcal{U} : \text{compact convex}, \\ f(\cdot, u) : \mathbb{R}^n_+\text{-quasiconvex } \forall u \in \mathcal{U}, \\ f(x, \cdot) : \text{proper quasi } \mathbb{R}^n_+\text{-concave } \forall x \in X. \end{array} \right\}.$$

In the following theorem, we investigate surrogate min-max duality for quasiconvex vector optimization with uncertain objective and constraints.

Theorem 4.4. *The following statements are equivalent:*

(i) *the following robust characteristic cone is closed and convex:*

$$\bigcup_{v\in\mathcal{V}, \lambda\in K^+} \mathrm{cl}\left\{\mathrm{cone}\,\mathrm{epi}\,(\lambda \circ g(\cdot, v))^* + \mathrm{epi}\delta_A^*\right\},$$

(ii) *for each $x_0 \in F$ and $f \in \mathcal{F}$, x_0 is a solution of (RC) if and only if for each $e \in \mathbb{R}^n_{++}$, there exists $z_0 \in \mathbb{R}^n$ such that*

$$\sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x_0, u) = \max_{v\in\mathcal{V}, \lambda\in K^+, u\in\mathcal{U}} \inf_{x\in A} \{\varphi_{e,z_0} \circ f(x, u) \mid \lambda \circ g(x, v) \leq 0\}.$$

Proof. By Theorem 3.2, $\varphi_{e,z_0} \circ f(\cdot, u)$ is continuous quasiconvex for each $u \in \mathcal{U}$. Since $\mathcal{U}$ is compact, and f is continuous, $\sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(\cdot, u)$ is continuous quasiconvex. Hence, we can prove the following equation by the similar way of the proof of equation (3.1),

$$\inf_{x\in F} \sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x, u) = \max_{v\in\mathcal{V}, \lambda\in K^+} \inf_{x\in A} \{\sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x, u) \mid \lambda \circ g(x, v) \leq 0\}.$$

Since f is continuous and $f(x, \cdot)$ is proper quasi $\mathbb{R}^n_+$-concave, by Theorem 3.2, $\varphi_{e,z_0} \circ f(x, \cdot)$ is continuous quasiconcave function. By Sion's min-max theorem,

$$\begin{aligned} & \inf_{x\in A} \{\sup_{u\in\mathcal{U}} \varphi_{e,z_0} \circ f(x, u) \mid \lambda \circ g(x, v) \leq 0\} \\ = \; & \max_{u\in\mathcal{U}} \inf_{x\in A} \{\varphi_{e,z_0} \circ f(x, u) \mid \lambda \circ g(x, v) \leq 0\}. \end{aligned}$$

This shows that (i) implies (ii). The proof of converse implication is similar to the proof of Theorem 3.7. □

Remark 4.5. In the definition of $\mathcal{F}$, we assume that $\mathcal{U}$ is compact convex. Exactly speaking, we assume that $\mathcal{U}$ is a compact convex subset of a topological vector space. It is not necessary that $\mathcal{U}$ is a subset of given topological vector spaces, X, Y, or $\mathbb{R}^n$.

5. Discussion

In this section, we discuss about our results in this paper. Especially, we investigate a necessary and sufficient constraint qualification for surrogate duality via robust optimization. Also, we compare a solution of (RC) in Section 4 with a solution of set optimization.

Recently, many types of necessary and sufficient constraint qualifications for duality theorems have been investigated, see [8, 10, 18–26]. In Theorem 3.7 and Theorem 4.4, we show surrogate min-max duality theorems for robust quasiconvex vector optimization with its necessary and sufficient constraint qualification. In the following theorem, we investigate surrogate duality theorem for real-valued optimization with uncertain objective and constraints. The proof is similar to Theorem 3.7 and Theorem 4.4, and will be omitted.

Theorem 5.1. *The following statements are equivalent:*

(i) *the following characteristic cone is closed and convex:*

$$\bigcup_{v \in \mathcal{V}, \lambda \in K^+} \mathrm{cl}\left\{\mathrm{cone}\,\mathrm{epi}\,(\lambda \circ g(\cdot, v))^* + \mathrm{epi}\delta_A^*\right\},$$

(ii) *for each upper semicontinuous quasiconvex function f from X to $\overline{\mathbb{R}}$,*

$$\inf_{x \in F} f(x) = \max_{v \in \mathcal{V}, \lambda \in K^+} \inf_{x \in A} \{f(x) \mid \lambda \circ g(x, v) \leq 0\},$$

(iii) *for each continuous function f from $X \times \mathcal{U}$ to $\overline{\mathbb{R}}$ with $\mathcal{U}$ is compact convex, $f(\cdot, u)$ is quasiconvex for each $u \in \mathcal{U}$, and $f(x, \cdot)$ is quasiconcave for each $x \in X$,*

$$\inf_{x \in F} \sup_{u \in \mathcal{U}} f(x, u) = \max_{v \in \mathcal{V}, \lambda \in K^+, u \in \mathcal{U}} \inf_{x \in A} \{f(x, u) \mid \lambda \circ g(x, v) \leq 0\},$$

(iv) *for each $x_0 \in F$ and continuous C-quasiconvex function f, x_0 is a weakly optimal solution of (RC) if and only if for each $e \in \mathrm{int}C$, there exists $z_0 \in Z$ such that*

$$\varphi_{e,z_0} \circ f(x_0) = \max_{v \in \mathcal{V}, \lambda \in K^+} \inf_{x \in A} \{\varphi_{e,z_0} \circ f(x) \mid \lambda \circ g(x, v) \leq 0\},$$

(v) *for each $x_0 \in F$ and $f \in \mathcal{F}$, x_0 is a solution of (RC) if and only if for each $e \in \mathbb{R}^n_{++}$, there exists $z_0 \in \mathbb{R}^n$ such that*

$$\sup_{u \in \mathcal{U}} \varphi_{e,z_0} \circ f(x_0, u) = \max_{v \in \mathcal{V}, \lambda \in K^+, u \in \mathcal{U}} \inf_{x \in A} \{\varphi_{e,z_0} \circ f(x, u) \mid \lambda \circ g(x, v) \leq 0\}.$$

In the present paper, we introduce a robust counterpart as a set-valued optimization problem. Unfortunately, it is not so evident which is the suitable robust counterpart of uncertain problem whenever the uncertainty aspects the vector-valued objective function. Many researchers introduce various types of robust approach for uncertain vector optimization problem. For example, in [5], Georgiev, Luc and

Pardalos define the efficient solutions of robust counterpart as those robust feasible solutions which are efficient for any possible value of $u \in \mathcal{U}$. In [7], Goberna, Jeyakumar, Li, and López investigate uncertain problems by transferring the uncertainty from the objective function to the constraints. This approach is a very elegant and important element for convex optimization problems. However, it is not always valid for quasiconvex optimization problems, see the following example.

Example 5.2. We consider the following counterpart:

$$\begin{array}{rl} \text{minimize} & t, \\ \text{subject to} & x \in F,\ f(x,u) - t \in -C,\ \forall u \in \mathcal{U}. \end{array}$$

If $f(\cdot,u)$ is C-convex for each $u \in \mathcal{U}$, then this counterpart is convex optimization problem. However, even if $f(\cdot,u)$ is C-quasiconvex, this counterpart is not always quasiconvex optimization problem.

Let f be the following real-valued quasiconvex function on $\mathbb{R}$:

$$f(x) := \begin{cases} 0, & x \geq 0, \\ 2x, & otherwise. \end{cases}$$

Let $\bar{f}(x,t) := f(x) - t$, $z_1 = (-1,1)$ and $z_2 = (1,3)$, then

$$\bar{f}\left(\frac{1}{2}z_1 + \frac{1}{2}z_2\right) = -2 > -3 = \max\{\bar{f}(z_1), \bar{f}(z_2)\}$$

This shows that $\bar{f}$ is not a quasiconvex function on $\mathbb{R}^2$.

In [14], Kuroiwa and Lee introduce a robust counterpart by a similar way in Section 4 of this paper. We consider this robust counterpart is one of the natural pessimistic approach to uncertain vector optimization problems.

In set-valued optimization, there are some definitions of solutions. Especially, Kuroiwa investigate set optimization approach, see [12, 13]. Next, we compare a solution of (RC) in Section 4 with a solution of set optimization. In set optimization, the following relation have been investigated. Let A and B be subsets of Z. The notation $A \leq^u B$ will mean $A \subset B - C$. Also, $A <^u B$ will mean $A \subset B - \mathrm{int}C$. An element $x_0 \in F$ is said to be a u-type weak minimal solution if there does not exist $x \in F$ such that $f(x,\mathcal{U}) <^u f(x_0,\mathcal{U})$. We can check easily that if $x_0 \in F$ is a solution of (RC), then x_0 is a u-type weak minimal solution. However, the converse is not generally true, see the following example.

Example 5.3. Let x_0, $x \in F$, $A = f(x_0,\mathcal{U}) = \{a \in \mathbb{R}^2 \mid \|a\| \leq 1\}$, and $B = f(x,\mathcal{U}) = \{a \in \mathbb{R}^2 \mid \|a - (\frac{1}{2}, \frac{1}{2})\| \leq \frac{3}{8}\}$. Then, $f(x,\mathcal{U}) \not\subset f(x_0,\mathcal{U}) - \mathbb{R}^2_{++}$, that is $f(x,\mathcal{U}) \not<^u f(x_0,\mathcal{U})$. If $f(y,\mathcal{U}) \not<^u f(x_0,\mathcal{U})$ for each $y \in F$, then x_0 is a u-type weak minimal solution. However, $\hat{f}(x_0) = (\max_{(x_1,x_2)\in A} x_1, \max_{(x_1,x_2)\in A} x_2) = (1,1)$ and $f(x,\mathcal{U}) \subset \hat{f}(x_0) - \mathbb{R}^2_{++}$. Hence, x_0 is not a solution of (RC). This indicates the existence of a u-type weak minimal solution which is not a solution of (RC).

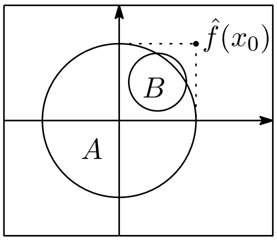

Fig. 1

In Theorem 4.3, we investigate a characterization of a solution of (RC) by the scalarizing function. We can check easily that if for each $e \in \mathbb{R}^n_{++}$, there exists $z_0 \in \mathbb{R}^n$ such that

$$\sup_{u \in \mathcal{U}} \varphi_{e,z_0} \circ f(x_0, u) = \min_{x \in F} \sup_{u \in \mathcal{U}} \varphi_{e,z_0} \circ f(x, u),$$

then $x_0 \in F$ is a u-type weak minimal solution. However, the converse is not generally true by Theorem 4.2 and Example 5.3. This means that a u-type weak minimal solution is not characterized by this type of scalarization. By using the notion of the solution of (RC) in this paper and the scalarizing function, we show surrogate min-max duality via robust quasiconvex vector optimization.

6. Conclusion

In this paper, we investigate surrogate duality for robust quasiconvex vector optimization. In Section 3 and Section 4, we investigate robust vector optimization by using scalarization. We investigate surrogate min-max duality for quasiconvex vector optimization with uncertain constraints and introduce a necessary and sufficient constraint qualification for surrogate duality. Also, we investigate vector optimization with uncertain objective and constraints. Because of the uncertainty of the objective function, the robust counterpart is a set-valued optimization problem. We introduce a notion of solutions of the robust counterpart, and we investigate surrogate min-max duality with its constraint qualification. In Section 5, we discuss about our results. We investigate surrogate duality for real-valued robust optimization with its constraint qualification. Also, we compare a solution of (RC) in Section 4 with a u-type weak minimal solution in set optimization.

References

[1] A. Beck and A. Ben-Tal, *Duality in robust optimization: Primal worst equals dual best*, Oper. Res. Lett. **37** (2009), 1–6.
[2] A. Ben-Tal, L. El Ghaoui and A. Nemirovski, *Robust Optimization*, Princeton Series in Applied Mathematics, Princeton, 2009.
[3] D. Dinh, A. M. Goberna and M. A. López, *From linear to convex systems: consistency, Farkas' lemma and applications*, J. Convex Anal. **13** (2006), 113–133.
[4] F. Ferro, *Minimax type theorems for n-valued functions*, Ann. Mat. Pura Appl. **132** (1982), 113–130.
[5] P. G. Georgiev, D. T. Luc and P. M. Pardalos, *Robust aspects of solutions in deterministic multiple objective linear programming*, European J. Oper. Res. **229** (2013), 29–36.
[6] C. Gerth and P. Weidner, *Nonconvex separation theorems and some applications in vector optimization*, J. Optim. Theory Appl. **67** (1990), 297–320.

[7] M. A. Goberna, V. Jeyakumar, G. Y. Li and M. A. López, *Robust linear semi-infinite programming duality under uncertainty.* Math. Program. **139** (2013), 185–203.
[8] M. A. Goberna, V. Jeyakumar and M. A. López, *Necessary and sufficient constraint qualifications for solvability of systems of infinite convex inequalities*, Nonlinear Anal. **68** (2008), 1184–1194.
[9] V. Jeyakumar, *Characterizing set containments involving infinite convex constraints and reverse-convex constraints*, SIAM J. Optim. **13** (2003), 947–959.
[10] V. Jeyakumar and G. Y. Li, *Strong duality in robust convex programming: complete characterizations*, SIAM J. Optim. **20** (2010), 3384–3407.
[11] H. Komiya, *Elementary proof for Sion's minimax theorem*, Kodai Math. J. **11** (1988), 5–7.
[12] D. Kuroiwa, *On set-valued optimization*, Nonlinear Anal. **47** (2001), 1395–1400.
[13] D. Kuroiwa, *On derivatives of set-valued maps and optimality conditions for set optimization*, J. Nonlinear Convex Anal. **10** (2009), 41–50.
[14] D. Kuroiwa and G. M. Lee, *On robust multiobjective optimization.* Vietnam J. Math. **40** (2012), 305–317.
[15] D. T. Luc, *Theory of vector optimization*, Lecture Notes in Econom. and Math. Systems, 319, Springer-Verlag, Berlin, 1989.
[16] D. G. Luenberger, *Quasi-convex programming*, SIAM J. Appl. Math. **16** (1968), 1090–1095.
[17] J. P. Penot and M. Volle, *On quasi-convex duality*, Math. Oper. Res. **15** (1990), 597–625.
[18] S. Suzuki, *Duality theorems for quasiconvex programming with a reverse quasiconvex constraint*, Taiwanese J. Math. **21** (2017), 489–503.
[19] S. Suzuki and D. Kuroiwa, *On set containment characterization and constraint qualification for quasiconvex programming*, J. Optim. Theory Appl. **149** (2011), 554–563.
[20] S. Suzuki and D. Kuroiwa, *Optimality conditions and the basic constraint qualification for quasiconvex programming*, Nonlinear Anal. **74** (2011), 1279–1285.
[21] S. Suzuki and D. Kuroiwa, *Necessary and sufficient conditions for some constraint qualifications in quasiconvex programming*, Nonlinear Anal. **75** (2012), 2851–2858.
[22] S. Suzuki and D. Kuroiwa, *Necessary and sufficient constraint qualification for surrogate duality*, J. Optim. Theory Appl. **152** (2012), 366–377.
[23] S. Suzuki and D. Kuroiwa, *Some constraint qualifications for quasiconvex vector-valued systems*, J. Global Optim. **55** (2013), 539–548.
[24] S. Suzuki and D. Kuroiwa, *A constraint qualification characterizing surrogate duality for quasiconvex programming*, Pacific. J. Optim. **12** (2016), 87–100.
[25] S. Suzuki and D. Kuroiwa, *Duality Theorems for Separable Convex Programming without Qualifications*, J. Optim. Theory Appl. **172** (2017), 669-683.
[26] S. Suzuki and D. Kuroiwa, *Generators and constraint qualifications for quasiconvex inequality systems*, J. Nonlinear Convex Anal. **18** (2017), 2101–2121.
[27] S. Suzuki, D. Kuroiwa and G. M. Lee, *Surrogate duality for robust optimization*, European J. Oper. Res. **231** (2013), 257–262.
[28] T. Tanaka, *Generalized quasiconvexities, cone saddle points, and minimax theorem for vector-valued functions*, J. Optim. Theory Appl. **81** (1994), 355–377.

Manuscript received October 15 2017
revised January 2 2018

S. Suzuki
Department of Mathematics, Shimane University, Shimane, Japan
E-mail address: `suzuki@riko.shimane-u.ac.jp`

D. Kuroiwa
Department of Mathematics, Shimane University, Shimane, Japan
E-mail address: `kuroiwa@riko.shimane-u.ac.jp`