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A CONFORMAL INVARIANT FOR DOMAINS IN THE HEISENBERG GROUP

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ABSTRACT. We construct a conformally invariant contact form θ_Ω for domains on the Heisenberg group $\mathscr{H}^n$, i.e. for a conformal diffeomorphism $f : \Omega \to \Omega'$ between bounded regular domains Ω and Ω', we have

$$f^*\theta_{\Omega'} = \theta_\Omega.$$

θ_Ω induces a quasi-hyperbolic Carnot-Carathéodory invariant metric on Ω.

1. Introduction

By the well known uniformization theorem, any simply connected domain in $\mathbb{C}$ is biholomorphic equivalence to the unit disc in $\mathbb{C}$ or $\mathbb{C}$. So when such a domain is equivalence to the unit disc, there exists an biholomorphic invariant hyperbolic metric on it. Its higher dimension generalization is to construct a conformally invariant metric on a domain in $\mathbb{R}^n$, which was given by Leutwiler [7]. But by Liouville's theorem, any conformal mapping restricted to some open set in $\mathbb{R}^n$ is an element of $\mathrm{SO}(n+1,1)$. So these metrics are invariant under $\mathrm{SO}(n+1,1)$. In this paper we will consider its CR version.

The simplest CR manifold, which plays the same role of Euclidean space in Riemannian geometry, is the *Heisenberg group* $\mathscr{H}^n = \mathbb{C}^n \oplus \mathbb{R}$. Its multiplication is given by

$$(z,t)\cdot(z',t') = (z+z', t+t'+2\mathrm{Im}(z\bar{z'})),$$

where $z, z' \in \mathbb{C}^n$ and $t, t' \in \mathbb{R}$. The neutral element is $(0,0)$ and the inverse of (z,t) is $(-z,-t)$. Let $X_1, \ldots, X_{2n}$ be the standard left invariant vector fields on $\mathscr{H}^n$. H = span $\{X_1, \ldots, X_{2n}\}$ is the horizontal space of $\mathscr{H}^n$. The standard *Carnot-Carathéodory metric* on $\mathscr{H}^n$ is given by

$$g_0(X,X) = \|X\|^2 = \sum_{j=1}^{n} a_j^2,$$

for $X = \sum_{j=1}^{n} a_j X_j \in H$. Let Ω, Ω' be domains in $\mathscr{H}^n$, $f : \Omega \to \Omega'$ is called *conformal* at point $\xi \in \Omega$ if

$$\|f_*X\| = \|f_*Y\|,$$

for any $X, Y \in H_\xi$ with $\|X\| = \|Y\|$.

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Let Δ_0 be the SubLaplacian operator defined in (2.8). Let G_Ω be the Green's function of Δ_0 on Ω, i.e. a continuous function $G_\Omega : (\bar{\Omega} \times \bar{\Omega}) \setminus \mathrm{diag}(\Omega) \to \mathbb{R}$ which satisfies $G_\Omega(x, y) = G_\Omega(y, x) = 0$ for $x \in \Omega$, $y \in \partial\Omega$, and

$$\int_\Omega G_\Omega(x,y)\Delta_0 u(y)\theta_0 \wedge (\mathrm{d}\theta_0)^n = u(x) \quad \text{for all } u \in C_0^\infty(\Omega), \tag{1.1}$$

where $\theta_0 \wedge (\mathrm{d}\theta_0)^n$ is the volume form with θ_0 the standard contact form on $\mathscr{H}^n$ defined in (2.7). Here, $\bar{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω. To promise the existence of Green's function, we assume domain Ω is bounded and regular. There exists some continuous function $H(x, \cdot)$ for each $x \in \Omega$, such that

$$G_\Omega(x,\cdot) = \Gamma(x,\cdot) - H(x,\cdot),\ x \in \Omega, \tag{1.2}$$

where $\Gamma(x, \cdot)$ is a fundamental solution of Δ_0 with pole at x. So the limit

$$\mathcal{A}_\Omega(x) := \lim_{y\to x} |H(x,y)|^{\frac{1}{Q-2}} = \lim_{y\to x} |G_\Omega(x,y) - \Gamma(x,y)|^{\frac{1}{Q-2}} \tag{1.3}$$

exists. We define

$$\theta_\Omega := \mathcal{A}_\Omega^2 \theta_0. \tag{1.4}$$

Then θ_Ω is an invariant.

Theorem 1.1. *θ_Ω is a C^∞ conformally invariant contact form, i.e. for any conformal diffeomorphism $f : \Omega \to \Omega'$ between two bounded regular domains Ω, Ω' in $\mathscr{H}^n$, we have*

$$f^*\theta_{\Omega'} = \theta_\Omega.$$

θ_Ω induces a Carnot-Carathéodory metric on Ω :

$$g_\Omega(X, Y) := \mathrm{d}\theta_\Omega(X, JY),$$

for any $X, Y \in H$. See section 4 for details. The Carnot-Carathéodory distance d_{cc} associated to a Carnot-Carathéodory metric on Ω is defined by $d_{cc}(x, y) = \inf_\gamma \int_0^1 |\gamma'(t)|dt$ for any $x, y \in \Omega$, where $\gamma : [0, 1] \to \Omega$ are Lipschitzian horizontal curves, i.e. $\gamma'(t) \in H_{\gamma(t)}$ almost everywhere. Let $d_x = d_{cc}(x, \partial\Omega)$ denote the Carnot-Carathéodory distance from $x \in \Omega$ to $\partial\Omega$.

Let Ω be a bounded subdomain of $\mathscr{H}^n$ and set

$$k_\Omega(x) = \frac{1}{d_x^{\frac{2}{Q-2}}}, \tag{1.5}$$

for $x \in \Omega$. Then

$$g_k|_x = k_\Omega^2(x) g_0|_x \tag{1.6}$$

defines a quasi-hyperbolic Carnot-Carathéodory metric which is not conformally invariant. We have the following comparison theorem.

Theorem 1.2. *Let Ω be a smooth regular domain in $\mathscr{H}^n$, we have*

$$c_1 g_k \le g_\Omega \le c_2 g_k,$$

for some constant $c_1, c_2 > 0$. Moreover,

$$\lim_{x\to\partial\Omega} \frac{g_\Omega|_x}{g_k|_x} = 1. \tag{1.7}$$

This domain version invariant was generalized to compact locally conformally flat manifolds in [4]. It also generalized to compact spherical CR manifolds in [10] and compact spherical qc manifolds in [9].

2. Some basic facts

The norm of the Heisenberg group $\mathscr{H}^n$ is defined by

$$\|(z,t)\| := (|z|^4 + |t|^2)^{\frac{1}{4}}. \tag{2.1}$$

We have the following automorphisms of $\mathscr{H}^n$:
(1) *dilations*:

$$D_\delta : (y,t) \longrightarrow (\delta z, \delta^2 t),\ \ \delta > 0; \tag{2.2}$$

(2) *left translations*:

$$\tau_{(z',t')} : (z,t) \longrightarrow (z',t') \cdot (z,t); \tag{2.3}$$

(3) *unitary transformations*:

$$U_A : (z,t) \longrightarrow (Az,t),\ \text{ for } A \in \mathrm{U}(n), \tag{2.4}$$

where

$$\mathrm{U}(n) = \{A \in \mathrm{GL}(n,\mathbb{C}) | A\bar{A}^t = I_n\};$$

(4) The *inversion*:

$$R : (z,t) \longrightarrow \left(-\frac{z}{|z|^2 - t}, \frac{-t}{|z|^4 + |t|^2} \right). \tag{2.5}$$

$\mathrm{SU}(n+1,1)$ is generated by these automorphisms.

The vector fields

$$Z_j = \frac{\partial}{\partial z_j} + i\bar{z}_j \frac{\partial}{\partial t}, \tag{2.6}$$

$j = 1, \ldots, n$, are left invariant vector fields on $\mathscr{H}^n$. The subbundle $T_{1,0}$ is $\mathrm{span}\{Z_1, \ldots, Z_n\}$. Let

$$\theta_0 = \mathrm{d}t + \sum_{j=1}^n i(z_j \mathrm{d}\bar{z}_j - \bar{z}_j \mathrm{d}z_j) \tag{2.7}$$

be the *standard contact* form on $\mathscr{H}^n$. The *SubLaplacian* on $\mathscr{H}^n$ is

$$\Delta_0 = -\frac{1}{2} \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j). \tag{2.8}$$

We also have the real left invariant vector fields:

$$X_j = \frac{1}{2}\frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t}, \quad X_{n+j} = \frac{1}{2}\frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

It is easy to verify that $\{X_1\ldots,X_{2n},T\}$ is a basis for the left invariant vector fields on $\mathscr{H}^n$ and $\mathrm{span}\{Z_j,\bar{Z}_j\}_{j=1}^n=\mathrm{span}\{X_j\}_{j=1}^{2n}$. Then we can write the SubLaplacian as

$$\Delta_0=-\sum_{j=1}^{2n}X_j^2. \tag{2.9}$$

We know the explicit form of the fundamental solution of the SubLaplacian of Heisenberg groups.

Proposition 2.1 (cf. p.180 in [6])**.** *The fundamental solution of Δ_0 on the Heisenberg group $\mathscr{H}^n$ with the pole at x is*

$$\Gamma(x,y):=\frac{C_Q}{\|x^{-1}y\|^{Q-2}},$$

for $x\neq y,\ x,y\in\mathscr{H}^n$, where $\|\cdot\|$ is the norm on $\mathscr{H}^n$ defined by (2.1) and

$$C_Q=\frac{2^{2-2n}\pi^{n+1}}{\Gamma\left(\frac{n}{2}\right)^2}. \tag{2.10}$$

Theorem 2.2 (**Liouville type theorem**) (cf. Theorem 2.5 in [10])**.** *If f is a local CR diffeomorphism form an open set $\Omega\subset\mathscr{H}^n$ to another open set $V\in\mathscr{H}^n$, then f is the restriction to Ω of an element in* $\mathrm{SU(n+1,1)}$.

A conformal mapping is either CR or anti-CR.

3. A canonical contact form on Heisenberg group domain

A domain Ω is called *regular* if for ϕ in $C(\partial\Omega)$, the Dirichlet problem $\Delta_0 u=0$ in Ω, $u=\phi$ in $\partial\Omega$ has a classical solution $u\in L^2(\Omega)\cap C(\bar{\Omega})$. $H(x,\cdot)$ defined in (1.2) is the classical solution of the Dirichlet problem:

$$\begin{cases}\Delta_0 H(x,\cdot)= & 0, \quad \text{in } \Omega\\ H(x,\cdot)= & \Gamma(x,\cdot), \quad \text{on } \partial\Omega.\end{cases} \tag{3.1}$$

Theorem 3.1 (The maximum principle) (cf. Lemma 3.1 in [3])**.** *Let $\Omega\subset\mathscr{H}^n$ be a bounded open set. For every $u\in C^2(\Omega)\cap C(\bar{\Omega})$ with $\sum_j^{2n}X_j^2u\geq 0$ (or ≤ 0) in Ω, we have*

$$\sup_{\bar{\Omega}}u=\sup_{\partial\Omega}u \quad (\text{or } \inf_{\bar{\Omega}}u=\inf_{\partial\Omega}u)$$

Then we have the following corollary.

Corollary 3.2. $\mathcal{A}_\Omega(x)>0$, *for any $x\in\Omega$.*

Proof. By the Dirichlet problem in (3.1), we have

$$\begin{cases}\sum_j^{2n}X_j^2H(x,\cdot)= & 0, \quad \text{in } \Omega\\ H(x,\cdot)= & \Gamma(x,\cdot), \quad \text{on } \partial\Omega,\end{cases}$$

By the maximum principle (Theorem 3.1), we have

$$H(x,y)\geq\min_{y\in\partial\Omega}(\Gamma(x,y))=\min_{y\in\partial\Omega}\frac{C_Q}{\|x^{-1}y\|^{Q-2}}>0.$$

Thus $\mathcal{A}_\Omega(x)>0$. The corollary is proved. □

Proposition 3.3. *Let Ω and Ω' be bounded regular domains in $\mathscr{H}^n$, and let $f : \Omega \to \Omega'$ be a conformal diffeomorphism. Then for all $u \in C^\infty(\mathscr{H})$, we have*

$$\phi^{\frac{Q+2}{Q-2}}\tilde{\Delta}_0 u = \Delta_0(\phi u), \tag{3.2}$$

if we write $f^*\theta_0 = \phi^{\frac{4}{Q-2}}\theta_0$ for smooth function ϕ on Ω, where $\tilde{\Delta}_0$ is the SubLaplacian with respect to the contact form $f^*\theta_0$.

Proof. By (3.6) in [11], we have

$$\phi^{\frac{Q+2}{Q-2}}\tilde{\Delta}_0 u = \Delta_0(\phi u) - \Delta_0(\phi)u.$$

By Liouville type theorem 2.2, f or $\overline{f} \in \mathrm{SU}(n+1,1)$, and is generated by dilations, left translations, unitary transformations and the inversion defined by (2.2)-(2.4). Recall the definition of θ_0 in (2.7), we have

$$\begin{aligned} D_\delta^*\theta_0 =& \delta^2\theta_0, \quad \text{for } \delta > 0, \\ \tau^*_{(z',t')}\theta_0 =& \theta_0, \quad \text{for } (z',t') \in \mathscr{H}^n, \\ U_A^*\theta_0 =& \theta_0, \quad \text{for } A \in \mathrm{U}(n), \end{aligned}$$

by directly calculation. So, if f is generated by dilations, left translations and unitary transformations, (3.2) follows. If we choose f to be the inversion R, we have

$$(R^*\theta_0)(z,t) = \phi^{\frac{4}{Q-2}}\theta_0(z,t) \quad \text{with } \phi = \frac{1}{\|(z,t)\|^{Q-2}},$$

for $(z,t) \neq (0,0)$. (cf. p.192 in [6]). We have

$$\begin{aligned} Z_j\frac{1}{\|(z,t)\|^{Q-2}} &= -\frac{Q-2}{4}\frac{Z_j\|(z,t)\|^4}{\|(z,t)\|^{Q+2}}, \\ \bar{Z}_j\frac{1}{\|(z,t)\|^{Q-2}} &= -\frac{Q-2}{4}\frac{\bar{Z}_j\|(z,t)\|^4}{\|(z,t)\|^{Q+2}}, \end{aligned} \tag{3.3}$$

with

$$Z_j\|(z,t)\|^4 = 2|z|^2\bar{z}_j + 2i\bar{z}_j t, \qquad \bar{Z}_j\|(z,t)\|^4 = 2|z|^2 z_j - 2iz_j t, \tag{3.4}$$

by using the expression of the vector field Z_j in (2.6). Then we get

$$\Delta_0\phi = -\frac{Q-2}{4}\left[\frac{\Delta_0\|(z,t)\|^4\|(z,t)\|^4 + \frac{Q+2}{4}\|(z,t)\|^4 Z_j\|(z,t)\|^4\bar{Z}_j\|(z,t)\|^4}{\|(z,t)\|^{Q+6}}\right], \tag{3.5}$$

where

$$\sum_{j=1}^n Z_j\|(z,t)\|^4\bar{Z}_j\|(z,t)\|^4 = 4|z|^2\|(z,t)\|^4. \tag{3.6}$$

By (3.4), we get

$$\Delta_0\|(z,t)\|^4 = -\frac{1}{2}\sum_{j=1}^n(Z_j\bar{Z}_j + \bar{Z}_jZ_j)\|(z,t)\|^4 = -(Q+2)|z|^2. \tag{3.7}$$

Then apply (3.6) and (3.7) to (3.5) to get

$$\Delta_0\phi = \Delta_0 \frac{1}{\|(z,t)\|^{Q-2}} = 0,$$

for $(z,t) \neq (0,0)$. Thus we have (3.2) holds for any conformal diffeomorphism f. □

Proposition 3.4. *Let Ω and Ω' be bounded regular domains in $\mathscr{H}^n$, and let $f : \Omega \to \Omega'$ be a conformal mapping. Let $G_\Omega : (\bar\Omega \times \bar\Omega) \setminus \operatorname{diag}(\Omega) \to \mathbb{R}$ be a Green's function of the SubLaplacian Δ_0 for the domain Ω. Then the Green's function for the domain Ω' satisfies*

$$G_{\Omega'}(f(x), f(y)) = \frac{1}{\phi(x)\phi(y)} G_\Omega(x,y), \tag{3.8}$$

for $x, y \in \Omega$, if we write the contact form $f^*\theta_0 = \phi^{\frac{4}{Q-2}}\theta_0$.

Proof. Recall that the definition of the SubLaplacian Δ_θ for a contact form θ on a CR manifold is independent of the choice of local coordinates, i.e.

$$f^*(\Delta_\theta u) = \Delta_{f^*\theta} f^* u. \tag{3.9}$$

Let $\tilde\theta_0 := f^*\theta_0 = \phi^{\frac{4}{Q-2}}\theta_0$, we have

$$\mathrm{d}\tilde\theta_0 = \mathrm{d}(\phi^{\frac{4}{Q-2}}\theta_0) = \frac{4}{Q-2}\phi^{\frac{6-Q}{Q-2}}\mathrm{d}\phi \wedge \theta_0 + \phi^{\frac{4}{Q-2}}\mathrm{d}\theta_0. \tag{3.10}$$

So we get

$$\tilde\theta_0 \wedge (\mathrm{d}\tilde\theta_0)^n = \phi^{\frac{2Q}{Q-2}}\theta_0 \wedge (\mathrm{d}\theta_0)^n. \tag{3.11}$$

Therefore, by the transformation law (3.2) and (3.9), we find that for $x' \in \Omega'$,

$$\begin{aligned}
&\int_{\Omega'} \frac{1}{\phi\left(f^{-1}(x')\right)\phi\left(f^{-1}(y')\right)} G_\Omega(f^{-1}(x'), f^{-1}(y'))\Delta_0 u(y')\theta_0 \wedge (\mathrm{d}\theta_0)^n \\
&= \int_\Omega \frac{1}{\phi\left(f^{-1}(x')\right)\phi\left(y\right)} G_\Omega(f^{-1}(x'), y) f^*(\Delta_0 u)(y) f^*\left(\theta_0 \wedge (\mathrm{d}\theta_0)^n\right) \\
&= \int_\Omega \frac{1}{\phi\left(f^{-1}(x')\right)\phi\left(y\right)} G_\Omega(f^{-1}(x'), y)\tilde\Delta_0(f^*u)(y)\tilde\theta_0 \wedge (\mathrm{d}\tilde\theta_0)^n \\
&= \frac{1}{\phi\left(f^{-1}(x')\right)} \int_\Omega G_\Omega(f^{-1}(x'), y)\Delta_0(\phi f^* u)(y)\theta_0 \wedge (\mathrm{d}\theta_0)^n \\
&= f^*u\left(f^{-1}(x')\right) = u(x')
\end{aligned}$$

for any $u \in C_0^\infty(\Omega')$. Here we take transform $y = f^{-1}(y')$ in the first identity. The proposition follows form the uniqueness of the Green's function. □

By the Liouville type theorem 2.2, any local conformal diffeomorphism f is the restriction of a conformal transformation of $\mathscr{H}^n$. Then we have the following proposition.

Proposition 3.5. *Let Ω and Ω' be bounded regular domains in $\mathscr{H}^n$, and let $f : \Omega \to \Omega'$ be a conformal diffeomorphism. We have*

$$\|f(x)^{-1}f(y)\| = \phi^{\frac{1}{Q-2}}(x)\phi^{\frac{1}{Q-2}}(y)\|x^{-1}y\|. \tag{3.12}$$

Proof. By (3.11) and taking transformation $f(y) \to y'$, we find that for any $u \in C_0^\infty(\Omega)$

$$\begin{aligned}&\int_\Omega \frac{C_Q\phi(x)\phi(y)}{\|f(x)^{-1}f(y)\|^{Q-2}}\Delta_0 u(y)\theta_0\wedge(\mathrm{d}\theta_0)^n\\ &=\int_\Omega \frac{C_Q\phi(x)}{\|f(x)^{-1}f(y)\|^{Q-2}}\,\tilde{\Delta}_0\left(\phi^{-1}u\right)\Big|_{f(y)}\tilde{\theta}_0\wedge(\mathrm{d}\tilde{\theta}_0)^n\\ &=\int_{\Omega'} \frac{C_Q\phi(x)}{\|f(x)^{-1}y'\|^{Q-2}}\,\Delta_0\left(\phi^{-1}u\right)\big|_{y'}\theta_0\wedge(\mathrm{d}\theta_0)^n=u(x).\end{aligned}$$

Now by the uniqueness of the fundamental solution of Δ_0 as before, we find that $\Gamma(x,y)=C_Q\phi(x)\phi(y)\|f(x)f(y)^{-1}\|^{2-Q}$. Thus the proposition is proved. □

See [8] for this proposition on the Euclidean space.

Proof of Theorem 1.1. Assume that

$$f^*\theta_0=\phi^{\frac{4}{Q-2}}\theta_0,$$

for some positive function $\phi\in C^\infty(\Omega)$. Then we have

$$f^*\theta_{\Omega'}=(\mathcal{A}_{\Omega'}\circ f)^2 f^*\theta_0=(\mathcal{A}_{\Omega'}\circ f)^2\phi^{\frac{4}{Q-2}}\theta_0, \tag{3.13}$$

where θ_Ω is defined in (1.4). Then, by (3.12), we have

$$\begin{aligned}\mathcal{A}_{\Omega'}(f(x))&=\lim_{y\to x}|\Gamma(f(x),f(y))-G_{\Omega'}(f(x),f(y))|^{\frac{1}{Q-2}}\\ &=\lim_{y\to x}\left|\frac{C_Q}{\phi(x)\phi(y)\|x^{-1}y\|^{Q-2}}-\frac{G_\Omega(x,y)}{\phi(x)\phi(y)}\right|^{\frac{1}{Q-2}}\\ &=\phi^{-\frac{2}{Q-2}}(\xi)\mathcal{A}_\Omega(x).\end{aligned}$$

Consequently, we have

$$\mathcal{A}_{\Omega'}^2(f(x))f^*\theta_0\big|_x=\mathcal{A}_\Omega^2(x)\theta_0|_x.$$

Since Corollary 3.2 ensures that $\mathcal{A}_\Omega$ is non-vanishing, θ_Ω is a contact form.

We can prove the Green's function symmetric in a way similar to the Euclidean case (cf. e.g. [1], Chapter 4)). Note that

$$\int_\Omega \Gamma(x,y)\Delta_0 u(y)\theta_0\wedge(\mathrm{d}\theta_0)^n(y)=u(x)$$

for each $u\in C_0^2(\Omega)$, and

$$\Delta_0\varphi(y)=\Delta_y\int_\Omega G_\Omega(x,y)\Delta_0\varphi(x)\theta_0\wedge(\mathrm{d}\theta_0)^n(x)$$

for $\varphi\in C_0^2(\Omega)$, where Δ_y means that the SubLaplacian is applied with respect to the variable y. Thus

$$\varphi(y)=\int_\Omega G_\Omega(x,y)\Delta_0\varphi(x)\theta_0\wedge(\mathrm{d}\theta_0)^n(x)+\mathrm{Const},$$

which yields

$$\int_\Omega\left(G_\Omega(x,y)-G_\Omega(y,x)\right)\Delta_0\varphi(y)\theta_0\wedge(\mathrm{d}\theta_0)^n(y)=\mathrm{Const}. \tag{3.14}$$

for all $\varphi \in C_0^2(\Omega)$. Integrating (3.14) proves that the constant is zero. Since

$$\int G_\Omega(y,x)\theta_0 \wedge (\mathrm{d}\theta_0)^n(x) = 0 \quad \text{and} \quad \int G_\Omega(x,y)\theta_0 \wedge (\mathrm{d}\theta_0)^n(x) = \text{Const}.$$

Thus $G_\Omega(x,y) - G_\Omega(y,x) = \text{Const}$. Interchanging x and y implies the second member is zero. Thus the Green's function G_Ω is symmetric. It follows that

$$\int_{\Omega\times\Omega} H(x,y)(\Delta_x + \Delta_y)w(x,y)\mathrm{d}V(x,y) = 0,$$

for each $w \in C_0^\infty(\Omega \times \Omega)$, where the $\mathrm{d}V$ is the associate volume form and Δ_x, Δ_y mean the SubLaplacian is applied with respect to the variables x and y. As for any weak solution u of $\Delta_0 u = 0$ on an open set $\Omega \times \Omega$, we have $u \in C^\infty(\Omega \times \Omega)$. Cf. Corollary 1.2.3 in [5] for the Euclidean case. So $H(x,x)$ is smooth in Ω. Then we obtain the limit

$$\mathcal{A}_\Omega(x) := \lim_{y\to x} |H(x,y)|^{\frac{1}{Q-2}} = \lim_{y\to x} |G_\Omega(x,y) - \Gamma_\Omega(x,y)|^{\frac{1}{Q-2}}$$

exists for each $x \in \Omega$. The theorem is proved.

4. The quasi-hyperbolic Carnot-Carathéodory metric

Let $J : H \to H$ be the standard CR structure satisfying $J^2 = -id_H$. Recall that g_0 satisfies the compatibility condition

$$g_0(JX, Y) = \mathrm{d}\theta_0(X, Y), \tag{4.1}$$

for any $X, Y \in H$. We have

$$\mathrm{d}\theta_\Omega = \mathcal{A}_\Omega^2 \mathrm{d}\theta_0 + \mathrm{d}(\mathcal{A}_\Omega^2) \wedge \theta_0.$$

So the associated Carnot-Carathéodory metric of θ_Ω is

$$g_\Omega(X,Y) = \mathrm{d}\theta_\Omega(X, JY) = \mathcal{A}_\Omega^2 g_0(X,Y)$$

for any $X, Y \in H$. We can easily verify that g_Ω is a conformally invariant Carnot-Carathéodory metric.

We have following comparison proposition about this Carnot-Carathéodory metric.

Proposition 4.1. *Let Ω_1, Ω_2 be bounded regular domains in $\mathscr{H}^n$. If $\Omega_1 \subset \Omega_2$, we have*

$$\mathcal{A}_{\Omega_1}(x) \geq \mathcal{A}_{\Omega_2}(x) \tag{4.2}$$

on Ω_1.

Proof. Denote $H_{\Omega_1}(x,\cdot)$ and $H_{\Omega_2}(x,\cdot)$ be the regular part of Green's function $G_{\Omega_1}(x,\cdot)$ and $G_{\Omega_2}(x,\cdot)$, respectively. Since

$$\left(X_1^2 + \cdots + X_{2n}^2\right)(-\Gamma(x,\cdot)) = \delta_x,$$

$-\Gamma(x,\cdot)$ is subharmonic with respect to the SubLaplacain $X_1^2 + \cdots + X_{2n}^2$ (cf. [2]). Thus

$$-H_{\Omega_2}(x,\cdot)|_{\partial\Omega_2} = -\Gamma(x,\cdot)|_{\partial\Omega_2}$$

implies that

$$-H_{\Omega_2}(x,\cdot) \geq -\Gamma(x,\cdot) \quad \text{on } \Omega_2.$$

On the other hand, we have

$$-H_{\Omega_1}(x,\cdot)|_{\partial\Omega_1} = -\Gamma(x,\cdot)|_{\partial\Omega_1}.$$

Hence

$$-H_{\Omega_2}(x,\cdot)|_{\partial\Omega_1} \geq -H_{\Omega_1}(x,\cdot)|_{\partial\Omega_1}.$$

Then the maximum principle implies

$$-H_{\Omega_2}(x,\cdot) \geq -H_{\Omega_1}(x,\cdot) \quad \text{on } \Omega_1.$$

The proposition is proved. □

Theorem 4.2. (cf. Theorem 1.1 in [3]) *For $x, y \in \Omega$, x near the boundary $\partial\Omega$, and H the regular part of Green's function at this point. We have*

$$H(x,x) = \frac{C_Q}{d_x^2} + o(d_x^{-2}). \tag{4.3}$$

There is a difference of factor $\frac{1}{4}$ from that in [3] because our definition of Δ_0 is different from that with a factor $-\frac{1}{4}$.

Proof of the Theorem 1.2. By Theorem 4.2, we have

$$\mathcal{A}_\Omega(x) = \lim_{y\to x}\left|H^{\frac{1}{Q-2}}(x,y)\right| = \left|\frac{C_Q}{d_x^2} + o(d_x^{-2})\right|^{\frac{1}{Q-2}}$$

Then

$$c_1 k_\Omega^2(x) \leq \mathcal{A}_\Omega^2(x) \leq c_2 k_\Omega^2(x) \tag{4.4}$$

for x near the boundary, for some constant $c_1, c_2 > 0$, where k_Ω is defined in (1.5). On a compact subset of Ω, (4.4) holds for some constant $c_1, c_2 > 0$, since $\mathcal{A}_\Omega^2/k_\Omega^2$ is positive and bounded. (1.7) follows directly by the definition of $H(\cdot,\cdot)$. The theorem is proved.

References

[1] T. Aubin, *Nonlinear analysis on manifolds. Monge-Ampère equations*, Springer Science and Business Media, vol. 252, Springer, 1982.

[2] A. Bonfiglioli, and E. Lanconelli, *Subharmonic functions on Carnot groups,* Math. Ann. **325** (2003), 97–122.

[3] N. Gamara, and H. Guemri, *Estimates of the Green's function and its regular part on Heisenberg group domains,* Adv. Nonlinear Stud. **11** (2011), 593–612.

[4] L. Habermann and J. Jost, *Green functions and conformal geometry,* J. Diff. Geom. **53** (1999), 405–442.

[5] L. Habermann, *Riemannian metrics of constant mass and moduli spaces of conformal structures,* Lect. Notes Math. vol. 1743, Berlin, Springer, 2000.

[6] D. Jerison and J. M. Lee, *The Yamabe problem on CR manifolds.* J. Diff. Geom. **25** (1987), 167–197.

[7] H. Leutwiler, *On a distance invariant under Möbius transformations in $\mathbb{R}^n$,* Ann. acad. sci. fenn. Ser. A I Math. **12** (1987), 3–17.

[8] H. Leutwiler, *A Riemannian metric invariant under Möbius transformations in* $\mathbb{R}^n$, Lect. Notes in Math. vol.1351, Berlin, Springer, Berlin, 1988, pp. 223–235.

[9] Y. Shi and W. Wang, *On conformal qc geometry, spherical qc manifolds and convex cocompact subgroups of* $\mathrm{Sp}(n+1,1)$*,* Ann. Global Anal. Geom. **49** (2016), 271–307.

[10] W. Wang, *Canonical contact forms on spherical CR manifolds,* J. Eur. Math. soc. **5** (2003), 245–273.

[11] W. Wang, *Representations of* $\mathrm{SU}(p,q)$ *and CR geometry I,* J. Math. Kyoto Univ. **45** (2005), 759–780.

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