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PAIRS TRADING

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Abstract. This survey paper is concerned with pairs trading strategies under mean reversion and geometric Brownian motion models. Pairs trading is about trading simultaneously a pair of highly correlated securities, typically stocks. The idea is to monitor the spread of their price movements over time. A pairs trade is triggered by their price divergence (e.g., one stock moves up a significant amount relative to the other) and consists of a short position in the strong stock and a long position in the weak one. Such a strategy bets on the reversal of their price strengths and the eventual convergence of the price spread. Pairs trading is a risk-neutral type strategy and it is popular among investment institutions. In practice, the trader needs to decide when to initiate a pairs position (how much divergence is enough) and when to close the position (how to take profits or cut losses). In this paper, we focus on two mathematical models and demonstrate how pairs trading strategies work under these models.

1. Introduction

This paper is about strategies for simultaneously trading a pair of stocks. The idea of pairs trading is to track the price movements of these two securities over time and compare their relative price strength. A pairs trade is triggered when their prices diverge, e.g., one stock moves up substantially relative to the other. A pairs trade is entered and consists of a short position in the stronger stock and a long position in the weaker one. Such a strategy bets on the reversal of their price strength and eventual convergence of their price spread.

A major advantage of pairs trading is its 'market neutral' nature in the sense that it helps to hedge market risks. For example, if the market crashes and takes both stocks with it, the trade would result in a gain on the short side and loss on the long side of the position. The gain and loss cancel out each other to some extent to reduce market risk.

In pairs trading, a crucial step is to determine when to initiate a pairs trade (i.e., how much spread divergence is sufficient to trigger a trade) and when to close the position (when to lock in profits). It is the focus of this paper. We formulate the pairs trading in terms of optimal stopping under two popular stochastic models and establish simple and yet mathematical optimal trading rules.

Pairs trading was introduced by G. Bamberger and followed by N. Tartaglia's quantitative group at Morgan Stanley in the 1980s. Tartaglia's group used advanced statistical tools and developed high tech trading systems by incorporating trader's intuition and disciplined filter rules. They were able to identify pairs of stocks and traded them with great success. See Gatev et al. [6] for additional background

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details. In addition, there are studies addressing why pairs trading works. For related in-depth discussions in connection with the cause of the price divergence and subsequent convergence, we refer the reader to the books by Vidyamurthy [17] and Whistler [18].

Empirical studies and related considerations can be found in papers by Do and Faff [3, 4], Gatev et al. [6], and books by Vidyamurthy [17] and Whistler [18]. Issues involved include statistical characterization of the spreading process, performance of pairs trading with various trading thresholds, and impact of trading costs in connection with pairs trading.

Following these empirical developments, increasing efforts were made addressing theoretical aspects of pairs trading. The main focus was devoted to develop mathematical models that capture the spread movements, filtering techniques, optimal entry and exit timings, and money management and risk control. For example, in Elliott et al. [5], the price spread is assumed to be a mean reversion process with additive noise. Several filtering techniques were explored to identify entry points. One exit rule with a fixed holding period was discussed in details. In Deshpande and Barmish [2], a general (mean-reversion based) framework was developed. Using a 'spread' function, they were able to determine the numbers of shares of each stock every moment and how to adjust them over time. They showed that such algorithm leads to a positive expected growth.

In this paper, we focus on optimal buying and selling rules for pairs trading. First, we consider an optimal selling rule. Assuming one entered a position based either on certain spread condition or on fundamental analysis, our goal is to determine when to exit the position in order to maximize an expected return or to cut losses short. Such decision making was treated in Kuo et al. [10]. In particular, given a fixed cut-loss level, the optimal target level can be determined by a mean reversion model. This approach will be presented in details in this survey.

Of course, from a trading system development point of view, a complete system with both entry and exit signals is more desirable. In Song and Zhang [15], advanced mathematical tools were developed to address such needs. In particular, under a mean reversion model, it is shown that the optimal trading rule can be determined by threshold levels. The calculation of these levels is shown in [15] only involves algebraic equations.

We would like to point out that almost all literature on pairs trading is mean reversion based one way or the other. This makes the trading more intuitive. In the meantime, such constraint adds a severe limitation on its potential applications. In order to meet the mean-reversion requirement, tradable pairs are typically selected among stocks from the same industrial sector. From a practical viewpoint, it is highly desirable to have a broad range of stock selections for pairs trading. Mathematically speaking, this amounts to the possibility of treating pairs trading under models other than mean reversion. In Tie et al. [16], they have developed a new method to treat the pairs-trading problem under general geometric Brownian motions.

In this paper, we mainly involve stocks. Nevertheless, the idea of pairs trading is not limited to stock trading. For example, the optimal timing of investments in irreversible projects can also be considered as a pairs-trading problem. Back in 1986, McDonald and Siegel [13] considered the optimal timing of investment in an irreversible project. Two factors are included in their model: The growth of the investment capital and the change in project cost. Greater capital growth potential and lesser future project cost will postpone the transaction. See also Hu and Øksendal [8] for more rigorous mathematical treatment. In terms of pairs trading, their results are about a pairs trading selling rule.

Mathematical trading rules have been studied for many years. In addition to the work by Hu and Øksendal [8] and Song and Zhang [15], Zhang [19] considered a selling rule determined by two threshold levels, a target price and a stop-loss limit. In [19], such optimal threshold levels are obtained by solving a set of two-point boundary value problems. Guo and Zhang [7] studied the optimal selling rule under a model with switching Geometric Brownian motion. Using a smooth-fit technique, they obtained the optimal threshold levels by solving a set of algebraic equations. These papers are concerned with the selling side of trading in which the underlying price models are of GBM type. Recently, Dai et al. [1] developed a trend following rule based on a conditional probability indicator. They showed that the optimal trading rule can be determined by two threshold curves which can be obtained by solving the associated Hamilton-Jacobi-Bellman (HJB) equations. A similar idea was developed following a confidence interval approach by Iwarere and Barmish [9]. In addition, Merhi and Zervos [14] studied an investment capacity expansion/reduction problem following a dynamic programming approach under a geometric Brownian motion market model. In connection with mean reversion trading, Zhang and Zhang [20] obtained a buy-low and sell-high policy by characterizing the 'low' and 'high' levels in terms of the mean reversion parameters.

In this paper, we focus on the mathematical aspects of pairs trading. In §2, we discuss pairs trading selling rule. Assuming one has an open pairs position, she needs to close it to maximize an expected return. In this section, the selling rule consists of two thresholds: target level and cut-loss level. Given a cut-loss level, the goal is to determine the best target to maximize an expected payoff function. In §3, we consider a complete trading system under a mean reversion model. The objective is to trade pairs over time to maximize a discounted reward function. In §4, we study pairs trading under geometric Brownian motions. It can be seen that pairs trading ideas are more general and they do not have to be cast under a mean reversion framework. Proofs of these results are omitted and can be found in [10, 15, 16]. Finally, some conclusion remarks are given in §5.

2. Mean Reversion Model: An Optimal Selling Rule

In this section, we consider pairs trading that involves two stocks $\mathbf{X}^1$ and $\mathbf{X}^2$. The pairs position consists of a long position in $\mathbf{X}^1$ and short position in $\mathbf{X}^2$. Let X_t^1 and X_t^2 denote their respective prices at time $t \geq 0$. For simplicity, we allow trading a fraction of a share and consider the pairs position consisting of $K_1 = 1/X_0^1$ shares of $\mathbf{X}^1$ in the long position and $K_2 = 1/X_0^2$ shares of $\mathbf{X}^2$ in the short position. The corresponding price spread of the position is given by $Z_t = K_1X_t^1 - K_2X_t^2$.

We assume that Z_t is a mean-reverting (Ornstein-Uhlenbeck) process governed by

$$dZ_t = \theta(\mu - Z_t)dt + \sigma dW_t,\ Z_0 = z, \tag{2.1}$$

where $\theta > 0$ is the rate of reversion, μ the equilibrium level, $\sigma > 0$ the volatility, and W_t a standard Brownian motion. In addition, the notation $\mathbf{Z}$ represents the corresponding pairs position. One share long in $\mathbf{Z}$ means the combination of K_1 shares of long position in $\mathbf{X}^1$ and K_2 shares of short position in $\mathbf{X}^2$. Similarly, for $i = 1, 2$, X_t^i represents the price of stock $\mathbf{X}^i$. Lastly, Z_t is the value of the pairs position at time t (which in this paper is allowed to be negative).

Assuming a pairs position was in place, the objective is to decide when to close the position. We consider the selling rule determined by two threshold levels: the target and a cut-loss level. In particular, let z_1 denote the cut-loss level and z_2 the target. The selling time is given by the exit time τ of Z_t from (z_1, z_2), i.e., $\tau = \inf\{t :\ Z_t \notin (z_1, z_2)\}$.

Here z_1 is the cut-loss level, which represents the risk tolerance of the investor per trade. It is determined by the investor. z_2 is the target which varies with each stock. In Gatev et al. [6], the threshold levels $z_1 = -\infty$ and $z_2 = \mu$ are used to determine when to close a pairs position. Note that in practice a cut-loss level is often imposed to limit possible undesirable events in the marketplace. It is a typical money management consideration. It can also be associated with a margin call due to substantial losses.

Given (z_1, z_2) and the initial state $Z_0 = z$, the corresponding reward function is

$$v(z) = v_{\{z_1,z_2\}}(z) = E[e^{-\rho\tau} Z_\tau | Z_0 = z].$$

Here $\rho > 0$ is a given discount (impatience) factor.

Following a similar approach as in Zhang [19], we can show that the reward function $v(z)$ satisfies the two-point-boundary-value differential equation

$$\begin{cases} \rho v(z) = \dfrac{\sigma^2}{2}\dfrac{d^2 v(z)}{dz^2} + \theta(\mu - z)\dfrac{dv(z)}{dz}, \\ v(z_1) = z_1,\ v(z_2) = z_2. \end{cases} \tag{2.2}$$

To solve the equation, let $\kappa = \sqrt{2\theta}/\sigma$ and $\eta(t) = t^{(\rho/\theta)-1}e^{-t^2/2}$. Then the general solution of (2) can be given in terms of a linear combination of independent solutions:

$$v(z) = C_1 \int_0^\infty \eta(t) e^{-\kappa(\mu - z)t} dt + C_2 \int_0^\infty \eta(t) e^{\kappa(\mu - z)t} dt,$$

for some constants C_1 and C_2. Note that these constants are (z_1, z_2) dependent, i.e., $C_1 = C_1(z_1, z_2)$ and $C_2 = C_2(z_1, z_2)$.

Taking $z = z_1$ and $z = z_2$ respectively, we have

$$\begin{pmatrix} v(z_1) \\ v(z_2) \end{pmatrix} = \begin{pmatrix} \int_0^\infty \eta(t) e^{-\kappa(\mu - z_1)t} dt & \int_0^\infty \eta(t) e^{\kappa(\mu - z_1)t} dt \\ \int_0^\infty \eta(t) e^{-\kappa(\mu - z_2)t} dt & \int_0^\infty \eta(t) e^{\kappa(\mu - z_2)t} dt \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}.$$

Let $\Phi(z_1, z_2)$ denote the above 2×2 matrix. It can be shown to be non-singular. Using the boundary conditions in (2), the constants C_1 and C_2 can be expressed in terms of z_1 and z_2 as follows:

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \Phi^{-1}(z_1, z_2) \begin{pmatrix} v(z_1) \\ v(z_2) \end{pmatrix} = \Phi^{-1}(z_1, z_2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

Given the initial value $Z_0 = z_0$, the corresponding reward function

$$v(z_0) = C_1 \int_0^\infty \eta(t) e^{-\kappa(\mu - z_0)t} dt + C_2 \int_0^\infty \eta(t) e^{\kappa(\mu - z_0)t} dt.$$

With z_1 fixed, the optimization problem is to choose $z_2 \geq z_0$ to maximize $v(z_0)$.

Let $\gamma(z) = \exp\left(\theta(z-\mu)^2/\sigma^2\right)$. Then, the expected holding time is given by

$$E[\tau_0 | Z_0 = z_0] = -\frac{2}{\sigma^2} \int_{z_1}^t \left(\gamma(t) \int_0^t \frac{du}{\gamma(u)} \right) dt + T_0 \int_{z_1}^{z_0} \left(\frac{\gamma(t)}{\gamma(0)} \right) dt,$$

where

$$T_0 = \frac{2 \displaystyle\int_{z_1}^{z_2} \left(\gamma(t) \int_0^t \frac{du}{\gamma(u)} \right) dt}{\sigma^2 \displaystyle\int_{z_1}^{z_2} \left(\frac{\gamma(t)}{\gamma(0)} \right) dt}.$$

Finally, the corresponding profit probability

$$P(Z_{\tau_0} = z_2 | Z_0 = z_0) = \frac{\displaystyle\int_{z_1}^{z_0} \exp\left(\frac{\theta}{\sigma^2}(u-\mu)^2 \right) du}{\displaystyle\int_{z_1}^{z_2} \exp\left(\frac{\theta}{\sigma^2}(u-\mu)^2 \right) du}.$$

3. Mean Reversion Model: An Optimal Trading Rule

In this section, we consider a pairs trading system with both buying and selling signals. Let Z_t be the price of the pairs position satisfying (1). In addition, we impose a state constraint and require $Z_t \geq M$. Here M is a given constant and it represents a stop-loss level. It is common in practice to limit losses to an acceptable level to account for unforeseeable events in the marketplace. A stop-loss limit is often enforced as part of money management. It can also be associated with a margin call due to substantial losses.

To accommodate such state constraint in our model, let τ_M denote the exit time of Z_t from (M, ∞), i.e., $\tau_M = \inf\{t : \; Z_t \notin (M, \infty)\}$.

Let

$$0 \leq \tau_1^b \leq \tau_1^s \leq \tau_2^b \leq \tau_2^s \leq \cdots \leq \tau_M \tag{3.1}$$

denote a sequence of stopping times. A buying decision is made at τ_n^b and a selling decision at τ_n^s, $n = 1, 2, \ldots$.

We consider the case that the net position at any time can be either long (with one share of $\mathbf{Z}$) or flat (no stock position of either $\mathbf{X}^1$ or $\mathbf{X}^2$). Let $i = 0, 1$ denote the initial net position. If initially the net position is long ($i = 1$), then one should sell $\mathbf{Z}$ before acquiring any future shares. The corresponding sequence of stopping times is denoted by $\Lambda_1 = (\tau_1^s, \tau_2^b, \tau_2^s, \tau_3^b, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then one should start to buy a share of $\mathbf{Z}$. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1^b, \tau_1^s, \tau_2^b, \tau_2^s, \ldots)$.

Let $K > 0$ denote the fixed transaction cost (e.g., slippage and/or commission) associated with buying or selling of $\mathbf{Z}$. Given the initial state $Z_0 = x$ and initial net position $i = 0, 1$, and the decision sequences, Λ_0 and Λ_1, the corresponding reward functions

$$J_i(x, \Lambda_i) = \begin{cases} E\left\{ \sum_{n=1}^{\infty} \left[e^{-\rho\tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(Z_{\tau_n^b} + K) \right] I_{\{\tau_n^b < \tau_M\}} \right\}, & \text{if } i = 0, \\ E\left\{ e^{-\rho\tau_1^s}(Z_{\tau_1^s} - K) \right. \\ \quad \left. + \sum_{n=2}^{\infty} \left[e^{-\rho\tau_n^s}(Z_{\tau_n^s} - K) - e^{-\rho\tau_n^b}(Z_{\tau_n^b} + K) \right] I_{\{\tau_n^b < \tau_M\}} \right\}, & \text{if } i = 1, \end{cases}$$

where $\rho > 0$ is a given discount factor.

In the reward function J_i, a buying decision has to be made before Z_t reaches M. When $t = \tau_M$ (or $Z_t = M$), only a selling can be done if $i = 1$.

For $i = 0, 1$, let $V_i(x)$ denote the value functions with the initial state $Z_0 = x$ and initial net positions $i = 0, 1$. That is,

$$V_i(x) = \sup_{\Lambda_i} J_i(x, \Lambda_i). \tag{3.2}$$

Note that

$$V_0(M) = 0 \text{ and } V_1(M) = M - K. \tag{3.3}$$

Remark 3.1. Note that we imposed the conditions $\tau_n^b \le \tau_M$ and $\tau_n^s \le \tau_M$, $n = 1, 2, \ldots$. If one has a share position of $\mathbf{Z}$ and $\tau_n^s = \tau_M$ for some n, then one has to sell the share to cut losses. On the other hand, if $\tau_n^b = \tau_M$, then one should not buy because she has to sell it right away, which only cause the round trip transaction fees.

Remark 3.2. In addition, we only consider the 'long' side trading in this paper. Actually, one can trade the 'short' side by simply reversing the trading rule obtained in this paper. For example, if the equilibrium $\mu = 0$, then we can trade both Z_t and $(-Z_t)$ simultaneously because they satisfy the same system equation (1).

Example 3.3. Typically a highly correlated pair can be found from the same industry sector. In this example, we choose Wal-Mart Stores Inc. (WMT) and Target Corp. (TGT). Both companies are from the retail industry and they have shared similar dips and highs. If the price of WMT were to go up a large amount while TGT stayed the same, a pairs trader would buy TGT and sell short WMT betting on the convergence of their prices. In Figure 1, the 'normalized' (dividing each price by its long term moving average) difference of WMT and TGT is plotted. In addition, the data (1992-2012) is divided into two sections. The first section (1992-2000) is used to calibrate the model and the second section (2001-2012) to backtest the performance of our results. Our construction of Z_t determines that the equilibrium level $\mu = 0$. By measuring the standard derivation of Z_t, we obtain

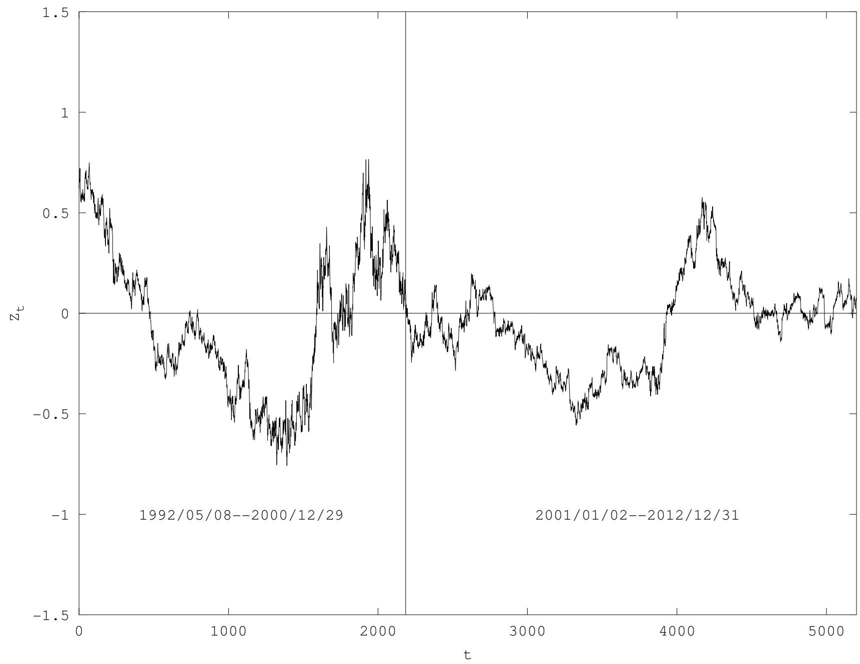

FIGURE 1. The normalized difference Z_t is based on WMT and TGT daily closing prices from 1992 to 2012. (Parameters: $\theta = 1.0$, $\mu = 0$, and $\sigma = 0.56$).

the historical volatility $\sigma = 0.56$. Finally, following the traditional least squares method, we obtain $\theta = 1.00$.

We can show that, for $x \geq M$, the following inequalities hold:

$$V_0(x) \geq V_1(x) - x - K, \quad V_1(x) \geq V_0(x) + x - K,$$
$$0 \leq V_0(x) \leq C_0, \quad x - K \leq V_1(x) \leq x + K + C_0,$$

where $C_0 = (\theta|\mu| + (\rho + \theta)|M|)/\rho$ and ρ is the discount factor.

Let $\mathcal{A}$ denote the generator of Z_t, i.e.,

$$\mathcal{A} = \theta(\mu - x)\frac{\partial}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}.$$

Formally, the associated HJB equations should have the form:

$$\begin{aligned} &\min\left\{\rho v_0(x) - \mathcal{A}v_0(x),\ v_0(x) - v_1(x) + x + K\right\} = 0, \\ &\min\left\{\rho v_1(x) - \mathcal{A}v_1(x),\ v_1(x) - v_0(x) - x + K\right\} = 0, \end{aligned} \tag{3.4}$$

for $x \in (M, \infty)$, with the boundary conditions $v_0(M) = 0$ and $v_1(M) = M - K$.

If $i = 0$, then one should only buy when the price is low (say less than or equal to x_1). In this case, $v_0(x) = v_1(x) - x - K$. The corresponding continuation region (given by $\rho v_0(x) - \mathcal{A}v_0(x) = 0$) should include (x_1, ∞). In addition, one should not establish any new position if Z_t is close to the stop-loss level M. In view of this,

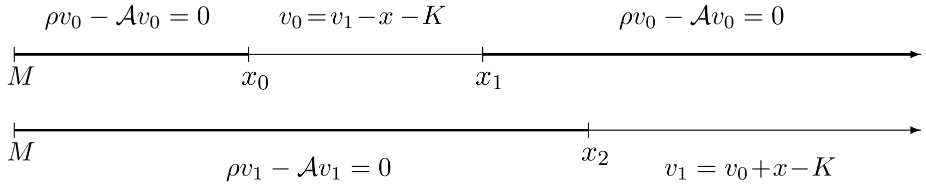

FIGURE 2. M is the stop-loss limit, x_0, x_1, and x_2 are transaction thresholds, and the continuation regions are bolded.

the continuation region should also include (M, x_0) for some $x_0 < x_1$. On the other hand, if $i = 1$, then one should only sell when the price is high (greater than or equal to $x_2 > x_1$), which implies $v_1(x) = v_0(x) + x - K$ and the continuation region (given by $\rho v_1(x) - \mathcal{A}v_1(x) = 0$) should be (M, x_2). These continuation regions are highlighted in Figure 2.

To solve the HJB equations in (6), we first solve the equations $\rho v_i(x) - \mathcal{A}v_i(x) = 0$ with $i = 0, 1$ on their continuation regions. Let

$$\begin{cases} \phi_1(x) = \displaystyle\int_0^\infty \eta(t) e^{-\kappa(\mu - x)t} dt, \\ \phi_2(x) = \displaystyle\int_0^\infty \eta(t) e^{\kappa(\mu - x)t} dt, \end{cases}$$

where $\eta(t) = t^{(\rho/\theta)-1} \exp\left(-t^2/2\right)$ and $\kappa = \sqrt{2\theta}/\sigma$. Then $\phi_1(x)$ and $\phi_2(x)$ are independent and the general solution is given by a linear combination of these functions.

First, consider the interval (x_1, ∞) and suppose the solution is given by $A_1\phi_1(x) + A_2\phi_2(x)$, for some A_1 and A_2. Recall the upper bound for $V_0(x)$, $v_0(\infty)$ should be bounded above. This implies that, $A_1 = 0$ and $v_0(x) = A_2\phi_2(x)$ on (x_1, ∞). Let B_1, B_2, C_1, and C_2 be constants such that $v_0(x) = B_1\phi_1(x) + B_2\phi_2(x)$ on (M, x_0) and $v_1(x) = C_1\phi_1(x) + C_2\phi_2(x)$ on (M, x_2).

It is easy to see that these functions are twice continuously differentiable on their continuation regions. We follow the smooth-fit method which requires the solutions to be continuously differentiable. In particular, it requires v_0 to be continuously differentiable at x_0. Therefore,

$$\begin{cases} B_1\phi_1(x_0) + B_2\phi_2(x_0) = C_1\phi_1(x_0) + C_2\phi_2(x_0) - x_0 - K, \\ B_1\phi_1'(x_0) + B_2\phi_2'(x_0) = C_1\phi_1'(x_0) + C_2\phi_2'(x_0) - 1. \end{cases} \tag{3.5}$$

Similarly, the smooth-fit conditions at x_1 and x_2 yield

$$\begin{cases} A_2\phi_2(x_1) = C_1\phi_1(x_1) + C_2\phi_2(x_1) - x_1 - K, \\ A_2\phi_2'(x_1) = C_1\phi_1'(x_1) + C_2\phi_2'(x_1) - 1, \end{cases} \tag{3.6}$$

and

$$\begin{cases} C_1\phi_1(x_2) + C_2\phi_2(x_2) = A_2\phi_2(x_2) + x_2 - K, \\ C_1\phi_1'(x_2) + C_2\phi_2'(x_2) = A_2\phi_2'(x_2) + 1. \end{cases} \tag{3.7}$$

Finally, the boundary conditions at $x = M$ lead to

$$\left\{\begin{array}{l} B_1\phi_1(M) + B_2\phi_2(M) = 0, \\ C_1\phi_1(M) + C_2\phi_2(M) = M - K. \end{array}\right. \tag{3.8}$$

For simplicity in notation, let

$$\Phi(x) = \begin{pmatrix} \phi_1(x) & \phi_2(x) \\ \phi_1'(x) & \phi_2'(x) \end{pmatrix},$$

which can be shown invertible for all x.

Also, let

$$\begin{aligned} R(x) &= \Phi^{-1}(x)\begin{pmatrix} \phi_2(x) \\ \phi_2'(x) \end{pmatrix},\ P_1(x) = \Phi^{-1}(x)\begin{pmatrix} x+K \\ 1 \end{pmatrix}, \\ P_2(x) &= \Phi^{-1}(x)\begin{pmatrix} x-K \\ 1 \end{pmatrix}, \end{aligned}$$

Rewrite the equations (7)-(10) in terms of these vectors. We have

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - P_1(x_0), \tag{3.9}$$

$$A_2R(x_1) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} - P_1(x_1), \tag{3.10}$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = A_2R(x_2) + P_2(x_2), \tag{3.11}$$

and

$$\left\{\begin{array}{l} (\phi_1(M), \phi_2(M))\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0, \\ (\phi_1(M), \phi_2(M))\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = M - K. \end{array}\right. \tag{3.12}$$

Multiplying both sides of (11) from the left by $(\phi_1(M), \phi_2(M))$ and using (14), we have

$$(\phi_1(M), \phi_2(M))P_1(x_0) = M - K. \tag{3.13}$$

We can show

$$A_2 = \frac{M - K - (\phi_1(M), \phi_2(M))P_2(x_2)}{(\phi_1(M), \phi_2(M))R(x_2)}, \tag{3.14}$$

and

$$(R(x_1) - R(x_2))\left(\frac{M - K - (\phi_1(M), \phi_2(M))P_2(x_2)}{(\phi_1(M), \phi_2(M))R(x_2)}\right) = P_2(x_2) - P_1(x_1). \tag{3.15}$$

Solving equations (15) and (17), we can obtain the triple (x_0, x_1, x_2). Then solving the equations (11), (12), and (16), to obtain A_2, (B_1, B_2), and (C_1, C_2).

Note that $v_i(x)$ has to satisfy the following inequalities for being solutions to the HJB equations (6):

$$\left\{\begin{array}{l} \rho v_0(x) - \mathcal{A}v_0(x) \geq 0, \\ \rho v_1(x) - \mathcal{A}v_1(x) \geq 0, \\ v_0(x) \geq v_1(x) - x - K, \\ v_1(x) \geq v_0(x) + x - K, \end{array}\right. \tag{3.16}$$

for all $x \geq M$.

We can show that these inequalities are equivalent to the following inequalities:

$$x_1 \leq \frac{\theta\mu - \rho K}{\rho + \theta}, \; x_2 \geq \frac{\theta\mu + \rho K}{\rho + \theta}, \tag{3.17}$$

and

$$\left\{\begin{array}{ll} |(C_1 - B_1)\phi_1(x) + (C_2 - B_2)\phi_2(x) - x| \leq K & \text{on } (M, x_0), \\ |C_1\phi_1(x) + (C_2 - A_2)\phi_2(x) - x| \leq K & \text{on } (x_1, x_2). \end{array}\right. \tag{3.18}$$

Theorem 3.4. *Let* (x_0, x_1, x_2) *be a solution to* (15) *and* (17) *satisfying* (19). *Let* A_2, B_1, B_2, C_1, *and* C_2 *be constants given by* (11), (13), *and* (16) *satisfying the inequalities in* (20).

Let

$$\left\{\begin{array}{l} v_0(x) = \left\{\begin{array}{ll} B_1\phi_1(x) + B_2\phi_2(x) & \text{if } x \in [M, x_0), \\ C_1\phi_1(x) + C_2\phi_2(x) - x - K & \text{if } x \in [x_0, x_1), \\ A_2\phi_2(x) & \text{if } x \in [x_1, \infty), \end{array}\right. \\ v_1(x) = \left\{\begin{array}{ll} C_1\phi_1(x) + C_2\phi_2(x) & \text{if } x \in [M, x_2), \\ A_2\phi_2(x) + x - K & \text{if } x \in [x_2, \infty). \end{array}\right. \end{array}\right.$$

Assume $v_0(x) \geq 0$. *Then,* $v_i(x) = V_i(x)$, $i = 0, 1$. *Moreover, if initially* $i = 0$, *let*

$$\Lambda_0^* = (\tau_1^{b*}, \tau_1^{s*}, \tau_2^{b*}, \tau_2^{s*}, \ldots),$$

such that the stopping times $\tau_1^{b*} = \inf\{t \geq 0 : \; Z_t \in [x_0, x_1]\} \wedge \tau_M$, $\tau_n^{s*} = \inf\{t > \tau_n^{b*} : \; Z_t \notin (M, x_2)\} \wedge \tau_M$, *and* $\tau_{n+1}^{b*} = \inf\{t > \tau_n^{s*} : \; Z_t \in [x_0, x_1]\} \wedge \tau_M$ *for* $n \geq 1$. *Similarly, if initially* $i = 1$, *let*

$$\Lambda_1^* = (\tau_1^{s*}, \tau_2^{b*}, \tau_2^{s*}, \tau_3^{b*}, \ldots),$$

such that $\tau_1^{s*} = \inf\{t \geq 0 : \; Z_t \notin (M, x_2)\} \wedge \tau_M$, $\tau_n^{b*} = \inf\{t > \tau_{n-1}^{s*} : \; Z_t \in [x_0, x_1]\} \wedge \tau_M$, *and* $\tau_n^{s*} = \inf\{t > \tau_n^{b*} : \; Z_t \notin (M, x_2)\} \wedge \tau_M$ *for* $n \geq 2$. *Then* Λ_0^* *and* Λ_1^* *are optimal.*

Next, we use the parameters of the WMT-TGT example, i.e.,

$$\theta = 1.0, \; \mu = 0, \; \sigma = 0.56, \; \rho = 0.10, \; K = 0.001, M = -0.2.$$

First, solving the equation (15), we have $x_0 = -0.142$. Then using this x_0 to find all (x_1, x_2) that satisfy both (17) and the inequalities (20). We obtain the pair $(x_1, x_2) = (-0.077, 0.077)$.

We backtest the pairs trading rule using the stock prices of WMT and TGT from 2001 to 2012. Let X_t^1 be the WMT stock divided by its 1000 day moving average and X_t^2 the TGT stock by its same period moving average. Taking $Z_t = X_t^1 - X_t^2$, a pairs trading is triggered when Z_t gets inside the buying interval $[x_0, x_1]$. The position is closed when Z_t exits the interval (M, x_2). Initially, we allocate trading

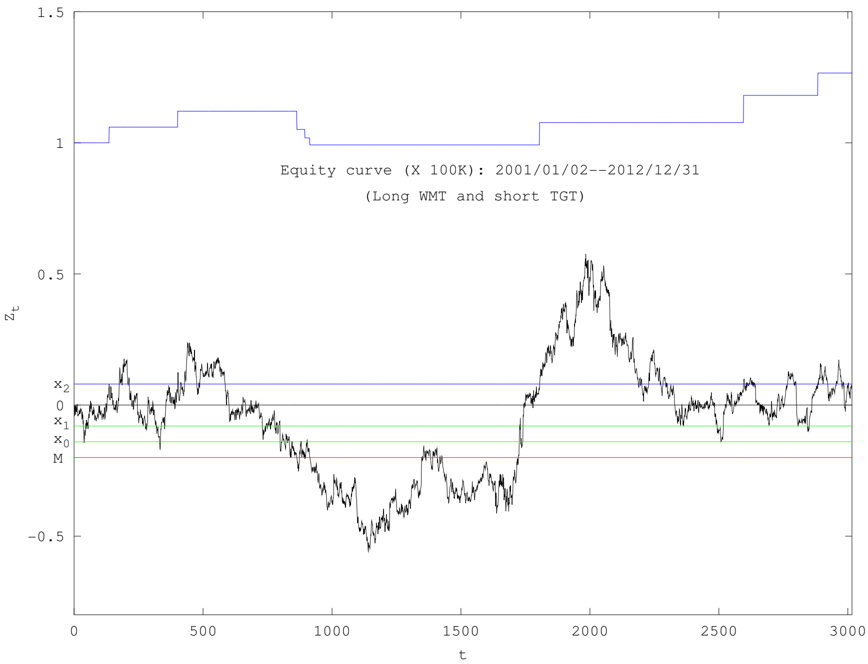

FIGURE 3. Trading Z_t: The threshold levels M, x_0, x_1, x_2 and the corresponding equity curve

the capital \$100K. When the first long signal is triggered, buy \$50K WMT stocks and short the same amount TGT. Close the position either when Z_t reaches the target x_2 or when it drops below the stop-loss level M. Such half-and-half capital allocation between long and short applies to all trades. In addition, each pairs transaction is charged \$5 commission fee. Furthermore, two variations from the assumptions prescribed in Theorem 3.4 in our 'actual' trading: (a) After the stop-loss level M is reached, the trading continues and a buying order is entered when Z_t goes back to the trading range; (b) All available capital will be used (half long and half short) for trading rather than following the 'single' share rule; Note that the choice of stop-loss level M can depend on many factors including the trader's risk tolerance level and margin requirements. Here our choice $M = -0.2$ corresponds to a 10% loss when WMT drops 10% and TGT stays the same.

In Figure 3, the corresponding Z_t, the threshold triple, and the corresponding equity curve are plotted. There are total 8 trades and the end balance is \$126.602K.

Note that Z_t is symmetric, i.e., $(-Z_t)$ satisfies the same equation (1). Naturally, one can reverse the pair and trade $(-Z_t)$ the same way. The reversed Z_t and equity curve is given in Figure 4. Such trade leads to the end balance \$114.935K. Note that both types of trades have no overlap, i.e., they do not compete for the same capital. The grand total profit is \$41547 which is a 41.54% gain.

The main advantage of pairs trading is its risk neutral nature, i.e., it can be profitable regardless the general market condition. In addition, there are only 2x8 trades in the eleven year period leaving the capital in cash most of the time. This

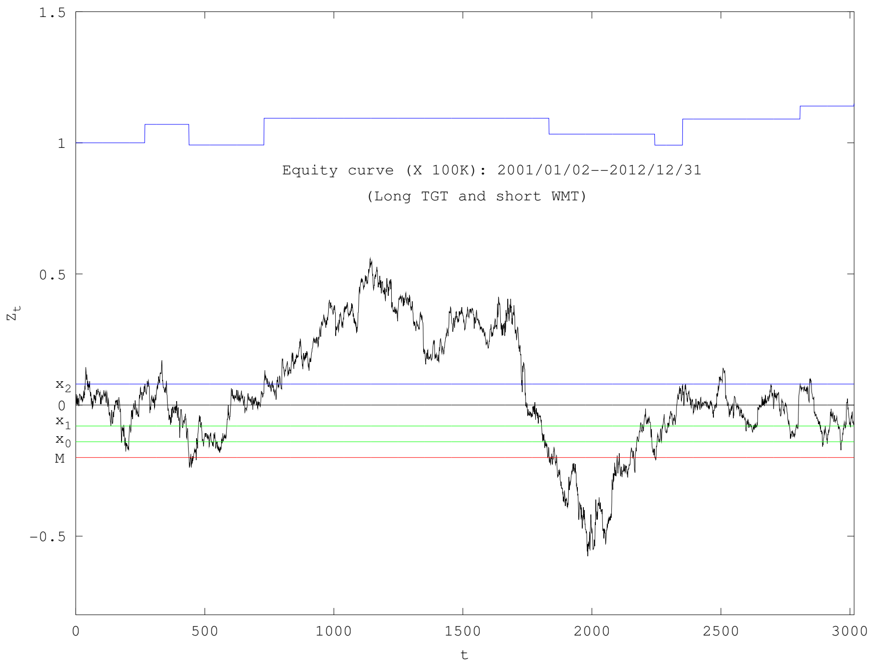

FIGURE 4. Trading $(-Z_t)$: The threshold levels M, x_0, x_1, x_2 and the corresponding equity curve

is desirable because the cash sitting in the account can be used for other types of shorter term trading in between, at least drawing interest over time.

4. GBM: An Optimal Trading Rule

In this section, we consider pairs trading under a geometric Brownian motion model. A share of pairs position $\mathbf{Z}$ consists of one share long position in stocks $\mathbf{X}^1$ and one share short position in $\mathbf{X}^2$. Let (X_t^1, X_t^2) denote their prices at t satisfying the following stochastic differential equation:

$$d\begin{pmatrix} X_t^1 \\ X_t^2 \end{pmatrix} = \begin{pmatrix} X_t^1 & \\ & X_t^2 \end{pmatrix}\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} d\begin{pmatrix} W_t^1 \\ W_t^2 \end{pmatrix}\right], \tag{4.1}$$

where μ_i, $i = 1, 2$, are the return rates, σ_{ij}, $i, j = 1, 2$, the volatility constants, and (W_t^1, W_t^2) a 2-dimensional standard Brownian motion.

We consider the case that the net position at any time can be either long (with one share of $\mathbf{Z}$) or flat (no stock position of either $\mathbf{X}^1$ or $\mathbf{X}^2$). Let $i = 0, 1$ denote the initial net position and let $\tau_0 \leq \tau_1 \leq \tau_2 \leq \cdots$ denote a sequence of stopping times. If initially the net position is long ($i = 1$), then one should sell $\mathbf{Z}$ before acquiring any future shares. That is, to first sell the pair at τ_0, then buy at τ_1, sell at τ_2, buy at τ_3, etc. The corresponding trading sequence is denoted by $\Lambda_1 = (\tau_0, \tau_1, \tau_2, \ldots)$. Likewise, if initially the net position is flat ($i = 0$), then one should start to buy a share of $\mathbf{Z}$. That is, to first buy at τ_1, sell at τ_2, then buy at τ_3, etc. The corresponding sequence of stopping times is denoted by $\Lambda_0 = (\tau_1, \tau_2, \ldots)$.

Let K denote the fixed percentage of transaction costs associated with buying or selling of stocks $\mathbf{X}^i$, $i = 1, 2$. For example, the cost to establish the pairs position $\mathbf{Z}$ at $t = t_1$ is $(1+K)X^1_{t_1} - (1-K)X^2_{t_2}$ and the proceeds to close it at a later time $t = t_2$ is $(1-K)X^1_{t_2} - (1+K)X^2_{t_2}$. For ease of notation, let $\beta_{\rm b} = 1+K$ and $\beta_{\rm s} = 1-K$.

Given the initial state (x_1, x_2), net position $i = 0, 1$, and the decision sequences Λ_0 and Λ_1, the corresponding reward functions

(4.2)

$$\begin{aligned}
&J_0(x_1,x_2,\Lambda_0) = \\
&\quad E\Big\{[e^{-\rho\tau_2}(\beta_{\rm s}X^1_{\tau_2} - \beta_{\rm b}X^2_{\tau_2})I_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}(\beta_{\rm b}X^1_{\tau_1} - \beta_{\rm s}X^2_{\tau_1})I_{\{\tau_1<\infty\}}] \\
&\quad +[e^{-\rho\tau_4}(\beta_{\rm s}X^1_{\tau_4} - \beta_{\rm b}X^2_{\tau_4})I_{\{\tau_4<\infty\}} - e^{-\rho\tau_3}(\beta_{\rm b}X^1_{\tau_3} - \beta_{\rm s}X^2_{\tau_3})I_{\{\tau_3<\infty\}}] + \cdots\Big\}, \\
&J_1(x_1,x_2,\Lambda_1) = \\
&\quad E\Big\{e^{-\rho\tau_0}(\beta_{\rm s}X^1_{\tau_0} - \beta_{\rm b}X^2_{\tau_0})I_{\{\tau_0<\infty\}} \\
&\quad +[e^{-\rho\tau_2}(\beta_{\rm s}X^1_{\tau_2} - \beta_{\rm b}X^2_{\tau_2})I_{\{\tau_2<\infty\}} - e^{-\rho\tau_1}(\beta_{\rm b}X^1_{\tau_1} - \beta_{\rm s}X^2_{\tau_1})I_{\{\tau_1<\infty\}}] \\
&\quad +[e^{-\rho\tau_4}(\beta_{\rm s}X^1_{\tau_4} - \beta_{\rm b}X^2_{\tau_4})I_{\{\tau_4<\infty\}} - e^{-\rho\tau_3}(\beta_{\rm b}X^1_{\tau_3} - \beta_{\rm s}X^2_{\tau_3})I_{\{\tau_3<\infty\}}] + \cdots\Big\},
\end{aligned}$$

where $\rho > 0$ is a given discount factor and I_A is the indicator function of an event A.

For $i = 0, 1$, let $V_i(x_1, x_2)$ denote the value functions with $(X^1_0, X^2_0) = (x_1, x_2)$ and initial net positions $i = 0, 1$. That is, $V_i(x_1, x_2) = \sup_{\Lambda_i} J_i(x_1, x_2, \Lambda_i)$, $i = 0, 1$.

Remark 4.1. Note that the 'one-share' assumption can be easily relaxed. For example, one can consider any pairs $\mathbf{Z}$ consisting of n_1 shares of long position in $\mathbf{X}^1$ and n_2 shares of short position in $\mathbf{X}^2$. This case can be treated by changing of the state variables $(X^1_t, X^2_t) \to (n_1X^1_t, n_2X^2_t)$. Due to the nature of GBMs, the corresponding system equation in (21) will stay the same. The new allocations will only affect the reward function in (22) implicitly. In addition, we only focus on the 'long' side of pairs trading and note that the 'short' side of trading can also be treated by simply switching the roles of the two stocks $\mathbf{X}^1$ and $\mathbf{X}^2$.

Example 4.2. In this example, we consider stock prices of Target Corp. (TGT) and Wal-Mart Stores Inc. (WMT). In Figure 5, daily closing prices of both stocks from 1985 to 2014 are plotted. The data is divided into two parts. The first part (1985-1999) will be used to calibrate the model and the second part (2000-2014) to backtest the performance of our results. Using the prices (1985-1999) and following the traditional least squares method, we obtain $\mu_1 = 0.2059$, $\mu_2 = 0.2459$, $\sigma_{11} = 0.3112$, $\sigma_{12} = 0.0729$, $\sigma_{21} = 0.0729$, $\sigma_{22} = 0.2943$.

In this section, we assume $\rho > \mu_1$ and $\rho > \mu_2$. Under these conditions, we can show that, for all $x_1, x_2 > 0$,

$$0 \le V_0(x_1,x_2) \le x_2, \quad \text{and} \quad \beta_{\rm s}x_1 - \beta_{\rm b}x_2 \le V_1(x_1,x_2) \le \beta_{\rm b}x_1 + Kx_2.$$

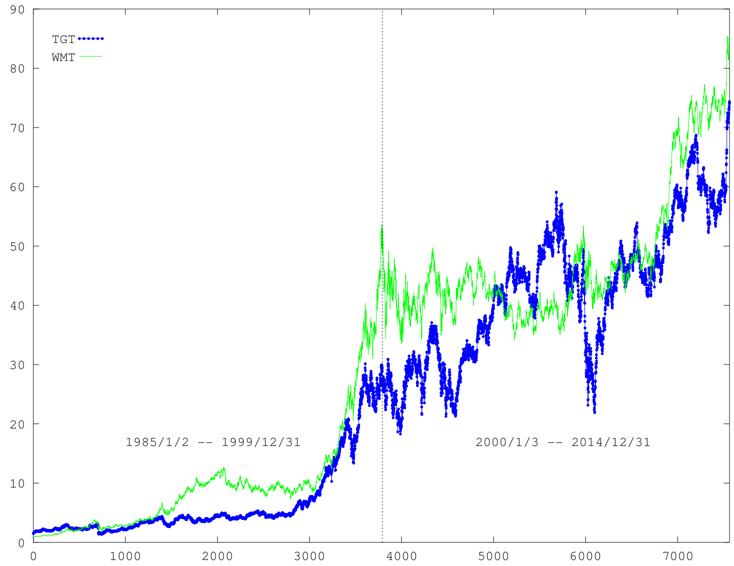

FIGURE 5. Daily Closing Prices of TGT and WMT from 1985 to 2014.

Next, we consider the associated HJB equations. Let

$$\mathcal{A} = \frac{1}{2}\left\{a_{11}x_1^2\frac{\partial^2}{\partial x_1^2} + 2a_{12}x_1x_2\frac{\partial^2}{\partial x_1\partial x_2} + a_{22}x_2^2\frac{\partial^2}{\partial x_2^2}\right\} + \mu_1x_1\frac{\partial}{\partial x_1} + \mu_2x_2\frac{\partial}{\partial x_2},$$

where $a_{11} = \sigma_{11}^2 + \sigma_{12}^2$, $a_{12} = \sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}$, and $a_{22} = \sigma_{21}^2 + \sigma_{22}^2$. Formally, the associated HJB equations have the form: For $x_1, x_2 > 0$,
(4.3)

$$\begin{aligned} &\min\left\{\rho v_0(x_1,x_2) - \mathcal{A}v_0(x_1,x_2),\ v_0(x_1,x_2) - v_1(x_1,x_2) + \beta_{\rm b}x_1 - \beta_{\rm s}x_2\right\} = 0,\\ &\min\left\{\rho v_1(x_1,x_2) - \mathcal{A}v_1(x_1,x_2),\ v_1(x_1,x_2) - v_0(x_1,x_2) - \beta_{\rm s}x_1 + \beta_{\rm b}x_2\right\} = 0. \end{aligned}$$

We divide the first quadrant $P = \{(x_1,x_2):\ x_1 > 0 \text{ and } x_2 > 0\}$ into three regions $\Gamma_1 = \{(x_1,x_2) \in P:\ x_2 \le k_1x_1\}$, $\Gamma_2 = \{(x_1,x_2) \in P:\ k_1x_1 < x_2 < k_2x_1\}$, and $\Gamma_3 = \{(x_1,x_2) \in P:\ x_2 \ge k_2x_1\}$. This is illustrated in Figure 6.

We can also solve the HJB equations and show the following theorem.

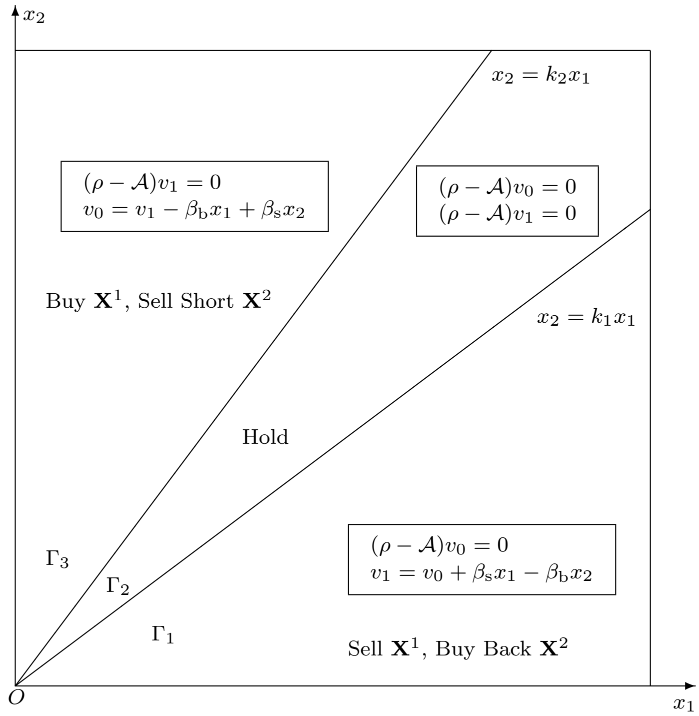

FIGURE 6. Regions Γ_1, Γ_2, and Γ_3

Theorem 4.3. *The functions* $v_0(x_1, x_2) = x_1 w_0(x_2/x_1)$ *and* $v_1(x_1, x_2) = x_1 w_1(x_2/x_1)$ *satisfy the original HJB equations (23) where*

$$w_0(y) = \begin{cases} \left(\dfrac{\beta_b(1-\delta_2)k_1^{1-\delta_1} + \beta_s\delta_2 k_1^{-\delta_1}}{\delta_1 - \delta_2}\right) y^{\delta_1}, & \text{if } 0 < y < k_2, \\ \left(\dfrac{\beta_b(1-\delta_1)k_1^{1-\delta_2} + \beta_s\delta_1 k_1^{-\delta_2}}{\delta_1 - \delta_2}\right) y^{\delta_2} + \beta_s y - \beta_b, & \text{if } y \geq k_2, \end{cases}$$

$$w_1(y) = \begin{cases} \left(\dfrac{\beta_b(1-\delta_2)k_1^{1-\delta_1} + \beta_s\delta_2 k_1^{-\delta_1}}{\delta_1 - \delta_2}\right) y^{\delta_1} + \beta_s - \beta_b y, & \text{if } 0 < y \leq k_1, \\ \left(\dfrac{\beta_b(1-\delta_1)k_1^{1-\delta_2} + \beta_s\delta_1 k_1^{-\delta_2}}{\delta_1 - \delta_2}\right) y^{\delta_2}, & \text{if } y > k_1, \end{cases}$$

$$\begin{aligned} \delta_1 &= \frac{1}{2}\left(1 + \frac{\mu_1 - \mu_2}{\lambda} + \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda}}\right) > 1, \\ \delta_2 &= \frac{1}{2}\left(1 + \frac{\mu_1 - \mu_2}{\lambda} - \sqrt{\left(1 + \frac{\mu_1 - \mu_2}{\lambda}\right)^2 + \frac{4\rho - 4\mu_1}{\lambda}}\right) < 0, \end{aligned} \tag{4.4}$$

$$k_1 = \frac{\delta_2(\beta_{\rm b} r_0^{-\delta_1} - \beta_{\rm s})}{(1-\delta_2)(\beta_{\rm b} - \beta_{\rm s} r_0^{1-\delta_1})}, k_2 = \frac{\delta_2(\beta_{\rm b} r_0^{1-\delta_1} - \beta_{\rm s} r_0)}{(1-\delta_2)(\beta_{\rm b} - \beta_{\rm s} r_0^{1-\delta_1})}, \tag{4.5}$$

and $r_0 > (\beta_{\rm b}/\beta_{\rm s})^2$

$$\begin{aligned} 0 &= f(r_0) \\ &= \delta_1(1-\delta_2)(\beta_{\rm b} r^{-\delta_2} - \beta_{\rm s})(\beta_{\rm b} - \beta_{\rm s} r^{1-\delta_1}) - \delta_2(1-\delta_1)(\beta_{\rm b} r^{-\delta_1} - \beta_{\rm s})(\beta_{\rm b} - \beta_{\rm s} r^{1-\delta_2}). \end{aligned}$$

The optimal trading rule can be determined by two threshold curves as follows:

Theorem 4.4. *We have* $v_i(x_1, x_2) = x_1 w_i(x_2/x_1) = V_i(x_1, x_2)$, $i = 0, 1$. *Moreover, if initially* $i = 0$, *let* $\Lambda_0^* = (\tau_1^*, \tau_2^*, \tau_3^*, \ldots)$ *such that* $\tau_1^* = \inf\{t \geq 0 : (X_t^1, X_t^2) \in \Gamma_3\}$, $\tau_2^* = \inf\{t \geq \tau_1^* : (X_t^1, X_t^2) \in \Gamma_1\}$, $\tau_3^* = \inf\{t \geq \tau_2^* : (X_t^1, X_t^2) \in \Gamma_3\}$, *and so on. Similarly, if initially* $i = 1$, *let* $\Lambda_1^* = (\tau_0^*, \tau_1^*, \tau_2^*, \ldots)$ *such that* $\tau_0^* = \inf\{t \geq 0 : (X_t^1, X_t^2) \in \Gamma_1\}$, $\tau_1^* = \inf\{t \geq \tau_0^* : (X_t^1, X_t^2) \in \Gamma_3\}$, $\tau_2^* = \inf\{t \geq \tau_1^* : (X_t^1, X_t^2) \in \Gamma_1\}$, *and so on. Then* Λ_0^* *and* Λ_1^* *are optimal.*

Next, we backtest our pairs trading rule using the stock prices of TGT and WMT from 2000 to 2014. Using the parameters obtained in Example 4.2 based on the historical prices from 1985 to 1999, we found the pair $(k_1, k_2) = (1.03905, 1.28219)$. A pairs trading (long $\mathbf{X}^1$ and short $\mathbf{X}^2$) is triggered when (X_t^1, X_t^2) enters Γ_3. The position is closed when (X_t^1, X_t^2) enters Γ_1. Initially, we allocate trading the capital \$100K. When the first long signal is triggered, buy \$50K TGT stocks and short the same amount of WMT. Such half-and-half capital allocation between long and short applies to all trades. In addition, each pairs transaction is charged \$5 commission. In Figure 8, the corresponding ratio X_t^2/X_t^1, the threshold levels k_1 and k_2, and the corresponding equity curve are plotted. There are total 3 trades and the end balance is \$155.914K.

We can also switch the roles of $\mathbf{X}^1$ and $\mathbf{X}^2$, i.e., to long WMT and short TGT by taking $\mathbf{X}^1$=WMT and $\mathbf{X}^2$=TGT. In this case, the new $(\tilde{k}_1, \tilde{k}_2) = (1/k_2, 1/k_1) = (1/1.28219, 1/1.03905)$. These levels and the corresponding equity curve is given in Figure 8. Such trade leads to the end balance \$132.340K. Note that both types of trades have no overlap, i.e., they do not compete for the same capital. The grand total profit is \$88254 which is a 88.25% gain.

Note also that there are only 5 trades in the fifteen year period leaving the capital in cash most of the time. This is desirable because the cash sitting in the account can be used for other types of shorter term trading in between, at least drawing interest over time.

5. Conclusions

In this paper, we have studied the pairs trading problems under both mean reversion and geometric Brownian motion models. We were able to obtain closed-form solutions. The trading rules are given in terms of threshold levels and are extremely simple in structure. The major advantage of pairs trading is its risk-neutral nature, i.e., it can be profitable regardless general market directions. Some initial efforts in connection with numerical computations and implementation have been done in Luu [12]. In particular, stochastic approximation techniques (see Kushner and Yin

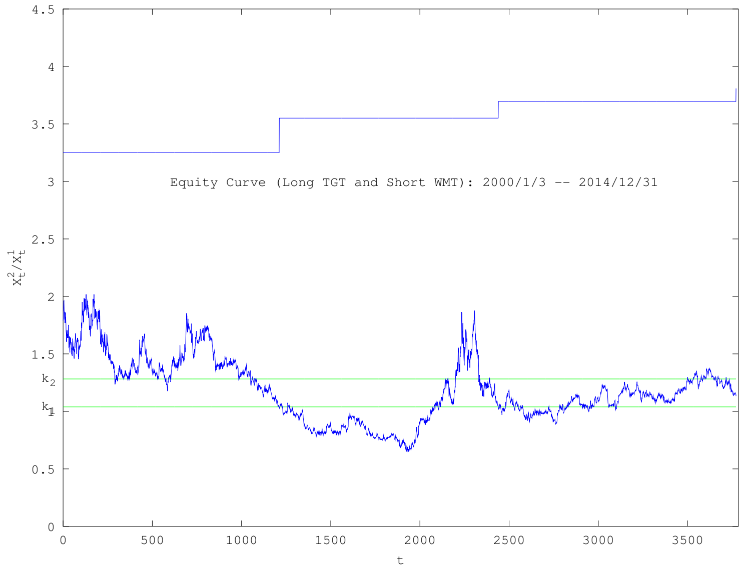

FIGURE 7. $\mathbf{X}^1$=TGT, $\mathbf{X}^2$=WMT: The threshold levels k_1, k_2 and the corresponding equity curve

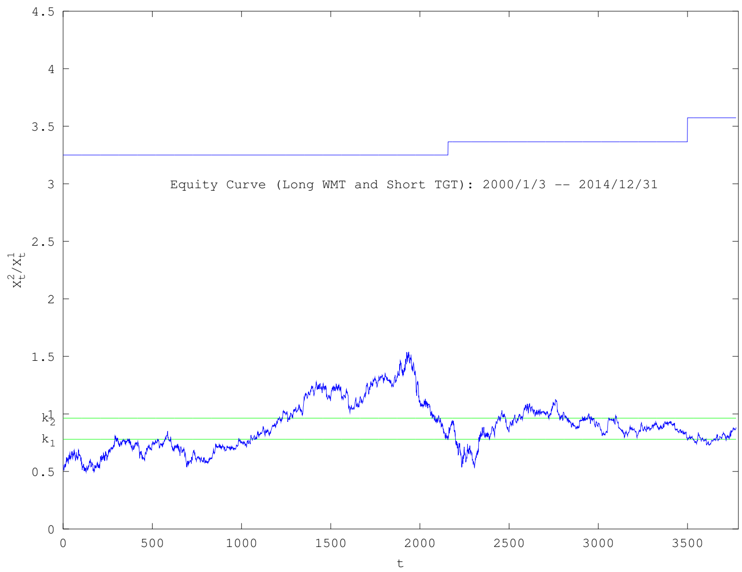

FIGURE 8. $\mathbf{X}^1$=WMT, $\mathbf{X}^2$=TGT: The threshold levels k_1, k_2 and the corresponding equity curve

[11]) can be used effectively to estimate these threshold levels directly. Finally, it would be interesting to examine how these methods work through backtests for a larger selection of stocks.

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