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CONTINUOUS DEPENDENCE OF THE SOLUTION OF OPTIMAL CONTROL PROBLEMS ON PARAMETER

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Dedicated to the memory of Jonathan M. Borwein

Abstract. We consider a parametrized optimal control problem for differential inclusions defined by a concave multivalued mapping. We proved that under some assumptions we can perturbed the cost function and the constraint multivalued mapping in such a way that the solution of the perturbed problem depends continuously on the parameter. The proof is based on the original non-parametric version introduced by F. Clarke and on a parametric version of the Borwein-Preiss smooth variational principle.

1. Introduction

One of the main tools in nonlinear and nonsmooth analysis are the variational principles in Banach spaces, as Ekeland's variational principle [12, 13] and its smooth generalizations, Borwein-Preiss' [3] and Deville-Godefroy-Zizler's variational principles [9, 10, 11]. Some interesting applications of these variational principles can be found in [4, 5, 10, 11, 16].

Parametric versions of the Ekeland and the Borwein-Preiss variational principles are developed in [14] and [15] respectively. The latter states that under some conditions we can perturb a convex function depending on a parameter with a smooth convex function in such a way, that the minimum point depends continuously on the parameter.

In this paper we present an application of this parametric variational principle to optimal control problems for differential inclusions, defined by concave multivalued mappings.

The main contribution in the paper states that we can perturbe the function and the multivalued mapping in such a way that the solution of the perturbed problem depends continuously on the parameter. We consider the following parameterized optimal control problem:

$$\begin{aligned}&\text{minimize} \quad f(v, x(b)) \;\; \text{for every } v \in V \;\; \text{(parameter space)}\\&\text{under constraints} \quad \dot{x}(t) \in F(t, v, x(t))\\&x(a) = c(v),\end{aligned}$$

where $f(v, .)$ is a convex function, the functions $\{f(., x) : x \in Y\}$ are equi-continuous for any bounded $Y \subset \mathbf{R}^n$, c is continuous mapping, the multivalued mapping $(t, v, y) \mapsto F(t, v, y)$ is measurable with respect to t, equi-Hausdorff continuous with

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respect to v and Lipschitz with respect to y with the following specific property: $F(t, v, .)$ is a concave multifunction, i.e. for every $x_1, x_2 \in E, t \in [a, b], v \in V$ and $l \in [0, 1]$ we have

$$\lambda F(t, v, x_1) + (1 - l)F(t, v, x_2) \subset F(t, v, \lambda x_1 + (1 - l)x_2)$$

This property is equivalent to convexity of the graph of $F(t, v, .)$. A particular case of this property is satisfied by so called linear relations [17, 18]. All conditions on F are given in Section 3. Then for any $\varepsilon > 0$ we can find a convex and smooth in x perturbation $\Delta(v, x)$ of f such that the perturbed problem:

$$\begin{aligned} &\text{minimize} \quad f(v, x(b)) + \varepsilon\Delta(v, x(b)) \;\; \text{for every } v \in V \\ &\text{under constraints} \quad \dot{x}(t) \in F(t, v, x(t)) + \varepsilon\overline{B}_{\mathbf{R}^n} \\ &x(a) = c(v), \end{aligned}$$

has a solution depending continuously of the parameter v. The proof is based on the original non-parametric version introduced by F. Clarke [7] and on a parametric version of the Borwein-Preiss smooth variational principle [14].

2. Parametric Borwein-Preiss variational principle with constarints

In this section we include, for completeness, the main results from [14], which are essential for obtaining the main result concerning perturbed optimal control problems.

Let $(E, \|.\|)$ be a Banach space, B the open unit ball in E and Y a convex subset of E.

Recall that a function $g : Y \to \mathbf{R}$ is quasi-convex if its sublevel sets $L(g, \alpha) := \{x \in Y : g(x) \leq \alpha\}$ are convex for every $\alpha \in \mathbf{R}$. Equivalently, g is quasi-convex if $g(lx + (1 - \lambda)y) \leq \max\{g(x), g(y)\}$, for every $x, y \in Y, \lambda \in [0, 1]$. Recall that a multuvalued mapping $F : T \to E$, where T is a topological space, is lower semi-continuous at x_0, if for every open V with $V \cap F(x_0) \neq \emptyset$ there exist an open $U \ni x_0$ such that $F(x) \cap V \neq \emptyset$ for every $x \in U$. Denote by $\mathbf{R}_+$ the set of all positive numbers and by 2^Y the set of all non-empty subsets of the set Y.

In the sequel we will use the following lemma, which appears to be a powerful instrument of variational analysis, since the parametric Borwein-Preiss variational principle (see below) is based on it, and it gives simple proofs of: Ky Fan's minimax inequality [2], extension to quasi-convex functions of minimax equalities (see [2], Theorems 6.3.2 and 6.3.4]), Sion's minimax theorem [22], etc.

Lemma 2.1 ([15])**.** *Suppose that X is a paracompact topological space, E is a Banach space, $Y \subset E$ is a closed, convex and nonempty subset, $F : X \to 2^Y$ is lower semicontinuous multivalued mapping with convex nonempty images, $\varepsilon : X \to \mathbf{R}_+$ is a continuous function and the functions $f : X \times Y \to \mathbf{R}$, $g : X \to \mathbf{R}$ satisfy the conditions:*

(i) *the function $f(x, .)$ is quasi-convex for every $x \in X$;*
(ii) *the function $f(., y)$ is upper semicontinuous for every $y \in Y$;*
(iii) *g is lower semicontinuous and $g(x) \geq \inf_{y \in (F(x) + \varepsilon(x)B) \cap Y} f(x, y) > -\infty$ for every $x \in X$.*

Then:

(a) *there exists a continuous selection* $\varphi_\varepsilon : X \to Y$ *of the mapping* F_ε *(i.e.* $\varphi_\varepsilon(x) \in F_\varepsilon(x)$ *for every* $x \in X$*), where* $F_\varepsilon(x) = (F(x) + \varepsilon(x)B) \cap Y$*, and*

$$f(x, \varphi_\varepsilon(x)) < g(x) + \varepsilon(x) \quad \forall x \in X. \tag{2.1}$$

(b) *If* $F(x)$ *is open for every* $x \in X$*, then there exists a continuous selection* φ_ε *of* F *satisfying (2.1).*

Below we present a particular case of a parametric Borwein-Preiss principle with constrauns, proved in [15] (Theorem 4.2).

Suppose that the following conditions, denoted collectively by hypothesis (H) are satisfied:

(H_a) P is a compact topological space,

(H_b) E is a Banach space, $C \subset E$ is a closed, convex and nonempty set, $\varepsilon_0 > 0$ is given, $C_0 = C + \varepsilon_0 B$,

(H_c) the function $f : P \times E \to \mathbf{R}$ is convex and continuous with respect to the second variable, and bounded on the bounded subsets of C_0,

(H_d) the family of functions $\{f(., x) : x \in Y\}$ is equi-continuous for any bounded subset $Y \subset C_0$,

(H_e) the multivalued mapping $\mathcal{C} : P \to 2^C$ has closed convex and nonempty values and is lower semi-continuous and Hausdorff upper semi-continuous and $\cup_{p\in P}\mathcal{C}(p)$ is bounded.

Consider the following parameterized minimization problem $\mathcal{P}$:

minimize $f(p, x)$ with respect to x for every $p \in P$,

under constraints: $x \in \mathcal{C}(p)$.

Denote $\text{dist}_Y(x) = \inf_{y\in Y} \|x - y\|$ - the distance from a point x to a set Y.

Theorem 2.2 ([15])**.** *Assume that the hypothesis (H) is satisfied,* $\inf f(p, C_0) > -\infty$ *for every* $p \in P$ *and the norm of the Banach space* E *is Fréchet differentiable on* $E \setminus \{0\}$*. Let the continuous functions* $\varepsilon, l : X \to \mathbf{R}_+$*,* $\gamma : X \to \mathbf{R}$ *and the numbers* $\alpha > 0, q \geq 1$ *be given,* l *be bounded,*

$$0 \leq \gamma(p) < \frac{l(p)}{2q5^{q-1}} \quad \forall p \in P, \tag{2.2}$$

$u : P \to C$ *is a continuous mapping satisfying*

$$u(p) \in \mathcal{C}(p) + \gamma(p)B \quad \text{and} \quad f(p, u(p)) < \inf_{x\in\mathcal{C}(p)+\gamma(p)B} f(p, x) + \varepsilon(p) \quad \forall p \in P. \tag{2.3}$$

Then there exist a continuous selection v *of* $\mathcal{C}$ *(solution set mapping) and a continuous function* $K : P \to \mathbf{R}_+$ *such that:*

(i) $\|v(p) - u(p)\| < l(p), \quad \forall p \in P,$

(ii) *the function* $f(p, x) + \Delta(p, x)$ *attains its minimum over* $\mathcal{C}(p)$ *at* $v(p)$*,* $\forall p \in P$.

(iii) $-\Delta'_x(p, v(p)) \in \partial_x L\big(p, v(p), K(p)\big)$,

where $\partial_x L(p, x, K)$ *is the subdifferential of* L *with respect to the second variable,* L *is the Lagrange function given by* $L(p, x, K) = f(p, x) + K\text{dist}_{\mathcal{C}(p)}(x)$,

$$\Delta(p,x) = \sum_{n=0}^{\infty} \mu_n(p)\|x - x_n(p)\|^q, \tag{2.4}$$

$$\mu_i(p) = \frac{\varepsilon(p) + \alpha + K(p)\gamma(p)}{l(p)^q}\nu_i(p), \quad \textit{for} \quad i \geq 0, \quad \sum_{i=0}^{\infty} \nu_i(p) = 1, \tag{2.5}$$

$\nu_i, x_i : P \to C_0$ *are continuous mappings, and* $x_i(p)$ *converges uniformly for* $p \in P$ *to* $v(p)$.

3. Parameterized optimal control problem for differential inclusions

Denote by $\mathcal{A}C([a,b],\mathbf{R}^n$ the space of all absolutely continuous functions from $[a,b]$ to $\mathbf{R}^n$ and define

$$G(v) = \Big\{x \in \mathcal{A}C([a,b],\mathbf{R}^n) : \dot{x}(t) \in F(t,v,x(t)), a.e., x(a) = c(v)\Big\}.$$

Consider the following parameterized minimization problem $\mathcal{P}(v)$:

minimize $f(v, x(b))$ for every $v \in V$ (parameter space)

under constraints: $x \in G(v)$.

The aim of this section is to show that under a small perturbation of the cost function f, there exists a solution of this problem, which depends continuously on the parameter v. Assume that:

(1) the function $f : V \times \mathbf{R}^n \to \mathbf{R}$, where V is a paracompact space has the properties:
  1.1. $f(v,.)$ is convex and differentiable for every $v \in V$,
  1.2. the functions $\{f(.,z) : z \in Y\}$ are equi-continuous for every bounded subset $Y \subset \mathbf{R}^n$;

(2) the multimap $F : V \times \mathbf{R} \times \mathbf{R}^n \to 2^{\mathbf{R}^n} \setminus \{\emptyset\}$ has convex compact images with smooth boundaries (i.e. at any boundary point of any image of F there is only one supporting hyperplane) and:
  2.1. the multimap $t \mapsto F(v,t,x)$ is measurable for any $x \in \mathbf{R}^n, v \in V$,
  2.2. $F(t,v,.)$ is Lipschitz with a Lipschitz constant $k(t,v)$, which is an integrable function with respect to t and the family $\{k(t,.) : t \in [a,b]\}$ is equi-continuous,
  2.3. $F(t,v,.)$ is a concave multifunction, i.e. for every $x_1, x_2 \in E, t \in [a,b], v \in V$ and $l \in [0,1]$ we have

  $$\lambda F(t,v,x_1) + (1-l)F(t,v,x_2) \subset F(t,v,\lambda x_1 + (1-l)x_2);$$

  This property is equivalent to the convexity of the graph of $F(t,v,.)$.
  2.4. the mappings $\{F(t,.,y) : t \in [a,b], y \in \mathbf{R}^n\}$ are equi-Hausdorff continuous,
  2.5. there is a continuous function $\gamma : V \to \mathbf{R}$ such that $\|z\| \leq \gamma(v)(\|y\|+1)$ for all $(t,y) \in [a,b] \times \mathbf{R}^n$, $z \in F(t,v,y)$ (linear growth condition).

(3) the mapping $c : V \to \mathbf{R}^n$ is continuous.

Denote by $L^2_{\|\cdot\|_{a,b}}$ the space $L^2([a,b],\mathbf{R}^n)$ furnished with the following norm:

$$\|x\|_{a,b} = \Big(\|x\|^2_{L^2([a,b],\mathbf{R}^n)} + ||x(a)||^2_{\mathbf{R}^n} + ||x(b)||^2_{\mathbf{R}^n}\Big)^{1/2},$$

which is Fréchet differentiable on $L^2([a,b],\mathbf{R}^n)\setminus\{0\}$, since the space $L^2([a,b],\mathbf{R}^n)$ is Hilbert.

By the above hypotheses and Theorem 1.11 of Chapter 4 in [8] it follows that $G(v)$ is nonempty compact and convex subset in $L^2([a,b],\mathbf{R}^n)$, and therefore, in $L^2_{\|\cdot\|_{a,b}}$ too.

Lemma 3.1. *The multivalued mapping $G : V \to L^2_{\|\cdot\|_{a,b}}$ is: (a) lower semi-continuous and (b) Hausdorff upper semi-continuous.*

Proof. (a). Let $v_0 \in V$, $x_0 \in G(v_0)$ and $\varepsilon > 0$ be given. Denote

$$K(v) = \exp\Big(\int_a^b k(t,v)dt\Big) \quad\text{and}\quad \rho_F(v,x) = \int_a^b \mathrm{dist}_{F(t,v,x(t))}(\dot{x}(t))dt.$$

Since the functions $\{k(t,.) : t \in [a,b]\}$ are equi-continuous, the function K is continuous, so there exists an open set $\mathcal{O}_1(v_0)$ such that $|K(v) - K(v_0)| < 1$ for every $v \in \mathcal{O}_1(v_0)$. Take $\gamma \in \Big(0, \varepsilon/\big(K(v_0)|a-b|(|a-b|+1)^{1/2}\big)\Big)$. Since the mappings $\{F(t,.,y) : t \in [a,b], y \in \mathbf{R}^n\}$ are Hausdorff equi lower semi-continuous and $c : V \to \mathbf{R}^n$ is continuous, there exists an open set $\mathcal{O}_2(v_0)$ such that

$$F(t,v_0,x_0(t)) \subset F(t,v,x_0(t)) + \gamma B,$$
$$\|c(v) - c(v_0)\| < \varepsilon, \quad \forall v \in \mathcal{O}_2(v_0), \forall t \in [a,b],$$

hence $\rho_F(v,x_0) < \gamma|a-b|$. Using Theorem 3.1.6 in [7], for any $v \in \mathcal{O}_1(v_0)\cap\mathcal{O}_2(v_0)$ we obtain existence of a trajectory x_v of $F(.,v,.)$ such that $x_v(a) = x_0(a)$ and

$$\begin{aligned}
\|x_v - x_0\|^2_{a,b} &= \int_a^b |x_v(t) - x_0(t)|^2 dt + \|x_v - x_0\|^2_{\mathbf{R}^n} \\
&\le (|a-b|+1)\max_{t\in[a,b]} |x_v(t) - x_0(t)|^2 \\
&\le (|a-b|+1)K(v)^2\rho_F(v,x_0)^2 \\
&\le (|a-b|+1)K(v)^2\gamma^2|a-b|^2 \\
&< \varepsilon^2.
\end{aligned}$$

Defining $\tilde{x}_v = x_v + c(v) - c(v_0)$, we have

$$\begin{aligned}
\|\tilde{x}_v - x_0\|_{a,b} &\le \|\tilde{x}_v - x_v\|_{a,b} + \|x_v - x_0\|_{a,b} \\
&\le \varepsilon\Big(1 + (|a-b|+2)^{1/2}\Big),
\end{aligned}$$

which proves the lower semi-continuity of G.

(b). Assume the contrary: G is not Hausdorff upper semi-continuous at some v_0. Then there exist $\varepsilon > 0$ and sequences $\{v_i\}_{i=1}^\infty$, $\{x_i\}_{i=1}^\infty$ such that $v_i \to v_0$ and $x_i \in G(v_i)\setminus(G(v_0)+\varepsilon B_{\|\cdot\|_{a,b}})$. Since the mappings $\{F(t,.,y) : t \in [a,b], y \in \mathbf{R}^n\}$ are Hausdorff equi upper semi-continuous, there exists a subsequence $\{i_k\}$ such that

$$\dot{x}_{i_k}(t) \in F(t,v_{i_k},x_{i_k}(t)) \subset F(t,v_0,x_{i_k}(t)) + B_{\|\cdot\|_{a,b}}/k.$$

The continuity of $\gamma(v)$ in the linear growth condition of $F(t,v,.)$ (see 2.5) and Gronwall's inequality (see Proposition 1.4 of Chapter 4 in [8]) guarantee the uniform boundedness of $\{x_{i_k}(t) : t \in [a,b], k = 1,...\}$, therefore, a uniform bound of $\{x_{i_k}\}$ with respect to the norm $\|.\|_{a,b}$. Further the proof is the same as the proof of Theorem 1.11 in Chapter 4 of [8], proving in such a way existence of subsequence of $\{x_{i_k}\}$ converging uniformly (therefore in the norm $\|.\|_{a,b}$ as well) to a continuous function x_0, which is a trajectory of $F(.,v_0,.)$ and satisfying $x_0(a) = c(v_0)$ (here we used the continuity of c). This means $x_0 \in G(v_0)$, a contradiction. □

Lemma 3.2. *Assume that V is a compact topological space. Then the function $\varphi(v) = \inf_{x\in G(v)+\varepsilon B_{\|\cdot\|_{a,b}}} f(v,x(b))$ is lower semi-continuous on V.*

Proof. Since G is Hausdorff upper semi-continuous and V is compact, $\overline{G(V)+\varepsilon B_{\|\cdot\|_{a,b}}}$ is compact too. Take $v_0 \in V$, $\varepsilon > 0$, an open set $U \ni v_0$ such that $G(v) \subset G(v_0)+\varepsilon B_{\|\cdot\|_{a,b}}$ and $f(v_0,x(b)) - f(v,x(b)) < \varepsilon, \forall x \in G(V)+\varepsilon B_{\|\cdot\|_{a,b}}, v \in U$ (this is possible by 1.2). For any $v \in U$, take $x_v \in G(v)+\varepsilon B_{\|\cdot\|_{a,b}}$ and $y_v \in G(v_0)+\varepsilon B_{\|\cdot\|_{a,b}}$ such that $f(v,x_v(b)) < \varphi(v)+\varepsilon$ and $\|x_v - y_v\|_{a,b} < \varepsilon$. Let K be the Lipschitz constant of $f(v_0,.)$ on $\{x(b) : x \in G(V)+\varepsilon B_{\|\cdot\|_{a,b}}\}$ (such a K exists by boundedness of $f(v_0,.)$ on the bounded subsets and by a basic lemma of convex analysis, see for instance [16], Lemma 5.23, page 742). Then

$$\begin{aligned}\varphi(v_0)-\varphi(v) &< f(v_0,y_v(b)) - f(v,x_v(b)) + \varepsilon\\ &< f(v_0,y_v(b)) - f(v_0,x_v(b)) + 2\varepsilon\\ &< K\|y_v(b)-x_v(b)\|_{\mathbf{R}^n} + 2\varepsilon\\ &< \varepsilon(K+2),\end{aligned}$$

which proves the lower semi-continuity of the function φ. □

Now we state the main result in this paper.

Theorem 3.3. *Assume that V is a compact topological space (a parameter space). Then*

(i) *for every $\varepsilon > 0, q > 1$ and $v \in V$ there exists a solution $x_\varepsilon(v)(.)$ of the perturbed minimization problem $\mathcal{P}_\varepsilon(v)$:*
minimize $f_\varepsilon(v,x(b) = f(v,x(b)) + \Delta(v,x)$ for every $v \in V$
under constraints: $x \in G(v)$, where

$$\Delta(v,x) = \sum_{n=0}^{\infty} \mu_n(v)\|x - x_n(v)\|^q_{a,b}, \tag{3.1}$$

$x_n : V \to L^2_{\|\cdot\|_{a,b}}$ are continuous mappings converging uniformly for V to the continuous mapping $x_\varepsilon : V \to L^2_{\|\cdot\|_{a,b}}$, $\mu_n : V \to [0,1]$ are continuous functions, $\sum_{n=0}^{\infty}\mu_n(v) < \varepsilon$.

(ii) *There exists a mapping $p_\varepsilon : V \to \mathcal{AC}([a,b],\mathbf{R}^n)$ (Hamilton multiplier mapping) such that the following necessary conditions for minimum are satisfied:*

$$1)\ \dot{p}_\varepsilon(v)(t) \in \partial\Big[-H_x\Big(t, x_\varepsilon(v)(t), p_\varepsilon(v)(t), v\Big)\Big] + \varepsilon B_{\|\cdot\|_{a,b}},$$

$\forall v \in V$, a.e. in $[a,b]$, where H is the Hamiltonian:

$$H(t,x,p,v) = \max\{\langle p,w\rangle : w \in F(t,x,v)\}.$$

$$2) \quad p_\varepsilon(v)(b) = -f'_\varepsilon(v, x_\varepsilon(v)(b)) \quad \forall v \in V.$$

(iii) *If for every $v \in V$, $\dot{x}_\varepsilon(v)(t) \in bdF(t, x_\varepsilon(v)(t)$ a.e., then p is continuous, as a mapping from V to $L^2([a,b], \mathbf{R}^n)$.*

Proof. We follow the proof of the nonparametric case (see [7], the proof of Theorem 3.2.6), with some modifications.

Put $l = 3, \alpha = \varepsilon$ and $\gamma(v) = \varepsilon/K(v)$, where $K(v)$ is given by Theorem 2.2. Then γ satisfies (2.2) and $(\varepsilon + K(v)\gamma(v) + \alpha)/l^q < \varepsilon$. We apply Lemma 2.1 (its conditions are satisfied due to Lemmas 2, 3) and obtain a continuous selection $\tilde{x}_\varepsilon$ of the multimap $v \mapsto G(v) + \gamma B$ such that

$$f(v, \tilde{x}_\varepsilon(v)(b)) < \inf_{x \in G(v)+\gamma(v)B} f(v, x(b)) + \gamma(v), \quad \forall v \in V.$$

Applying Theorem 2.2, we obtain a continuous selection x_ε of the mapping $v \mapsto G(v)$ such that

$$f(v, x_\varepsilon(v)(b)) + \Delta(v, x_\varepsilon) = \min_{y \in G(v)} \Big(f(v, y(b)) + \Delta(v,y) \Big) \quad \forall v \in V,$$

where $\Delta(v,x)$ is given by (3.1) and $\sum_{n=0}^{\infty} \mu_n(v) < \varepsilon$. This proves (i).

Further we follow the proof of Theorem 3.2.6 of [7]: the proof of Lemma 2 there is the same in our parametric situation, but in our case the constants involved are continuous functions of the parameter v. Define

$$K_{exp}(v) = \exp\Big(\int_a^b k(t,v)dt \Big) \tag{3.2}$$

$$K_1(v) = K(v)\Big[K_{exp}(v) ln(K_{exp}(v)) + 1 \Big], \tag{3.3}$$

$$K_2(v) = (b-a+1)K_1(v)/K(v), \tag{3.4}$$

$$K_3(v) = K_{exp}(v)K(v), \tag{3.5}$$

$$K_4(v) = (b-a+1)K_{exp}(v), \tag{3.6}$$

$$f_1(v,y) = f(v, y(b)) - f(v, x(v)(b)) + \gamma(v), \tag{3.7}$$

$$f_2(v,y) = \Delta(v,y), \tag{3.8}$$

$\Delta(v,y)$ is given by (3.1),

$$f_3(v,y) = (K_1(v) + \gamma(v)K_2(v)) \Big\| y(a) - c(v) \Big\|_{\mathbf{R}^n}, \tag{3.9}$$

$$f_4(v,y) = \Big(K_3(v) + \gamma(v)K_4(v)\Big)\rho_F(v,y), \tag{3.10}$$

$$\rho_F(v,x) = \int_a^b \operatorname{dist}_{F(t,v,x(t))}(\dot{x}(t))dt, \tag{3.11}$$

which is a continuous function.

The functions $f_i(v,.), i = 1,...,4$ are convex (the convexity of $f_4(v,.)$ is non-trivial and follows from the concavity of $F(t,v,.)$ (condition 2.3) and convexity of the images of F).

So, applying Lemma 2 in the proof or Theorem 3.2.6 of [7] we obtain that $x_\varepsilon(v)(.)$ is a global minimum of the function

$$f_1(v,.) + f_2(v,.) + f_3(v,.) + f_4(v,.).$$

Therefore

$$0 \in \partial\Big(f_1(v,x_\varepsilon(v)) + f_2(v,x_\varepsilon(v)) + f_3(v,x_\varepsilon(v)) + f_4(v,x_\varepsilon(v))\Big). \tag{3.12}$$

The functions $f_i(v,.), i = 1,2,3$ are Fréchet differentiable and we calculate (as in the proof of Theorem 3.2.6 in [7]): for any $y \in L^2([a,b],\mathbf{R}^n)$,

$$f_3'(v,x_\varepsilon(v))(y) = \Big(K_1(v) + \gamma(v)K_2(v)\Big)\Big\langle \|x_\varepsilon(a) - c(v)\|'_{\mathbf{R}^n}, y(a)\Big\rangle_{\mathbf{R}^n}; \tag{3.13}$$

$$f_1'(v,x_\varepsilon(v))(y) = \langle f'\big(v,x_\varepsilon(v)(b)\big), y(b)\rangle_{\mathbf{R}^n}, \tag{3.14}$$

this follows from Theorem 2.3.10 in [7];

$$f_2'(v,x_\varepsilon(v))(y) = \sum_{n=1}^{\infty}\mu_n(v)q\Big\|x_n(v) - x_\varepsilon(v)\Big\|^{q-1}_{L^2_{a,b}}\Big\langle \|x_n(v) - x_\varepsilon(v)\|'_{L^2_{a,b}}, y\Big\rangle_{L^2_{a,b}} \tag{3.15}$$

We put $r(v) = f_2'(v,x_\varepsilon(v))$ and note that $\|r(v)(t)\| < \varepsilon$ for $t \in [a,b]$.

In the calculation of the subdifferential of $f_4(v,.)$ we use Example 2.7.4 in [7] and Proposition 2.5.3 in [7]. The distance function $\operatorname{dist}_{F(t,v,x(t))}(\dot{x}(t))$ is Fréchet differentiable at $\dot{x}(t)$ and is convex separately with respect to $x(t)$ and $\dot{x}(t)$.

Let us denote the derivative of $(K_3(v) + \gamma(v)K_4(v))\operatorname{dist}_{F(t,v,x_\varepsilon(t))}(.)$ at $\dot{x}_\varepsilon(t)$ by $s_v(t)$ and

$$\xi_v = -f_1'(v,x_\varepsilon(v)) - f_2'(v,x_\varepsilon(v)) - f_3'(v,x_\varepsilon(v)).$$

From (3.12) we obtain $\xi_v \in \partial f_4(v,x_\varepsilon(v))$, so by Example 2.7.4 in [7] there exist measurable functions $(\tilde{q}_v,\tilde{s}_v)$ such that

$$(\tilde{q}_v(t),\tilde{s}_v(t)) \in \partial\Big(K_3(v) + \gamma(v)K_4(v)\Big)\operatorname{dist}_{F(t,v,x(t))}(\dot{x}(t)), \quad \text{a.e.},$$

where ∂ is the Clarke subdifferential with respect to $(x_\varepsilon(t),\dot{x}_\varepsilon(t))$, and

$$\xi_v(y) = \int_a^b \Big[\big\langle\tilde{q}_v(t),y(t)\big\rangle + \big\langle\tilde{s}_v(t),\dot{y}(t)\big\rangle\Big]dt, \quad \forall y \in C^1([a,b],\mathbf{R}^n). \tag{3.16}$$

By Proposition 2.5.3 of [7] it follows that $\tilde{s}_v(t) = s_v(t)$ a.e. and

$$\tilde{q}_v(t) \in \partial_x\Big[\big(K_3(v) + \gamma(v)K_4(v)\big)\operatorname{dist}_{F(t,v,x(t))}(\dot{x}(t))\Big], \quad \text{a.e.},$$

Here ∂_x is the subdifferential from convex analysis, with respect to x.

Further the proof is the same as those of Theorem 3.2.6 in [7], as it follows that $p_\varepsilon(v) = s_v$ - see Lemma 3, page 127 of [7] and the subsequent part of the proof. So we prove (ii).

Note that $p_\varepsilon(v)$ is zero, if $p_\varepsilon(v)(t)$ is an interior point of $F(t, v, x(t)$.

By (3.13), (3.14), (3.15), ξ is continuous as a mapping from V to $L^2([a, b], \mathbf{R}^n)$, and if $\dot{x}_\varepsilon(v)(t) \in bdF(t, x_\varepsilon(v)(t)$ a.e., then s is continuous too, as a mapping from V to $L^2([a, b], \mathbf{R}^n)$. Then by (3.16), $\tilde{q}$ is continuous too, which proves (iii), since $\dot{p}_\varepsilon(v)(.) = \tilde{q}_v(.) + r(v)(.)$. □

Remark 3.4. The condition for smooth boundaries of the images of F is satisfied on the perturbed mapping F_ε defined by $F_\varepsilon(t, v, x) = F(t, v, x) + \varepsilon \overline{B}_{\mathbf{R}^n}$.

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