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THE SPLIT COMMON FIXED POINT PROBLEM AND STRONG CONVERGENCE THEOREMS BY HYBRID METHODS FOR NEW DEMIMETRIC MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we consider the split common fixed point problem in Hilbert spaces. Then using the hybrid method and the shrinking projection method, we prove strong convergence theorems for new demimetric mappings in Hilbert spaces. Using these theorems, we obtain well-known and new strong convergence theorems in Hilbert spaces.

1. Introduction

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [30] if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2.$$

Then we have from [30] that the set $F(U)$ of fixed points of U is nonempty, closed and convex. Using this property, we proved weak and strong convergence theorems in Hilbert spaces and Banach spaces; see [15, 26, 30, 33]. Very recently, Kawasaki and Takahashi [11] generalized the concept of demimetric mappings as follows: Let θ be a real number with $\theta \neq 0$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called generalized demimetric [11] if

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2 \tag{1.1}$$

for all $x \in C$ and $q \in F(U)$. This mapping U is called θ-generalized demimetric. We can also prove that the set $F(U)$ of fixed points of such a mapping U is nonempty, closed and convex; see [11].

Let H_1 and H_2 be two real Hilbert spaces. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [7] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Byrne, Censor, Gibali and Reich [6] considered the following problem: Given two set-valued mappings $G : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$, and a bounded linear operator $A : H_1 \to H_2$, the *split common null point problem* [6] is to find a point $z \in H_1$ such that

$$z \in G^{-1}0 \cap A^{-1}(B^{-1}0),$$

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where $G^{-1}0$ and $B^{-1}0$ are null point sets of G and B, respectively. Given two mappings $T : H_1 \to H_1$ and $U : H_2 \to H_2$, and a bounded linear operator $A : H_1 \to H_2$, the *split common fixed point problem* [8, 19] is to find a point $z \in H_1$ such that $z \in F(T) \cap A^{-1}F(U)$, where $F(T)$ and $F(U)$ are fixed point sets of T and U, respectively.

Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [3], where A^* is the adjoint operator of A and P_Q is the metric projection of H_2 onto Q. Furthermore, if $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.2}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Using such results regarding nonlinear operators and fixed points, many authors have studied the split feasibility problem in Hilbert spaces; see, for instance, [1, 3, 6, 8, 19, 35].

On the other hand, by using the hybrid method by Nakajo and Takahashi [20] and the shrinking projection method by Takahashi, Takeuchi and Kubota [31], many authors have obtained strong convergence theorems in Hilbert spaces and Banach spaces; see, for instance, [2, 9, 21, 26, 27, 28, 29, 32].

In this paper, motivated by these problems and results in Hilbert spaces and Banach spaces, we consider the split common fixed point problem for generalized demimetric mappings in Hilbert spaces. Then using the hybrid method and the shrinking projection method, we prove two strong convergence theorems for finding a solution of the split common point problem in Hilbert spaces. Using these theorems, we obtain well-known and new strong convergence theorems in Hilbert spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot\ \cdot \rangle$ and norm $\|\cdot\|$, respectively. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have from [22, 24] that

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle; \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.2}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_Cx\| \le \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_Cx - P_Cy\|^2 \le \langle P_Cx - P_Cy, x - y\rangle \tag{2.4}$$

for all $x, y \in H$. Furthermore $\langle x - P_Cx, y - P_Cx\rangle \le 0$ holds for all $x \in H$ and $y \in C$; see [22]. Using this inequality and (2.3), we have that

$$\|P_Cx - y\|^2 + \|P_Cx - x\|^2 \le \|x - y\|^2, \quad \forall x \in H,\ y \in C. \tag{2.5}$$

Let E be a Banach space and let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^* \rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u^* \in Bx$, and $v^* \in By$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [4]; see also [23, Theorem 3.5.4].

Theorem 2.1 ([4]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rBx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of B.

Let B be a maximal monotone operator on a Hilbert space H. In a Hilbert space H, the metric resolvent J_r of B is called the resolvent of A simply. It is known that the resolvent J_r of B for $r > 0$ is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y \rangle, \quad \forall x, y \in H.$$

The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [23].

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ was called generalized demimetric [11] if it satisfies (1.1), i.e.,

$$\theta \langle x - q, J(x - Ux) \rangle \geq \|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$, where J is the duality mapping on E.

Examples 2.2. We know examples of generalized demimetric mappings.

(1) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a k-strict pseudo-contraction [5] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is $\frac{2}{1-k}$-generalized demimetric; see [11].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called generalized hybrid [12] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{2.6}$$

for all $x, y \in C$. Such a mapping U is called $(\alpha,\ \beta)$-generalized hybrid. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 2-generalized demimetric; see [11]. In fact, setting $x = u \in F(U)$ and $y = x \in C$ in (2.6), we have that

$$\alpha\|u - Ux\|^2 + (1-\alpha)\|u - Ux\|^2 \leq \beta\|u - x\|^2 + (1-\beta)\|u - x\|^2$$

and hence

$$\|Ux - u\|^2 \leq \|x - u\|^2.$$

From $\|Ux - x + x - u\|^2 \leq \|x - u\|^2$, we have that

$$2\langle x - u, x - Ux\rangle \geq \|x - Ux\|^2$$

for all $x \in C$ and $u \in F(U)$. This means that U is 2-generalized demimetric.

Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [13, 14] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [25] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [10].

(3) Let E be a mooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is 1-generalized demimetric; see [11].

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ for $\lambda > 0$ is 1-generalized demimetric; see [11].

(5) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let T be a mapping from C into H. Suppose that T is Lipschitzian, that is, there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$. Let $S = (L+1)I - T$. If $F(\frac{T}{L}) \neq \emptyset$, then S is $(-2L)$-generalized demimetric; see [11].

(6) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $\alpha > 0$. If B be an α-inverse strongly monotone mapping from C into H with $B^{-1}0 \neq \emptyset$, then $T = I + B$ is $\left(-\frac{1}{\alpha}\right)$-generalized demimetric; see [11].

The following lemmas are important and crucial in the proofs of our main results.

Lemma 2.3 ([11]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. If a mapping $U : C \to E$ is θ-generalized demimetric and $\theta > 0$, then U is $\left(1 - \frac{2}{\theta}\right)$-demimetric.*

Lemma 2.4 ([11]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping of C into E. Then $F(T)$ is closed and convex.*

Lemma 2.5 ([11]). *Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping from C into E and let $k \in \mathbb{R}$ with $k \neq 0$. Then $(1-k)I + kT$ is θk-generalized demimetric from C into E.*

We also know the following lemma from [33]:

Lemma 2.6 ([33]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 1)$ and let T be a k-demimetric mapping of C into H such that $F(T)$ is nonempty. Let λ be a real number with $0 < \lambda \leq 1-k$ and define $S = (1-\lambda)I + \lambda T$. Then S is a quasi-nonexpansive mapping of C into H.*

3. Main results

In this section, using the hybrid method by Nakajo and Takahashi [20], we first prove a strong convergence theorem for finding a solution of the split common fixed point problem in Hilbert spaces.

Theorem 3.1. *Let H_1 and H_2 be Hilbert spaces. Let θ and τ be real numbers with $\theta, \tau \neq 0$. Let $S : H_1 \to H_1$ be a θ-generalized demimetric and demiclosed mapping with $F(S) \neq \emptyset$ and let $T : H_2 \to H_2$ be a τ-generalized demimetric and demiclosed mapping with $F(T) \neq \emptyset$. Let k and h be real numbers with $\theta k > 0$ and $\tau h > 0$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $x_1 \in H_1$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = ((1-\lambda)I + \lambda S)\Big(x_n - rhA^*(Ax_n - TAx_n)\Big), \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_n = \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H_1 : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$, $r \in (0,\infty)$ and $\lambda \in \mathbb{R}$ satisfy the following:

$$0 < a \leq \alpha_n \leq 1, \ 0 < r < \frac{2}{\tau h \|A\|^2} \quad \text{and} \quad 0 < \frac{\lambda}{k} \leq \frac{2}{\theta k}$$

for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(S) \cap A^{-1}F(T)$, where $z_0 = P_{F(S)\cap A^{-1}F(T)} x_1$.

Proof. We first show that $\{x_n\}$ is well defined. Since

$$\begin{aligned} \|y_n - z\| \leq \|x_n - z\| &\iff \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ &\iff \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \leq 0, \end{aligned}$$

it follows that C_n is closed and convex for all $n \in \mathbb{N}$. It is obvious that D_n is closed and convex. Then $C_n \cap D_n$ are closed and convex for all $n \in \mathbb{N}$. Let us show that $F(S) \cap A^{-1}F(T) \subset C_n$ for all $n \in \mathbb{N}$. Let $z \in F(S) \cap A^{-1}F(T)$. Then $z = Sz$ and $Az = TAz$. Since $T : H_2 \to H_2$ is τ-generalized demimetric, we have from Lemma 2.5 that $(1-h)I + hT$ is τh-generalized demimetric. Since $S : H_1 \to H_1$ is θ-generalized demimetric, we also have from Lemma 2.5 that

$(1-k)I + kS$ is θk-generalized demimetric. Furthermore, from Lemma 2.3 and $\theta k > 0$, we have that $(1-k)I+kS$ is $\left(1-\frac{2}{\theta k}\right)$-demimetric in the sense of [30]. Since $0 < \frac{\lambda}{k} \leq \frac{2}{\theta k} = 1 - \left(1 - \frac{2}{\theta k}\right)$ and

$$(1-\lambda)I + \lambda S = \left(1 - \frac{\lambda}{k}\right) I + \frac{\lambda}{k}((1-k)I + kS),$$

we have from Lemma 2.6 that $(1-\lambda)I+\lambda S$ is quasi-nonexpansive. Since $(1-\lambda)I+\lambda S$ is quasi-nonexpansive, we have that for $z \in F(S) \cap A^{-1}F(T)$,

$$\begin{aligned}
\|z_n - z\|^2 &= \|((1-\lambda)I + \lambda S)\Big(x_n - rhA^*(Ax_n - TAx_n)\Big) - ((1-\lambda)I + \lambda S)z\|^2 \\
&\leq \|x_n - rhA^*(Ax_n - TAx_n) - z\|^2 \\
&= \|x_n - z - rhA^*(Ax_n - TAx_n)\|^2 \\
&= \|x_n - z\|^2 - 2\langle x_n - z, rhA^*(Ax_n - TAx_n)\rangle \\
&\qquad + \|rhA^*(Ax_n - TAx_n)\|^2 \\
&\leq \|x_n - z\|^2 - 2rh\langle Ax_n - Az, Ax_n - TAx_n\rangle \\
&\qquad + r^2h^2\|A\|^2\|Ax_n - TAx_n\|^2 \qquad (3.1) \\
&= \|x_n - z\|^2 - 2r\langle Ax_n - Az, Ax_n - ((1-h)I + hT)Ax_n\rangle \\
&\qquad + r^2h^2\|A\|^2\|Ax_n - TAx_n\|^2 \\
&\leq \|x_n - z\|^2 - 2r\frac{1}{\tau h}\|Ax_n - ((1-h)I + hT)Ax_n\|^2 \\
&\qquad + r^2h^2\|A\|^2\|Ax_n - TAx_n\|^2 \\
&\leq \|x_n - z\|^2 - 2rh^2\frac{1}{\tau h}\|Ax_n - TAx_n\|^2 + r^2h^2\|A\|^2\|Ax_n - TAx_n\|^2 \\
&= \|x_n - z\|^2 + rh^2\Big(r\|A\|^2 - \frac{2}{\tau h}\Big)\|Ax_n - TAx_n\|^2 \\
&\leq \|x_n - z\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\|y_n - z\| &= \|\alpha_n x_n + (1-\alpha_n)z_n - z\| \\
&\leq \alpha_n\|x_n - z\| + (1-\alpha_n)\|z_n - z\| \\
&\leq \alpha_n\|x_n - z\| + (1-\alpha_n)\|x_n - z\| \\
&\leq \|x_n - z\|.
\end{aligned}$$

Therefore, $F(S)\cap A^{-1}F(T) \subset C_n$ for all $n \in \mathbb{N}$. Let us show that $F(S)\cap A^{-1}F(T) \subset D_n$ for all $n \in \mathbb{N}$. It is obvious that $F(S) \cap A^{-1}F(T) \subset D_1$. Suppose that $F(S) \cap A^{-1}F(T) \subset D_j$ for some $j \in \mathbb{N}$. Then $F(S) \cap A^{-1}F(T) \subset C_j \cap D_j$. From $x_{j+1} = P_{C_j\cap D_j}x_1$, we have that

$$\langle x_{j+1} - z, x_1 - x_{j+1}\rangle \geq 0, \quad \forall z \in C_j \cap D_j$$

and hence

$$\langle x_{j+1} - z, x_1 - x_{j+1}\rangle \geq 0, \quad \forall z \in F(S) \cap A^{-1}F(T).$$

Then $F(S) \cap A^{-1}F(T) \subset D_{j+1}$. We have by induction that $F(S) \cap A^{-1}F(T) \subset D_n$ for all $n \in \mathbb{N}$. Thus, we have that $F(S) \cap A^{-1}F(T) \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $F(S) \cap A^{-1}F(T)$ is nonempty, closed and convex, there exists $z_0 \in F(S) \cap A^{-1}F(T)$ such that $z_0 = P_{F(S)\cap A^{-1}F(T)}x_1$. From $x_{n+1} = P_{C_n \cap D_n}x_1$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|$$

for all $y \in C_n \cap D_n$. Since $z_0 \in F(S) \cap A^{-1}F(T) \subset C_n \cap D_n$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z_0\|. \tag{3.2}$$

This means that $\{x_n\}$ is bounded.

Next we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From the definition of D_n, we have that $x_n = P_{D_n}x_1$. From $x_{n+1} = P_{C_n \cap D_n}x_1$ we have $x_{n+1} \in D_n$. Thus

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$$

for all $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. From $x_{n+1} \in D_n$ we have that

$$\langle x_n - x_{n+1}, x_1 - x_n\rangle \geq 0.$$

This implies from (2.3) that

$$0 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - \|x_{n+1} - x_n\|^2$$

and hence

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.$$

Since there exists the limit of $\{\|x_1 - x_n\|\}$, we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3.3}$$

From $x_{n+1} \in C_n$, we also have that $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Then we get from (3.3) that $\|y_n - x_{n+1}\| \to 0$. Using this, we have that

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{3.4}$$

We have from (3.1) that for any $z \in F(S) \cap A^{-1}F(T)$,

$$\begin{aligned}
\|y_n - z\|^2 &= \|(1-\alpha_n)x_n + \alpha_n z_n - z\|^2 \\
&\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n \|z_n - z\|^2 \\
&\leq (1-\alpha_n)\|x_n - z\|^2 + \alpha_n \|x_n - z\|^2 \\
&\qquad + \alpha_n r h^2 \Big(r\|A\|^2 - \frac{2}{\tau h}\Big)\|Ax_n - TAx_n\|^2 \\
&\leq \|x_n - z\|^2 + \alpha_n r h^2 \Big(r\|A\|^2 - \frac{2}{\tau h}\Big)\|Ax_n - TAx_n\|^2.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
\alpha_n r h^2 \Big(\frac{2}{\tau h} - r\|A\|^2\Big)\|Ax_n - TAx_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\
&= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\
&\leq (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|.
\end{aligned}$$

From $\|y_n - x_n\| \to 0$, $0 < a \leq \alpha_n \leq 1$ and $0 < r\|A\|^2 < \frac{2}{\tau h}$, we have that

$$\lim_{n\to\infty} \|Ax_n - TAx_n\|^2 = 0. \tag{3.5}$$

We also have that

$$\|y_n - x_n\| = \|(1-\alpha_n)x_n + \alpha_n z_n - x_n\| = \alpha_n\|z_n - x_n\| \geq a\|z_n - x_n\|.$$

From $\|y_n - x_n\| \to 0$, we have that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.6}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. From (3.4) $\{y_{n_i}\}$ converges weakly to w. Furthermore, from (3.6) $\{z_{n_i}\}$ converges weakly to w. Since A is bounded and linear, we also have that $\{Ax_{n_i}\}$ converges weakly to Aw. Using this and $\lim_{n\to\infty} \|Ax_n - TAx_n\| = 0$, we have from the demiclosedness of T that $Aw = TAw$. This implies that $Aw \in F(T)$ and hence $w \in A^{-1}F(T)$. We also prove $w \in F(S)$. Putting $t_n = x_n - rhA^*(Ax_n - TAx_n)$, we have that

$$\|t_n - z_n\| = \|t_n - ((1-\lambda)I + \lambda S)t_n\| = \|\lambda(t_n - St_n)\| = |\lambda|\|t_n - St_n\|.$$

Furthemore, we have that $\|t_n - x_n\| = \|rhA^*(Ax_n - TAx_n)\| \to 0$. We have from $\|t_n - z_n\| \leq \|t_n - x_n\| + \|x_n - z_n\|$ that $\|t_n - z_n\| \to 0$. This implies that

$$\lim_{n\to\infty} \|t_n - St_n\| = 0. \tag{3.7}$$

Since $\|t_n - x_n\| \to 0$, we also have that $\{t_{n_i}\}$ converges weakly to w. From the demiclosedness of S, we have that $w = Sw$ and hence $w \in F(S)$. This implies that $w \in F(S) \cap A^{-1}F(T)$.

From $z_0 = P_{F(S)\cap A^{-1}F(T)}x_1$, $w \in F(S) \cap A^{-1}F(T)$ and (3.2), we have that

$$\begin{aligned}\|x_1 - z_0\| \leq \|x_1 - w\| &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - z_0\|.\end{aligned}$$

Then we get that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_0\|$$

and hence $w = z_0$. Furthermore, from the Kadec-Klee property of H_1, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence

$$x_{n_i} \to w = z_0.$$

Therefore, we have $x_n \to z_0$. This completes the proof. □

Next, using the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [31], we prove a strong convergence theorem for finding a solution of the split common fixed point problem in Hilbert spaces.

Theorem 3.2. *Let H_1 and H_2 be Hilbert spaces. Let θ and τ be real numbers with $\theta, \tau \neq 0$. Let $S : H_1 \to H_1$ be a θ-generalized demimetric and demiclosed mapping with $F(S) \neq \emptyset$ and let $T : H_2 \to H_2$ be a τ-generalized demimetric and demiclosed mapping with $F(T) \neq \emptyset$. Let k and h be real numbers with $\theta k > 0$ and $\tau h > 0$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$*

be a sequence in H_1 such that $u_n \to u$. Let $x_1 \in H_1$ and $C_1 = H_1$. Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = ((1-\lambda)I + \lambda S)\Big(x_n - rhA^*(Ax_n - TAx_n)\Big), \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_{n+1} = \{z \in H_1 : \|y_n - z\| \le \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$, $r \in (0,\infty)$ and $\lambda \in \mathbb{R}$ satisfy the following:

$$0 < a \le \alpha_n \le 1, \ 0 < r < \frac{2}{\tau h\|A\|^2} \quad \textit{and} \quad 0 < \frac{\lambda}{k} \le \frac{2}{\theta k}$$

for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $w_0 \in F(S) \cap A^{-1}F(T)$, where $w_0 = P_{F(S)\cap A^{-1}F(T)}u$.

Proof. We first show that the sequence $\{x_n\}$ is well defined. Let $x_1 \in H_1$. We have that $C_1 = H_1$ is closed and convex and $F(S) \cap A^{-1}F(T) \subset C_1$. Suppose that $F(S) \cap A^{-1}F(T) \subset C_j$, C_j is closed and convex and x_j is defined for some $j \in \mathbb{N}$. Let $z_j = ((1-\lambda)I + \lambda S)\Big(x_j - rhA^*(Ax_i - TAx_j)\Big)$ and let $y_j = (1-\alpha_j)x_i + \alpha_j z_j$,. Since

$$\begin{aligned} \{z \in H_1 : \|y_j - z\| \le \|x_j - z\|\} &= \{z \in H_1 : \|y_j - z\|^2 \le \|x_j - z\|^2\} \\ &= \{z \in H_1 : \|y_j\|^2 - \|x_j\|^2 \le 2\langle y_j - x_j, z\rangle\}, \end{aligned}$$

it is closed and convex. We show that $F(S) \cap A^{-1}F(T) \subset C_{j+1}$ for all $n \in \mathbb{N}$. It is obvious that From $0 < r\|A\|^2 < \frac{2}{\tau h}$, we have that for $z \in F(S) \cap A^{-1}F(T)$,

$$\begin{aligned} \|z_j - z\|^2 &= \|((1-\lambda)I + \lambda S)\Big(x_j - rhA^*(Ax_j - TAx_j)\Big) - ((1-\lambda)I + \lambda S)z\|^2 \\ &\le \|x_j - rhA^*(Ax_j - TAx_j) - z\|^2 \\ &= \|x_j - z - rhA^*(Ax_j - TAx_j)\|^2 \\ &= \|x_j - z\|^2 - 2\langle x_j - z, rhA^*(Ax_j - TAx_j)\rangle + \|rhA^*(Ax_j - TAx_j)\|^2 \\ &\le \|x_j - z\|^2 - 2r\langle Ax_j - Az, Ax_j - ((1-h)I + hT)Ax_j\rangle \\ &\qquad + r^2h^2\|A\|^2\|Ax_n - TAx_n)\|^2 \qquad (3.8) \\ &\le \|x_j - z\|^2 - 2r\frac{1}{\tau h}\|Ax_j - ((1-h)I + hT)Ax_j\|^2 \\ &\qquad + r^2h^2\|A\|^2\|Ax_n - TAx_n)\|^2 \\ &= \|x_j - z\|^2 - 2r\frac{1}{\tau h}h^2\|Ax_j - TAx_j\|^2 + r^2h^2\|A\|^2\|Ax_j - TAx_j)\|^2 \\ &= \|x_j - z\|^2 + rh^2(r\|A\|^2 - \frac{2}{\tau h})\|Ax_j - TAx_j\|^2 \\ &\le \|x_j - z\|^2 \end{aligned}$$

and hence

$$\begin{aligned}\|y_j - z\|^2 &= \|(1-\alpha_j)x_j + \alpha_j z_j - z\|^2 \\ &\le (1-\alpha_j)\|x_j - z\|^2 + \alpha_j\|z_j - z\|^2 \\ &\le (1-\alpha_j)\|x_j - z\|^2 + \alpha_j\|x_j - z\|^2 \\ &\le \|x_j - z\|^2.\end{aligned}$$

Therefore, $F(S) \cap A^{-1}F(T) \subset C_{j+1}$. Applying these facts inductively, we obtain that C_n are nonempty, closed and convex for all $n \in \mathbb{N}$, and hence $\{x_n\}$ is well defined.

Since $F(S) \cap A^{-1}F(T)$ is nonempty, closed and convex, there exists $w_0 \in F(S) \cap A^{-1}F(T)$ such that $w_0 = P_{F(S)\cap A^{-1}F(T)}u$. From $w_n = P_{C_n}u$, we have that

$$\|u - w_n\| \le \|u - y\|$$

for all $y \in C_n$. Since $w_0 \in F(S) \cap A^{-1}F(T) \subset C_n$, we have that

$$\|u - w_n\| \le \|u - w_0\|. \tag{3.9}$$

This means that $\{w_n\}$ is bounded. From $w_n = P_{C_n}u$ and $w_{n+1} \in C_{n+1} \subset C_n$, we have that

$$\|u - w_n\| \le \|u - w_{n+1}\|.$$

Thus $\{\|u - w_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|u - w_n\|\}$. Put $\lim_{n\to\infty}\|w_n - u\| = c$. For any $m, n \in \mathbb{N}$ with $m \ge n$, we have $C_m \subset C_n$. From $w_m = P_{C_m}u \in C_m \subset C_n$ and (2.5), we have that

$$\|x_m - P_{C_n}u\|^2 + \|P_{C_n}u - u\|^2 \le \|u - w_m\|^2.$$

This implies that

$$\|w_m - w_n\|^2 \le \|u - w_m\|^2 - \|w_n - u\|^2 \le c^2 - \|w_n - u\|^2. \tag{3.10}$$

Since $c^2 - \|w_n - u\|^2 \to 0$ as $n \to \infty$, we have that $\{w_n\}$ is a Caushy sequence. By the completeness of H_1, there exists a point $z_0 \in H_1$ such that $\lim_{n\to\infty} w_n = z_0$. Since the metric projection P_{C_n} is nonexpansive, it follows that

$$\begin{aligned}\|x_n - z_0\| &\le \|x_n - w_n\| + \|w_n - z_0\| \\ &= \|P_{C_n}u_n - P_{C_n}u\| + \|w_n - z_0\| \\ &\le \|u_n - u\| + \|w_n - z_0\|\end{aligned}$$

and hence

$$x_n \to z_0. \tag{3.11}$$

From $x_{n+1} \in C_{n+1}$, we have that $\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$. We also get from $x_n \to z_0$ that $\|x_{n+1} - x_n\| \to 0$. Then $\|y_n - x_{n+1}\| \to 0$. Using this, we have that

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0.$$

From $y_n - x_n = \alpha_n x_n + (1-\alpha_n)z_n - x_n = (1-\alpha_n)(z_n - x_n)$, we also have that

$$\|y_n - x_n\| = (1-\alpha_n)\|z_n - x_n\| \ge a\|z_n - x_n\|$$

and hence

$$\|z_n - x_n\| \to 0. \tag{3.12}$$

We have that for any $z \in F(S) \cap A^{-1}F(T)$,

$$\begin{aligned}
\|y_n - z\|^2 &= \|(1-\alpha_n)x_n + \alpha_n z_n - z\|^2 \\
&\le (1-\alpha_n)\|x_n - z\|^2 + \alpha_n \|z_n - z\|^2 \\
&\le (1-\alpha_n)\|x_n - z\|^2 + \alpha_n \|x_n - z\|^2 \\
&\quad + \alpha_n r h^2 \Big(r\|A\|^2 - \frac{2}{\tau h}\Big)\|Ax_n - TAx_n\|^2 \\
&\le \|x_n - z\|^2 + \alpha_n r h^2 \Big(r\|A\|^2 - \frac{2}{\tau h}\Big)\|Ax_n - TAx_n\|^2.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
\alpha_n r h^2 \Big(\frac{2}{\tau h} - r\|A\|^2\Big)\|Ax_n - TAx_n\|^2 &\le \|x_n - z\|^2 - \|y_n - z\|^2 \\
&= (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\
&\le (\|x_n - z\| + \|y_n - z\|)\|x_n - y_n\|.
\end{aligned}$$

From $\|y_n - x_n\| \to 0$, $0 < a \le \alpha_n \le 1$ and $0 < r\|A\|^2 < \frac{2}{\tau h}$, we have that

$$\lim_{n\to\infty} \|Ax_n - TAx_n\|^2 = 0. \tag{3.13}$$

Since $x_n \to z_0$ and A is continuous, $Ax_n \to Az_0$ and hence $Ax_n \rightharpoonup Az_0$. Since T is demiclosed and $\lim_{n\to\infty}\|Ax_n - TAx_n\| = 0$, we have $Az_0 = TAz_0$. We show that $z_0 \in F(S)$. Putting $t_n = x_n - rhA^*(Ax_n - TAx_n)$, we have that

$$\|t_n - z_n\| = \|t_n - ((1-\lambda)I + \lambda S)t_n\| = \|\lambda(t_n - St_n)\| = |\lambda|\|t_n - St_n\|.$$

Furthemore, we have that $\|t_n - x_n\| = \|rhA^*(Ax_n - TAx_n)\| \to 0$. We have from $\|t_n - z_n\| \le \|t_n - x_n\| + \|x_n - z_n\|$ and (3.12) that $\|t_n - z_n\| \to 0$. This implies that

$$\lim_{n\to\infty} \|t_n - St_n\| = 0. \tag{3.14}$$

Since $\|t_n - x_n\| \to 0$, we also have that $\{t_{n_i}\}$ converges strongly to z_0 and hence $\{t_{n_i}\}$ converges weakly to z_0. From the demiclosedness of S, we have that $z_0 = Sz_0$ and hence $z_0 \in F(S)$. This implies that $z_0 \in F(S) \cap A^{-1}F(T)$.

From $w_0 = P_{F(S)\cap A^{-1}F(T)}u$, $z_0 \in F(S) \cap A^{-1}F(T)$ and (3.9), we have that

$$\|u - w_0\| \le \|u - z_0\| = \lim_{n\to\infty}\|u - x_n\| = \lim_{n\to\infty}\|u - w_n\| \le \|u - w_0\|.$$

Then we get that $\|u - w_0\| = \|u - z_0\|$ and hence $z_0 = w_0$. Therefore, we have $x_n \to z_0 = w_0$. This completes the proof. □

4. Applications

In this section, using Theorems 3.1 and 3.2, we get new strong convergence theorems which are connected with the split common fixed point problem in Hilbert spaces. We know the following result obtained by Marino and Xu [18]; see also [34].

Lemma 4.1 ([18]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \le k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following result from Kocourek, Takahashi and Yao [12]; see also [36].

Lemma 4.2 ([12]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Theorem 4.3. *Let H_1 and H_2 be Hilbert spaces. Let t be a real number with $t \in [0, 1)$. Let $S : H_1 \to H_1$ be a generalized hybrid mapping with $F(S) \neq \emptyset$ and let $T : H_2 \to H_2$ be a t-strict pseud-contraction with $F(T) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $x_1 \in H_1$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = S\Big(x_n - rA^*(Ax_n - TAx_n)\Big), \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_n = \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H_1 : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ and $r \in (0, \infty)$ satisfy the conditions:

$$0 < a \leq \alpha_n \leq 1 \quad \text{and} \quad 0 < r\|A\|^2 < 1 - t$$

for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap A^{-1}F(T)$, where $z_0 = P_{F(S) \cap A^{-1}F(T)} x_1$.

Proof. Since S is a generalized mapping with $F(S) \neq \emptyset$, from (2) in Examples, S is 2-generalized demimetric. We also have from Lemma 4.2 that S is demiclosed. On the other hand, since T is a t-strict pseud-contraction with $F(T) \neq \emptyset$, from (1) in Examples, T is $\frac{2}{1-t}$-generalized demimetric. It follows from Lemma 4.1 that T is demiclosed. Therefore, we have the desired result from Theorem 3.1. $\square$

Using Theorem 3.1, we have the following strong convergence theorem for the split common null point problem in Hilbert spaces.

Theorem 4.4. *Let H_1 and H_2 be Hilbert spaces. Let G and B be maximal monotone operators of H_1 and H_2, respectively. Let J_s and Q_t be the resolvents of G for $s > 0$ and B for $t > 0$, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_s\big(x_n - rA^*(Ax_n - Q_tAx_n)\big), \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < r\|A\|^2 < 1$ and $s, t > 0$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = P_{G^{-1}0 \cap A^{-1}(B^{-1}0)} x_1$.

Proof. Since Q_t is the resolvent of B for $t > 0$, from (4) in Examples, Q_t is 1-generalized demimetric. We also have that if $\{u_n\}$ is a sequence in H_2 such that

$u_n \rightharpoonup z$ and $u_n - Q_t u_n \to 0$, then $z \in F(T) = B^{-1}0$. In fact, since Q_t is the resolvent of B, we have that

$$u_n - Q_t u_n)/t \in BQ_t u_n$$

for all $n \in \mathbb{N}$; see [23]. From the monotonicity of B, we have

$$0 \leq \left\langle u - Q_t u_n, v - \frac{u_n - Q_t u_n}{t} \right\rangle$$

for all $(u, v) \in B$ and $i \in \mathbb{N}$. Taking $n \to \infty$, we get that $\langle u - z, v^* \rangle \geq 0$ for all $(u, v^*) \in B$. Since B is a maximal monotone operator, we have

$$z \in B^{-1}0 = F(Q_t).$$

This means that Q_t is demiclosed. Similarly, since J_s is the resolvent of G, it is 1-generalized demimetric and demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Using Theorem 3.1, we obtain the following strong convergence theorem for demimetric mappings in Hilbert spaces.

Theorem 4.5. *Let H_1 and H_2 be Hilbert spaces. Let ξ and η be real numbers with $\xi, \eta \in (-\infty, 1)$. Let $S : H_1 \to H_1$ be a ξ-demimetric and demiclosed mapping with $F(S) \neq \emptyset$ and let $T : H_2 \to H_2$ be an η-demimetric and demiclosed mapping with $F(T) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $x_1 \in H_1$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = ((1-\lambda)I + \lambda S)\Big(x_n - rA^*(Ax_n - TAx_n)\Big), \\ y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ C_n = \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H_1 : \langle x_n - z, x_1 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$, $r \in (0, \infty)$ and $\lambda \in \mathbb{R}$ satisfy the following:

$$0 < a \leq \alpha_n \leq 1, \ 0 < r\|A\|^2 < 1 - \eta \ \text{ and } \ 0 < \lambda \leq 1 - \xi$$

for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to a point $z_0 \in F(S) \cap A^{-1}F(T)$, where $z_0 = P_{F(S) \cap A^{-1}F(T)} x_1$.

Proof. Since S is a ξ-demimetric mapping of H_1 into H_1 such that $F(S) \neq \emptyset$, S is $\frac{2}{1-\xi}$-generalized demimetric. Take $k = 1$ in Theorem 3.1. Then we get that $\frac{2}{\theta k} = 1 - \xi$ in Theorem 3.1. Furthermore, since T is an η-demimetric mapping of H_2 into H_2 such that $F(T) \neq \emptyset$, T is $\frac{2}{1-\eta}$-generalized demimetric. Take $h = 1$ in Theorem 3.1. Then we get that $\frac{2}{\tau h} = 1 - \eta$ in Theorem 3.1. Therefore, we have the desired result from Theorem 3.1. □

Similarly, using Theorem 3.2, we have the following results.

Theorem 4.6. *Let H_1 and H_2 be Hilbert spaces. Let t be a real number with $t \in [0, 1)$. Let $S : H_1 \to H_1$ be a generalized hybrid mapping and let $T : H_2 \to H_2$ be a t-strict pseud-contraction such that $F(T) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = S\Big(x_n - rA^*(Ax_n - TAx_n)\Big), \\ y_n = (1 - \alpha_n)x_n + \alpha_n z_n, \\ C_{n+1} = \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ and $r \in (0, \infty)$ satisfy the conditions:

$$0 < a \leq \alpha_n \leq 1 \quad \textit{and} \quad 0 < r\|A\|^2 < 1 - t$$

for some $a \in \mathbb{R}$. Then $\{x_n\}$ converges strongly to $z_0 \in F(S) \cap A^{-1}F(T)$, where $z_0 = P_{F(S)\cap A^{-1}F(T)}x_1$.

Theorem 4.7. *Let H_1 and H_2 be Hilbert spaces. Let G and B be maximal monotone operators of H_1 and H_2, respectively. Let J_s and Q_t be the resolvents of G for $s > 0$ and B for $t > 0$, respectively. Let $A : H \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H such that $u_n \to u$. For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_s\big(x_n - \lambda_n A^*(Ax_n - Q_t Ax_n)\big), \\ y_n = (1 - \alpha_n)x_n + \alpha_n z_n, \\ C_{n+1} = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $0 < a \leq \alpha_n \leq 1$, $0 < r\|A\|^2 < 1$ and $s, t > 0$ for some $a \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = P_{G^{-1}0\cap A^{-1}(B^{-1}0)}x_1$.

Theorem 4.8. *Let H_1 and H_2 be Hilbert spaces. Let ξ and η be real numbers with $\xi, \eta \in (-\infty, 1)$. Let $S : H_1 \to H_1$ be a ξ-demimetric and demiclosed mapping with $F(S) \neq \emptyset$ and let $T : H_2 \to H_2$ be an η-demimetric and demiclosed mapping with $F(T) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(S) \cap A^{-1}F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in H_1 such that $u_n \to u$. Let $x_1 \in H_1$ and $C_1 = H_1$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = ((1 - \lambda)I + \lambda S)\Big(x_n - rA^*(Ax_n - TAx_n)\Big), \\ y_n = (1 - \alpha_n)x_n + \alpha_n z_n, \\ C_{n+1} = \{z \in H_1 : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$, $r \in (0,\infty)$ *and* $\lambda \in \mathbb{R}$ *satisfy the following:*

$$0 < a \leq \alpha_n \leq 1,\ 0 < r\|A\|^2 < 1-\eta \quad and \quad 0 < \lambda \leq 1-\xi$$

for some $a \in \mathbb{R}$. *Then* $\{x_n\}$ *converges strongly to a point* $w_0 \in F(S) \cap A^{-1}F(T)$, *where* $w_0 = P_{F(S)\cap A^{-1}F(T)}u$.

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