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SOME FIXED POINT RESULTS FOR WEAK CONTRACTION MAPPINGS IN ORDERED 2-METRIC SPACES

PAKEETA SUKPRASERT, POOM KUMAM*, ARSLAN H. ANSARI, SUMIT CHANDOK, AND NAWAB HUSSAIN

Abstract. In this paper, we establish certain new fixed point results for generalized weak contraction mappings using the concept of a triangular $2-\alpha-\eta$ admissible mapping in the framework of 2-metric spaces. As an application of the obtained results, we prove some fixed point results in partially ordered 2-metric spaces. The presented theorems generalize certain recent results in the literature.

1. Introduction

Several attempts have been made in order to generalize the concept of metric space. Many important and interesting results have been obtained, for instance, quasi metric spaces, 2-metric spaces [20], G-metric spaces [34], b-metric spaces, fuzzy metric spaces, partial metric spaces, probabilistic metric spaces and many more (see e.g. [1-41]and references cited therein). The notion of 2-metric was introduced by Gahler in [20]. Note that a 2-metric is not a continuous function of its variables, whereas an ordinary metric is continuous. It is well known that in several situations fixed point results in G-metric spaces can be deduced from fixed point theorems in metric or quasi-metric spaces. It has also been shown by various authors that in several cases the fixed point results in cone metric spaces can be obtained by reducing them to their standard metric counterparts. It is worth to note that in the above generalizations, a 2-metric space was not known to be topologically equivalent to an ordinary metric.

In this paper, we establish certain new fixed point results for generalized weak contraction mappings using the concept of a triangular $2-\alpha-\eta$ admissible mapping in the framework of (ordered) 2-metric spaces.

2. Preliminaries

To begin with we give some notations, definitions and primary results which will be used in the sequel.

Definition 2.1 ([20]). Let X be a non-empty set and let $d : X \times X \times X \to \mathbb{R}^+$ be a mapping and satisfying the following assertions:

(1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$;
(2) If at least two of three points x, y, z are the same, then $d(x, y, z) = 0$;

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(3) The symmetry:

$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$

for all $x,y,z \in X$;

(4) The rectangle inequality: $d(x,y,z) \leq d(x,y,t) + d(y,z,t) + d(z,x,t)$ for all $x,y,z,t \in X$.

Then d is called a 2-metric on X and (X,d) is called a 2-metric space which will be sometimes denoted by X if there is no confusion. Every member $x \in X$ is called a point in X.

Definition 2.2 ([20])**.** Let (X,d) be a 2-metric space and $a,b \in X$, $r \geq 0$. The set

$$B(a,b,r) = \{x \in X : d(a,b,x) < r\}$$

is called a 2-ball centered at a and b with radius r. The topology generated by the collection of all a 2-balls as a subbase is called a 2-metric topology on X.

Definition 2.3 ([25])**.** Let $\{x_n\}$ be a sequence in a 2-metric space (X,d).

(1) $\{x_n\}$ is said to be convergent to x in (X,d), written $\lim_{n\to\infty} x_n = x$, if for all $a \in X$, $\lim_{n\to\infty} d(x_n,x,a) = 0$;

(2) $\{x_n\}$ is said to be Cauchy in X if for all $a \in X$, $\lim_{n,m\to\infty} d(x_n,x_m,a) = 0$, that is, for each $\epsilon > 0$, there exists n_0 such that $d(x_n,x_m,a) < \epsilon$ for all $n,m \geq n_0$ and $a \in X$;

(3) (X,d) is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 2.4 ([25])**.** A 2-metric space (X,d) is said to be compact if every sequence in X has a convergent subsequence.

Lemma 2.5 ([25])**.** *Every a 2-metric space is a T_1-space.*

Lemma 2.6 ([25])**.** $\lim_{n\to\infty} x_n = x$ *in a 2-metric space (X,d) if and only if* $\lim_{n\to\infty} x_n = x$ *in the 2-metric topological space X.*

Lemma 2.7 ([25])**.** *If $T : X \to Y$ is a continuous map from a 2-metric space X to a 2-metric space Y, then* $\lim_{n\to\infty} x_n = x$ *in X implies* $\lim_{n\to\infty} Tx_n = Tx$ *in Y.*

It is straight forward from above definitions, that every 2-metric is non-negative and every 2-metric space contains at least three distinct points. A 2-metric $d(x,y,z)$ is sequentially continuous in one argument. Moreover, if a 2-metric $d(x,y,z)$ is sequentially continuous in two arguments, then it is sequentially continuous in all three arguments (see [32]). A convergent sequence in a 2-metric space need not be a Cauchy sequence (see [32]). In a 2-metric space (X,d), every convergent sequence is a Cauchy sequence if d is continuous (see [32]). There exists a 2-metric space (X,d) such that every convergent sequence is a Cauchy sequence but d is not continuous (see [32]).

Chatterjea in [12] introduced the notion of a C-contraction as follows.

Definition 2.8. Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is called a C-contraction if there exists $a \in [0,1)$ such that for all $x,y \in X$,

$$d(Tx,Ty) \leq \frac{a}{2}[d(x,Ty) + d(y,Tx)].$$

This notion was generalized to a weak C-contraction by Choudhury in [13].

Definition 2.9. Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then T is called a weak C-contraction if there exists $\psi : [0, \infty)^2 \to [0, \infty)$ which is continuous and $\psi(s, t) = 0$ if and only if $s = t = 0$ such that

$$d(Tx, Ty) \leq \frac{1}{2}[d(x, Ty) + d(y, Tx)] - \psi(d(x, Ty), d(y, Tx))$$

for all $x, y \in X$.

Samet et al. [38] defined the notion of an α-admissible mapping as follows.

Definition 2.10. Let T be a self-mapping on a non-empty set X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

In [38] the authors consider the family Ψ of non-decreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ and give the following theorem.

Theorem 2.11. *Let (X, d) be a complete metric space and T be an α-admissible mapping such that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y)) \tag{2.1}$$

for all $x, y \in X$, where $\psi \in \Psi$. Suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,*
(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

Salimi et al. [39] modified and generalized the notions of α-ψ-contractive and α-admissible mappings as follows.

Definition 2.12 ([39])**.** Let T be a self-mapping on a non-empty set X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that if we take $\eta(x, y) = 1$ then this definition reduces to Definition 2.10. Also, if we take $\alpha(x, y) = 1$ then we say that T is an η-subadmissible mapping.

Recently Karapinar et al. [27] introduced the notion of triangular α-admissible mapping as follows.

Definition 2.13 ([27])**.** Let X be a non-empty set and $T : X \to X$ and $\alpha : X \times X \to (-\infty, +\infty)$ be two given mappings. We say that T is a triangular α-admissible mapping if

(1) $x, y \in X, \quad \alpha(x, y) \geq 1 \quad \text{implies} \quad \alpha(Tx, Ty) \geq 1, \quad x, y \in X,$

(2) $x, y \in X, \quad \begin{cases} \alpha(x,z) \geq 1 \\ \alpha(z,y) \geq 1 \end{cases}$ implies $\alpha(x,y) \geq 1$.

Motivated by Karapinar et al. [27] and Salimi et al. [39], Fathollahi et al. [19] introduced the following notion.

Definition 2.14. Let (X, d) be a 2-metric space and $T : X \to X$ and $\alpha, \eta : X \times X \times X \to [0, +\infty)$ be mappings. We say that T is a triangular 2-α-η-admissible mapping if for all $a \in X$,

(1) $x, y \in X$, $\alpha(x, y, a) \geq \eta(x, y, a)$ implies $\alpha(Tx, Ty, a) \geq \eta(Tx, Ty, a)$,

(2) $x, y \in X, \quad \begin{cases} \alpha(x,z,a) \geq \eta(x,z,a) \\ \alpha(z,y,a) \geq \eta(z,y,a) \end{cases}$ implies $\alpha(x,y,a) \geq \eta(x,y,a)$.

If we take $\eta(x, y, a) = 1$, then we say that T is a triangular 2-α-admissible mapping. Also, if we take $\alpha(x, y, a) = 1$, then we say T is a triangular 2-η- subadmissible mapping.

Example 2.15. Let $X = [0, \infty)$. Define $T : X \to X$ and $\alpha, \eta : X \times X \times X \to [0, +\infty)$ by $Tx = \frac{1}{4}x$,

$$\alpha(x,y,a) = \begin{cases} a^2 + 2, & \text{if } x, y \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

and $\eta(x, y, a) = a^2 + 1$. Then T is a triangular 2-α-η-admissible mapping.

Lemma 2.16. *Let (X, d) be a 2-metric space and $T : X \to X$ be a triangular 2-α-η-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, a) \geq \eta(x_0, Tx_0, a)$ for all $a \in X$. Define sequence $\{x_n\}$ by $x_n = T^n x_0$. Then*

$$\alpha(x_m, x_n, a) \geq \eta(x_m, x_n, a) \text{ for all } m, n \in \mathbb{N} \text{ with } m < n \text{ and for all } a \in X$$

Definition 2.17. Let (X, d) be a 2-metric space. Let $\alpha, \eta : X \times X \times X \to [0, \infty)$ and $T : X \to X$. We say T is a 2-α-η-continuous on (X, d), if

$$x_n \to x \quad \text{as} \quad n \to \infty, \ \alpha(x_n, x_{n+1}, a) \geq \eta(x_n, x_{n+1}, a) \text{ for all } n \in \mathbb{N} \text{ and}$$

$a \in X \implies Tx_n \to Tx$. If we take $\eta(x, y, a) = 1$, then we say that T is a 2-α-continuous mapping. Also, if we take $\alpha(x, y, a) = 1$, then we say T is a 2-η-continuous mapping.

Example 2.18. Let $X = [0, \infty)$ and $d(x, y, a) = \min\{|x - y|, |y - a|, |x - a|\}$. Assume that the mapping $T : X \to X$ and $\alpha, \eta : X^3 \to [0, +\infty)$ are defined by

$$Tx = \begin{cases} x^2, & \text{if} \quad x \in [0,1] \\ \\ \ln x + 2, & \text{if} \quad x \in (1, \infty), \end{cases}$$

$$\alpha(x,y,a) = \begin{cases} 1, & \text{if} \quad x, y \in [0,1] \\ \\ 0, & \text{otherwise} \end{cases} \quad \text{and } \eta(x,y,a) = 1.$$

Clearly T is not continuous, but T is a 2-α-η-continuous on (X, d). Indeed, if $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}, a) \geq \eta(x_n, x_{n+1}, a) = 1$, then $x_n \in [0, 1]$ and $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_n^2 = x^2 = Tx$.

Motivated by the above mentioned developments, we first generalize the concepts of a 2-α-η-admissible mapping with weak and rational α-η-ψ-contractions using some auxiliary functions and establish the existence and uniqueness of fixed points for such mappings in complete 2-metric spaces. As an application of obtained results, we prove some fixed point theorems in partially ordered 2-metric spaces. The presented theorems generalize and improve many existing results in the literature. Moreover, some examples and an application to integral equations are provided to illustrate the usability of the proved results.

3. Fixed point results for weak α-η-C-contraction mappings

Definition 3.1. We say that $f : [0,\infty)^2 \longrightarrow \mathbb{R}$ is a $\mathcal{C}$-class function if it is continuous and satisfies

(1) $f(s,t) \leq s$;

(2) $f(s,t) = s \Longrightarrow s = 0$ or $t = 0$.

Example 3.2. The following functions $f : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$. For all $s,t \in [0,\infty)$, we have

(1) $f(s,t) = s - t$, $f(s,t) = s \Longrightarrow t = 0$;

(2) $f(s,t) = \frac{(s-t)}{(1+t)}$, $f(s,t) = s \Longrightarrow t = 0$;

(3)$f(s,t) = \frac{s}{(1+t)}$, $f(s,t) = s \Longrightarrow s = 0$ or $t = 0$;

(4) $f(s,t) = \log\frac{(t+a^s)}{(1+t)}$, $a > 1$, $f(s,t) = s \Longrightarrow s = 0$ or $t = 0$;

(5) $f(s,1) = \ln\frac{(1+a^s)}{2}$, $a > e$, $f(s,1) = s \Longrightarrow s = 0$;

(6) $f(s,t) = (s+l)^{(1/(1+t))} - l, l > 1, f(s,t) = s \Longrightarrow t = 0$.

Denote with Ψ the family of continuous functions $\psi : [0,\infty)^2 \to [0,\infty)$ such that $\psi(s,t) > 0$ if $s \neq 0$ or $t \neq 0$ and $\psi(0,0) \geq 0$.

Definition 3.3. Let (X,d) be a 2-metric space and $T : X \to X$, $\alpha,\eta : X\times X\times X \to [0,+\infty)$ and $f \in \mathcal{C}$. Then

- T is a weak $\alpha - \eta - f - C$-contraction mapping if
$$x,y \in X,\ \alpha(x,y,a) \geq \eta(x,y,a) \Longrightarrow d(Tx,Ty,a)$$
$$\leq f\Big(\frac{1}{2}[d(x,Ty,a) + d(y,Tx,a)], \psi(d(x,Ty,a), d(y,Tx,a))\Big)$$
for all $a \in X$ where $\psi \in \Psi$.
- T is a modified weak $\alpha - f - C$-contraction mapping if
$$x,y \in X,\ \alpha(x,y,a) \geq 1 \Longrightarrow d(Tx,Ty,a)$$
$$\leq f\Big(\frac{1}{2}[d(x,Ty,a) + d(y,Tx,a)], \psi(d(x,Ty,a), d(y,Tx,a))\Big)$$
for all $a \in X$ where $\psi \in \Psi$.
- T is a modified weak $\eta - f - C$-contraction mapping if
$$x,y \in X,\ \eta(x,y,a) \leq 1 \Longrightarrow d(Tx,Ty,a)$$
$$\leq f\Big(\frac{1}{2}[d(x,Ty,a) + d(y,Tx,a)], \psi(d(x,Ty,a), d(y,Tx,a))\Big)$$
for all $a \in X$ where $\psi \in \Psi$.

- T is a weak $\alpha - f - C$-contraction mapping of type (I) if
$$\alpha(x,y,a)d(Tx,Ty,a) \leq f\Big(\frac{1}{2}[d(x,Ty,a)+d(y,Tx,a)], \psi(d(x,Ty,a), d(y,Tx,a))\Big)$$
for all $x,y,a \in X$ where $\psi \in \Psi$.
- T is a weak $\eta - f - C$-contraction mapping of type (I) if$f$ is increasing with respect to first variable and
$$d(Tx,Ty,a) \leq f\Big(\frac{\eta(x,y,a)}{2}[d(x,Ty,a)+d(y,Tx,a)], \psi(d(x,Ty,a), d(y,Tx,a))\Big)$$
for all $x,y,a \in X$ where $\psi \in \Psi$.
- T is a weak $\alpha - f - C$-contraction mapping of type (II) if
$$(\alpha(x,y,a)+\ell)^{d(Tx,Ty,a)} \leq (1+\ell)^{f\Big(\frac{1}{2}[d(x,Ty,a)+d(y,Tx,a)], \psi(d(x,Ty,a), d(y,Tx,a))\Big)}$$
for all $x,y,a \in X$ where $\psi \in \Psi$ and $\ell > 0$,
- T is a weak $\eta - f - C$-contraction mapping of type (II) if
$$(1+\ell)^{d(Tx,Ty,a)} \leq (\eta(x,y,a)+\ell)^{f\Big(\frac{1}{2}[d(x,Ty,a)+d(y,Tx,a)], \psi(d(x,Ty,a), d(y,Tx,a))\Big)}$$
for all $x,y,a \in X$, $\ell > 0$ and $\psi \in \Psi$.

Now we are ready to state and prove our first main result of this section.

Theorem 3.4. *Let (X,d) be a complete 2-metric space. Assume, $T : X \to X$ is a weak α-η-$f - C$-contraction mapping satisfying the following assertions:*

(i) *T is a triangular 2-α-η-admissible mapping;*
(ii) *there exists x_0 in X such that, $\alpha(x_0, Tx_0, a) \geq \eta(x_0, Tx_0, a)$ for all $a \in X$;*
(iii) *T is continuous or 2-α-η-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq \eta(x_n, x_{n+1}, a)$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, a) \geq \eta(x_n, x, a)$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0, a) \geq \eta(x_0, Tx_0, a)$ for all $a \in X$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Now, since T is a triangular 2-α-η-admissible mapping, so by Lemma 2.16, we have

$$\alpha(x_m, x_n, a) \geq \eta(x_m, x_n, a) \text{ for all } m,n \in \mathbb{N} \text{ with } m < n \text{ and for all } a \in X. \tag{3.1}$$

From (3.1), we deduce

(3.2)

$$\begin{aligned}
&d(x_{n+1}, x_n, a)\\
&= d(Tx_n, Tx_{n-1}, a)\\
&\leq f\Big(\frac{1}{2}[d(x_n, Tx_{n-1}, a) + d(x_{n-1}, Tx_n, a)], \psi(d(x_n, Tx_{n-1}, a), d(x_{n-1}, Tx_n, a))\Big)\\
&= f\Big(\frac{1}{2}[d(x_n, x_n, a) + d(x_{n-1}, x_{n+1}, a)], \psi(d(x_n, x_n, a), d(x_{n-1}, x_{n+1}, a))\Big)\\
&= f\Big(\frac{1}{2}d(x_{n-1}, x_{n+1}, a), \psi(0, d(x_{n-1}, x_{n+1}, a))\Big)\\
&\leq \frac{1}{2}d(x_{n-1}, x_{n+1}, a),
\end{aligned}$$

for all $a \in X$. By taking $a = x_{n-1}$ in (3.2), we get $d(x_{n+1}, x_n, x_{n-1}) \leq 0$. i.e.,

$$d(x_{n+1}, x_n, x_{n-1}) = 0 \tag{3.3}$$

and so by (3.2) and (3.3), we have

$$\begin{aligned}
d(x_{n+1}, x_n, a) &\leq \frac{1}{2}d(x_{n-1}, x_{n+1}, a)\\
&\leq \frac{1}{2}[d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, x_{n+1})] \qquad (3.4)\\
&\leq \frac{1}{2}[d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a)],
\end{aligned}$$

which implies

$$d(x_{n+1}, x_n, a) \leq d(x_{n-1}, x_n, a). \tag{3.5}$$

Hence, the sequence $\{d(x_{n+1}, x_n, a)\}$ is decreasing in $\mathbb{R}_+$ and so it is convergent to $r \in \mathbb{R}_+$. i.e., $\lim_{n\to\infty} d(x_{n+1}, x_n, a) = r$. Taking limit in (3.4), we get

$$r \leq \frac{1}{2}\lim_{n\to\infty} d(x_{n-1}, x_{n+1}, a) \leq \frac{1}{2}(r + r) = r$$

and then

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}, a) = 2r. \tag{3.6}$$

By taking limit as $n \to \infty$ in (3.2) and applying (3.6), we get

$$r \leq f(\frac{1}{2}(2r), \psi(0, 2r)) \leq r$$

this implies $r = 0$ or $\psi(0, 2r) = 0$. i.e., $r = 0$. Hence

$$\lim_{n\to\infty} d(x_{n+1}, x_n, a) = 0. \tag{3.7}$$

If $d(x_{n-1}, x_n, a) = 0$, then by (3.5), we have $d(x_{n+1}, x_n, a) = 0$. Since, $d(x_0, x_1, x_0) = 0$, we have $d(x_n, x_{n+1}, x_0) = 0$ for all $n \in \mathbb{N}$.
Since $d(x_{m-1}, x_m, x_m) = 0$, we have

$$d(x_n, x_{n+1}, x_m) = 0 \tag{3.8}$$

for all $n \geq m-1$. For $0 \leq n < m-1$, noting that $m-1 \geq n+1$, from (3.8), we have

$$d(x_{m-1}, x_m, x_{n+1}) = d(x_{m-1}, x_m, x_n) = 0$$

which implies

$$\begin{aligned} d(x_n, x_{n+1}, x_m) &\leq d(x_n, x_{n+1}, x_{m-1}) + d(x_{n+1}, x_m, x_{m-1}) + d(x_m, x_n, x_{m-1}) \\ &= d(x_n, x_{n+1}, x_{m-1}). \end{aligned} \tag{3.9}$$

Now, since, $d(x_n, x_{n+1}, x_{n+1}) = 0$, from (3.9), we get

$$d(x_n, x_{n+1}, x_m) = 0 \tag{3.10}$$

for all $0 \leq n < m-1$. Hence, from (3.8) and (3.10), we have $d(x_n, x_{n+1}, x_m) = 0$ for all $m, n \in \mathbb{N}$. Now for all $i, j, k \in \mathbb{N}$ with $i < j$,, we have $d(x_{j-1}, x_j, x_i) = d(x_{j-1}, x_j, x_k) = 0$. Hence, we obtain

$$\begin{aligned} d(x_i, x_j, x_k) &\leq d(x_i, x_j, x_{j-1}) + d(x_j, x_k, x_{j-1}) + d(x_k, x_i, x_{j-1}) \\ &= d(x_i, x_{j-1}, x_k) \leq \ldots \leq d(x_i, x_i, x_k) = 0. \end{aligned}$$

That is, for all $i, j, k \in \mathbb{N}$, we have

$$d(x_i, x_j, x_k) = 0. \tag{3.11}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad d(x_{n(k)}, x_{m(k)}, a) \geq \varepsilon \quad \text{and}\, d(x_{n(k)-1}, x_{m(k)}, a) < \varepsilon. \tag{3.12}$$

From (3.11) and (3.12), we deduce

$$\begin{aligned} \epsilon &\leq d(x_{n(k)}, x_{m(k)}, a) \\ &\leq d(x_{n(k)}, x_{n(k)-1}, a) + d(x_{n(k)-1}, x_{m(k)}, a) \\ &\quad + d(x_{n(k)}, x_{m(k)}, x_{n(k)-1}) \\ &= d(x_{n(k)}, x_{n(k)-1}, a) + d(x_{n(k)-1}, x_{m(k)}, a) \\ &< d(x_{n(k)}, x_{n(k)-1}, a) + \epsilon. \end{aligned} \tag{3.13}$$

Taking limit as $k \to \infty$ in the above inequality and applying (3.7), we get

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}, a) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}, a) = \epsilon. \tag{3.14}$$

Also by (3.11), we get

$$\begin{aligned} &d(x_{m(k)}, x_{n(k)-1}, a) \\ &\leq d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)-1}, a) + d(x_{m(k)}, x_{n(k)-1}, x_{m(k)-1}) \\ &= d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)-1}, a) \\ &\leq d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)}, a) + d(x_{n(k)-1}, x_{n(k)}, a) \\ &+ d(x_{m(k)-1}, x_{n(k)-1}, x_{n(k)}) \\ &= d(x_{m(k)}, x_{m(k)-1}, a) + d(x_{m(k)-1}, x_{n(k)}, a) + d(x_{n(k)-1}, x_{n(k)}, a), \end{aligned} \tag{3.15}$$

and

$$
\begin{aligned}
&d(x_{m(k)-1}, x_{n(k)}, a)\\
&\leq d(x_{m(k)-1}, x_{m(k)}, a) + d(x_{n(k)}, x_{m(k)}, a) + d(x_{m(k)-1}, x_{n(k)}, x_{m(k)})\\
&= d(x_{m(k)-1}, x_{m(k)}, a) + d(x_{n(k)}, x_{m(k)}, a).
\end{aligned} \tag{3.16}
$$

By taking limit as $k \to \infty$ in (3.15) and (3.16) and applying (3.7) and (3.14), we have

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}, a) = \epsilon. \tag{3.17}$$

Now since, $n(k) > m(k)$ then by (3.1), we have

$$\alpha(x_{m(k)-1}, x_{n(k)-1}, a) \geq \eta(x_{m(k)-1}, x_{n(k)-1}, a)$$

for all $a \in X$. So by (3.1), we get

$$
\begin{aligned}
\epsilon &\leq d(x_{m(k)}, x_{n(k)}, a)\\
&= d(Tx_{m(k)-1}, Tx_{n(k)-1}, a)\\
&\leq f\Big(\frac{1}{2}[d(x_{m(k)-1}, Tx_{n(k)-1}, a) + d(x_{n(k)-1}, Tx_{m(k)-1}, a)],\\
&\quad \psi(d(x_{m(k)-1}, Tx_{n(k)-1}, a), d(x_{n(k)-1}, Tx_{m(k)-1}, a))\Big)\\
&= f\Big(\frac{1}{2}[d(x_{m(k)-1}, x_{n(k)}, a) + d(x_{n(k)-1}, x_{m(k)}, a)],\\
&\quad \psi(d(x_{m(k)-1}, x_{n(k)}, a), d(x_{n(k)-1}, x_{m(k)}, a))\Big).
\end{aligned}
$$

Taking limit as $k \to \infty$ in above inequality and applying (3.14), (3.17) and continuity of ψ, we deduce

$$\epsilon \leq f\Big(\frac{1}{2}[\epsilon + \epsilon], \psi(\epsilon, \epsilon)\Big) \leq \epsilon$$

and so $\epsilon = 0$, or $\psi(\epsilon, \epsilon) = 0$. That is $\epsilon = 0$ which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Now, since (X, d) is a complete 2-metric space, then there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. At first we assume (iii) holds. That is, T is continuous. Then

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*.$$

That is, x^* is a fixed point of T. If T is 2-α-η-continuous on X, $x_n \to x^*$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}, a) \geq \eta(x_n, x_{n+1}, a)$, then we have

$$Tx^* = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^*.$$

So x^* is a fixed point of T. Next we assume (iv) holds. That is, $\alpha(x_n, x^*, a) \geq \eta(x_n, x^*, a)$ for all $n \in \mathbb{N}$ and all $a \in X$. Then by (3.1), we get

$$\begin{aligned}
&d(x_{n+1}, Tx^*, a)\\
&= d(Tx_n, Tx^*, a)\\
&\leq f\Big(\frac{1}{2}[d(x_n, Tx^*, a) + d(x^*, Tx_n, a)], \psi(d(x_n, Tx^*, a), d(x^*, Tx_n, a))\Big)\\
&= f\Big(\frac{1}{2}[d(x_n, Tx^*, a) + d(x^*, x_{n+1}, a)], \psi(d(x_n, Tx^*, a), d(x^*, x_{n+1}, a))\Big).
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality we get

$$\begin{aligned}
&d(x^*, Tx^*, a)\\
&\leq f\Big(\frac{1}{2}[d(x^*, Tx^*, a) + d(x^*, x^*, a)], \psi(d(x^*, Tx^*, a), d(x^*, x^*, a))\Big)\\
&\leq f\Big(\frac{1}{2}d(x^*, Tx^*, a), \psi(d(x^*, Tx^*, a), 0)\Big)\\
&\leq \frac{1}{2}d(x^*, Tx^*, a)
\end{aligned}$$

which implies, $d(x^*, Tx^*, a) = 0$. i.e., $x^* = Tx^*$. □

If we take $f(s, t) = s - t$ in Theorem 3.4, we obtain following main result in [19] as corollary.

Corollary 3.5 (Theorem 2.1 [19]). *Let (X, d) be a complete 2-metric space. Assume that $T : X \to X$ is a weak α-η-C-contraction mapping satisfying the following assertions:*

(i) *T is a triangular 2-α-η-admissible mapping;*
(ii) *there exists x_0 in X such that, $\alpha(x_0, Tx_0, a) \geq \eta(x_0, Tx_0, a)$ for all $a \in X$;*
(iii) *T is continuous or 2-α-η-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq \eta(x_n, x_{n+1}, a)$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, a) \geq \eta(x_n, x, a)$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

By taking $\eta(x, y, a) = 1$ in Theorem 3.4, we obtain following results.

Corollary 3.6. *Let (X, d) be a complete 2-metric space. Assume that $T : X \to X$ is a modified weak α-$f - C$-contraction mapping satisfying the following assertions:*

(i) *T is a triangular 2-α-admissible mapping;*
(ii) *there exists x_0 in X such that, $\alpha(x_0, Tx_0, a) \geq 1$ for all $a \in X$;*
(iii) *T is continuous or 2-α-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, a) \geq 1$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

Example 3.7. Let $X = [0, \infty)$. We define a 2-metric d on X by

$$d(x, y, a) = \min\{|x - y|, |y - a|, |x - a|\}.$$

Clearly (X, d) is a complete 2-metric space. Define, $T : X \to X$, $\psi : [0,\infty)^2 \to [0,\infty)$, $\alpha : X \times X \times X \to [0,\infty)$, and $f : [0,\infty)^2 \to \mathbb{R}$ as follows:

$$Tx = \begin{cases} \frac{e}{5} & \text{if } x \in [0,2] \\ x^7 + x^5 + 1, & \text{if } x \in (2,8] \\ \log_3 x + \cos x + 2, & \text{if } x \in (8,14] \\ |x^2 + x - 225|, & \text{if } x \in (14,\infty) \end{cases}$$

and $\alpha(x, y, a) = \begin{cases} 1, & \text{if } x, y \in [0,2] \\ \frac{1}{2}, & \text{otherwise,} \end{cases}$

$$f(s,t) = \frac{s}{1+t}, \quad \psi(s,t) = 1.$$

Now, we prove that all the hypotheses of Corollary 3.6 (Theorem 3.4) are satisfied and hence T has a fixed point.

Proof. Let $x, y, a \in X$, if $\alpha(x, y, a) \geq 1$ then $x, y \in [0, 2]$. On the other hand for all $w \in [0, 2]$, we have $Tw \leq 2$. Hence $\alpha(Tx, Ty, a) \geq 1$ for all $a \in X$. This implies that T is a 2-α-admissible mapping. Clearly, $\alpha(0, T0, a) \geq 1$ for all $a \in X$. Now, if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $a \in X$ and $x_n \to x$ as $n \to +\infty$. Then $\{x_n\} \subseteq [0, 2]$ and hence $x \in [0, 2]$. This implies that $\alpha(x_n, x, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and all $a \in X$.

Let $\alpha(x, y, a) \geq 1$. Then $x, y \in [0, 2]$ and hence

$$\begin{aligned} d(Tx, Ty, a) = 0 &\leq \frac{1}{4}[d(x, Ty, a) + d(y, Tx, a)] \\ &= \frac{\frac{1}{2}[d(x, Ty, a) + d(y, Tx, a)]}{1+1} \\ &= \frac{\frac{1}{2}[d(x, Ty, a) + d(y, Tx, a)]}{1 + \psi(d(x, Ty, a), d(y, Tx, a))}. \end{aligned}$$

That is,

$$\alpha(x, y, a) \geq 1 \Longrightarrow d(Tx, Ty, a) \leq \frac{\frac{1}{2}[d(x, Ty, a) + d(y, Tx, a)]}{1 + \psi(d(x, Ty, a), d(y, Tx, a))}$$

for all $a \in X$. Hence, T is a modified weak α-$f - C$-contraction mapping. Then, all the hypotheses of the Corollary 3.6 (Theorem 3.4) are satisfied and hence T has a fixed point. □

By taking $\alpha(x, y, a) = 1$ in Theorem 3.4 we have the following Corollary.

Corollary 3.8. *Let (X, d) be a complete 2-metric space. Assume that $T : X \to X$ is a modified weak η-$f - C$-contraction mapping such that satisfying the following assertions:*

(i) *T is a triangular 2-η-subadmissible mapping;*
(ii) *there exists x_0 in X such that, $\eta(x_0, Tx_0, a) \leq 1$ for all $a \in X$;*

(iii) *T is continuous or 2-η-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}, a) \leq 1$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\eta(x_n, x, a) \leq 1$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

Corollary 3.9. *Let (X, d) be a complete 2-metric space. Assume that $T : X \to X$ is a weak α-$f - C$-contraction mapping of type (I) or weak α-$f - C$-contraction mapping of type (II) satisfying the following assertions:*

(i) *T is a triangular 2-α-admissible mapping;*
(ii) *there exists x_0 in X such that, $\alpha(x_0, Tx_0, a) \geq 1$ for all $a \in X$;*
(iii) *T is continuous or 2-α-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, a) \geq 1$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

Corollary 3.10. *Let (X, d) be a complete 2-metric space. Assume that $T : X \to X$ is a weak η-$f - C$-contraction mapping of type (I) or weak α-$f - C$-contraction mapping of type (II) satisfying the following assertions:*

(i) *T is a triangular 2-η-admissible mapping;*
(ii) *there exists x_0 in X such that, $\eta(x_0, Tx_0, a) \leq 1$ for all $a \in X$;*
(iii) *T is continuous or 2-η-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}, a) \leq 1$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\eta(x_n, x, a) \leq 1$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

(A) for all $x, y \in X$ where $\alpha(x, y, a) < \eta(x, y, a)$ and $\alpha(y, x, a) < \eta(y, x, a)$ for all $a \in X$, there exists $z \in X$ such that $\alpha(x, z, a) \geq \eta(x, z, a)$ or $\alpha(z, x, a) \geq \eta(z, x, a)$ and $\alpha(y, z, a) \geq \eta(y, z, a)$ or $\alpha(z, y, a) \geq \eta(z, y, a)$ for all $a \in X$.

Theorem 3.11. *Adding condition (A) to the hypotheses of Theorem 3.4 (resp. Corollary 3.6, 3.8, 3.9 and 3.10) we obtain uniqueness of the fixed point of T.*

Proof. Assume that x^* and y^* are two fixed points of T. We consider to following cases:
Case 1: Let, $\alpha(x^*, y^*, a) \geq \eta(x^*, y^*, a)$ or $\alpha(y^*, x^*, a) \geq \eta(y^*, x^*, a)$ for all $a \in X$. Then from (3.1), we have

$$\begin{aligned} &d(Tx^*, Ty^*, a) \\ &\leq f(\frac{1}{2}[d(x^*, Ty^*, a) + d(y^*, Tx^*, a)], \psi(d(x^*, Ty^*, a), d(y^*, Tx^*, a))) \end{aligned}$$

for all $a \in X$. This implies that

$$d(x^*, y^*, a) \leq f(d(x^*, y^*, a), \psi(d(x^*, y^*, a), d(x^*, y^*, a))) \leq d(x^*, y^*, a).$$

That is, $\psi(d(x^*, y^*, a), d(x^*, y^*, a)) = 0$ for all $a \in X$. So, $d(x^*, y^*, a) = 0$ for all $a \in X$. Hence $x^* = y^*$.
Case 2: Let $\alpha(x^*, y^*, a) < \eta(x^*, y^*, a)$ and $\alpha(y^*, x^*, a) < \eta(y^*, x^*, a)$ for all $a \in X$. From (A) there exists $z \in X$ such that

$$\alpha(x^*, z, a) \geq \eta(x^*, z, a) \text{ or } \alpha(z, x^*, a) \geq \eta(z, x^*, a)$$

and

$$\alpha(y^*, z, a) \geq \eta(y^*, z, a) \text{ or } \alpha(z, y^*, a) \geq \eta(z, y^*, a)$$

Without loss of generality, we can assume

$$\alpha(x^*, z, a) \geq \eta(x^*, z, a) \text{ and } \alpha(y^*, z, a) \geq \eta(y^*, z, a)$$

Now, since T is a triangular $2-\alpha-\eta$-admissible mapping, then

$$\begin{aligned}\alpha(Tx^*, T(T^{n-1}z), a) \geq \eta(Tx^*, T(T^{n-1}z), a),\ \alpha(Ty^*, T(T^{n-1}z), a)\\ \geq \eta(Ty^*, T(T^{n-1}z), a)\end{aligned}$$

for all $n \in \mathbb{N} \cup 0$ and all $a \in X$. Then from (3.1), we get

$$\begin{aligned}d(Tx^*, T(T^{n-1}z), a) \leq f(\frac{1}{2}[d(x^*, T(T^{n-1}z), a) + d(T^{n-1}z, Tx^*, a)],\\ \psi(d(x^*, T(T^{n-1}z), a), d(T^{n-1}z, Tx^*, a)))\end{aligned}$$

which implies that

$$\begin{aligned}&d(x^*, T^n z, a)\\ &\leq f\Big(\frac{1}{2}[d(x^*, T^n z, a) + d(x^*, T^{n-1}z, a)], \psi(d(x^*, T^n z, a), d(x^*, T^{n-1}z, a))\Big)\\ &\leq \frac{1}{2}[d(x^*, T^n z, a) + d(x^*, T^{n-1}z, a)]\end{aligned} \tag{3.18}$$

which also implies that $d(x^*, T^n z, a) \leq d(x^*, T^{n-1}z, a)$. Then there exists $\ell \in \mathbb{R}$ such that $\lim_{n\to\infty} d(x^*, T^n z, a) = \ell$. By taking limit as $n \to \infty$ in (3.18), we get

$$\ell \leq f\Big(\frac{1}{2}(\ell + \ell), \psi(\ell, \ell)\Big) \leq \ell$$

and so $\ell = 0$ or $\psi(\ell, \ell) = 0$. Therefore $\ell = 0$. That is, $\lim_{n\to\infty} T^n z = x^*$. Similarly, we can deduce that $\lim_{n\to\infty} T^n z = y^*$. Then by Lemma 2.5, we get $x^* = y^*$. $\square$

Remark 3.12. Several more consequences can be obtained using the other functions $f \in \mathcal{C}$ given in Example 3.12, and/or some other concrete choices of control functions involved here.

4. Fixed point results for rational contraction in 2-metric spaces

In this section, we prove certain fixed point theorems for rational contraction mapping via a triangular 2-α-η-admissible mapping.
Denote with Ψ_φ the family of continuous functions $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) > 0$ if $t \neq 0$ and $\varphi(0) \geq 0$.

Definition 4.1. Let (X, d) be a 2-metric space and $\alpha, \eta : X \times X \times X \to [0, +\infty)$ be three mappings and $f \in \mathcal{C}$. A mapping $T : X \to X$ is called

- T is a modified rational α-η-φ-f-contraction mapping if

$$x, y \in X, \quad \alpha(x, y, a) \geq \eta(x, y, a) \Longrightarrow d(Tx, Ty, a) \leq f(M(x, y, a), \varphi(M(x, y, a))) \tag{4.1}$$

for all $a \in X$ where $\varphi \in \Psi_\varphi$ and

$$M(x,y,a) = \max\left\{d(x,y,a), d(x,Tx,a), d(y,Ty,a), \frac{d(x,Ty,a)+d(y,Tx,a)}{2}, \frac{d(x,Tx,a)d(y,Ty,a)}{1+d(Tx,Ty,a)}\right\}$$

- T is a modified rational α-φ-f-contraction mapping if

$$x,y \in X,\ \alpha(x,y,a) \geq 1 \Longrightarrow d(Tx,Ty,a) \leq f(M(x,y,a), \varphi(M(x,y,a)))$$

  for all $a \in X$ where $\varphi \in \Psi_\varphi$.
- T is a modified rational η-φ-f-contraction mapping if

$$x,y \in X,\ \eta(x,y,a) \leq 1 \Longrightarrow d(Tx,Ty,a) \leq f(M(x,y,a), \varphi(M(x,y,a)))$$

  for all $a \in X$ where $\varphi \in \Psi_\varphi$.
- T is a rational α-φ-f-contraction mapping if

$$\alpha(x,y,a)d(Tx,Ty,a) \leq f(M(x,y,a), \varphi(M(x,y,a)))$$

  for all $x,y,a \in X$ where $\varphi \in \Psi_\varphi$.
- We say that T is a rational η-φ-f-contraction mapping if f is increasing with respect to first variable and

$$d(Tx,Ty,a) \leq f(\eta(x,y,a)M(x,y,a), \varphi(M(x,y,a)))$$

  for all $x,y,a \in X$ where $\varphi \in \Psi_\varphi$.

Theorem 4.2. *Let (X,d) be a complete 2-metric space. Assume that $T : X \to X$ is a modified rational α-η-φ-f-contraction mapping satisfying the following assertions:*

(i) *T is a triangular 2-α-η-admissible mapping;*
(ii) *there exists x_0 in X such that, $\alpha(x_0,Tx_0,a) \geq \eta(x_0,Tx_0,a)$ for all $a \in X$;*
(iii) *T is continuous or 2-α-η-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1},a) \geq \eta(x_n,x_{n+1},a)$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n,x,a) \geq \eta(x_n,x,a)$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0,Tx_0,a) \geq \eta(x_0,Tx_0,a)$ for all $a \in X$. Define a sequence $\{x_n\}$ by $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Now, since T is a triangular 2-α-η-admissible mapping, so by Lemma 2.16, we have

$$\alpha(x_m,x_n,a) \geq \eta(x_m,x_n,a) \text{ for all } m,n \in \mathbb{N} \text{ with } m < n \text{ and for all } a \in X. \tag{4.2}$$

From (4.1), we deduce

$$d(x_{n+1},x_n,a) = d(Tx_n,Tx_{n-1},a) \leq f(M(x_n,x_{n-1},a), \varphi(M(x_n,x_{n-1},a))) \tag{4.3}$$

where,

$$
\begin{aligned}
&M(x_n, x_{n-1}, a)\\
&= \max\Big\{d(x_n, x_{n-1}, a), d(x_n, Tx_n, a), d(x_{n-1}, Tx_{n-1}, a),\\
&\frac{d(x_n, Tx_{n-1}, a), d(x_{n-1}, Tx_n, a)}{2}, \frac{d(x_n, Tx_n, a)d(x_{n-1}, Tx_{n-1}, a)}{1 + d(Tx_n, Tx_{n-1}, a)}\Big\}\\
&= \max\Big\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a), d(x_{n-1}, x_n, a),\\
&\frac{d(x_n, x_n, a) + d(x_{n-1}, x_{n+1}, a)}{2}, \frac{d(x_n, x_{n+1}, a)d(x_{n-1}, x_n, a)}{1 + d(x_{n+1}, x_n, a)}\Big\}\\
&= \max\Big\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a), \frac{d(x_{n-1}, x_{n+1}, a)}{2},\\
&\frac{d(x_n, x_{n+1}, a)d(x_{n-1}, x_n, a)}{1 + d(x_{n+1}, x_n, a)}\Big\}\\
&\leq \max\Big\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a), \frac{d(x_{n-1}, x_{n+1}, a)}{2},\\
&\frac{(1 + d(x_n, x_{n+1}, a))d(x_{n-1}, x_n, a)}{1 + d(x_{n+1}, x_n, a)}\Big\}\\
&= \max\Big\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a), \frac{d(x_{n-1}, x_{n+1}, a)}{2}\Big\}
\end{aligned}
$$

and so

$$M(x_n, x_{n-1}, a) = \max\Big\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a), \frac{d(x_{n-1}, x_{n+1}, a)}{2}\Big\}.$$

By taking $a = x_{n-1}$ in (4.3), we have

$$d(x_{n+1}, x_n, x_{n-1}) \leq f(d(x_n, x_{n+1}, x_{n-1}), \varphi(d(x_n, x_{n+1}, x_{n-1})))$$

and then $\varphi(d(x_n, x_{n+1}, x_{n-1})) = 0$. i.e., $d(x_n, x_{n+1}, x_{n-1}) = 0$. Hence

$$
\begin{aligned}
&\max\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a)\} \leq M(x_n, x_{n-1}, a)\\
&= \max\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a), \frac{d(x_{n-1}, x_{n+1}, a)}{2}\}\\
&\leq \max\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a),\\
&\ \frac{1}{2}[d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, x_{n+1})]\}\\
&= \max\Big\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a), \frac{1}{2}[d(x_{n-1}, x_n, a) + d(x_n, x_{n+1}, a)]\Big\}\\
&= \max\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a)\}.
\end{aligned}
$$

Therefore, $M(x_n, x_{n-1}, a) = \max\{d(x_n, x_{n-1}, a), d(x_n, x_{n+1}, a)\}$.

If $M(x_n, x_{n-1}, a) = d(x_n, x_{n+1}, a)$, then from (4.3), we get

$$d(x_{n+1}, x_n, a) \leq f(d(x_n, x_{n+1}, a), \varphi(d(x_n, x_{n+1}, a))).$$

Thus, $d(x_n, x_{n+1}, a) = 0$,or $\varphi(d(x_n, x_{n+1}, a)) = 0$. i.e., $d(x_n, x_{n+1}, a) = 0$ for all $a \in X$. Hence by Definition 2.1 (d1), $x_n = x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. Then $x = x_n$ is a fixed point of T. Now, if $M(x_n, x_{n-1}, a) = d(x_n, x_{n-1}, a)$, then from (4.3), we get

$$d(x_{n+1}, x_n, a) \leq f(d(x_n, x_{n-1}, a), \varphi(d(x_n, x_{n-1}, a))) \leq d(x_n, x_{n-1}, a). \tag{4.4}$$

So, the sequence $\{d(x_{n+1}, x_n, a)\}$ is decreasing in $\mathbb{R}_+$ and so it is convergent to $r \in \mathbb{R}_+$. i.e., $\lim_{n\to\infty} d(x_{n+1}, x_n, a) = r$. Taking limit in (4.4), we get

$$r \leq f(r, \varphi(r))$$

which implies that $r = 0$. Hence

$$\lim_{n\to\infty} d(x_{n+1}, x_n, a) = 0. \tag{4.5}$$

From (3.11) in Theorem 3.4, we have $d(x_i, x_j, x_k) = 0$ for all $i, j, k \in \mathbb{N}$. We now show that $\{x_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there is $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k,

$$n(k) > m(k) > k, \quad d(x_{n(k)}, x_{m(k)}, a) \geq \varepsilon \quad \text{and}\, d(x_{n(k)-1}, x_{m(k)}, a) < \varepsilon.$$

As in the proof of Theorem 3.4, we get

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}, a) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}, a) = \epsilon; \tag{4.6}$$

$$\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}, a) = \epsilon. \tag{4.7}$$

Now since, $n(k) > m(k)$ so by (4.2), we have

$$\alpha(x_{m(k)-1}, x_{n(k)-1}, a) \geq \eta(x_{m(k)-1}, x_{n(k)-1}, a)$$

for all $a \in X$. So by (4.1), we get

$$\begin{aligned} d(x_{m(k)}, x_{n(k)}, a) &= d(Tx_{m(k)-1}, Tx_{n(k)-1}, a) \\ &\leq f(M(x_{m(k)-1}, x_{n(k)-1}, a), \varphi(M(x_{m(k)-1}, x_{n(k)-1}, a))) \end{aligned} \tag{4.8}$$

where

$$\begin{aligned}
M(x_{m(k)-1}, x_{n(k)-1}, a) = \max\Big\{ & d(x_{m(k)-1}, x_{n(k)-1}, a), d(x_{m(k)-1}, Tx_{m(k)-1}, a), \\
& d(x_{n(k)-1}, Tx_{n(k)-1}, a), \\
& \frac{d(x_{m(k)-1}, Tx_{n(k)-1}, a) + d(x_{n(k)-1}, Tx_{m(k)-1}, a)}{2}, \\
& \frac{d(x_{m(k)-1}, Tx_{m(k)-1}, a) d(x_{n(k)-1}, Tx_{n(k)-1}, a)}{1 + d(Tx_{m(k)-1}, Tx_{n(k)-1}, a)} \Big\} \\
= \max\Big\{ & d(x_{m(k)-1}, x_{n(k)-1}, a), d(x_{m(k)-1}, x_{m(k)}, a), \\
& d(x_{n(k)-1}, x_{n(k)}, a), \\
& \frac{d(x_{m(k)-1}, x_{n(k)}, a) + d(x_{n(k)-1}, x_{m(k)}, a)}{2}, \\
& \frac{d(x_{m(k)-1}, x_{m(k)}, a) d(x_{n(k)-1}, x_{n(k)}, a)}{1 + d(x_{m(k)}, x_{n(k)}, a)} \Big\}.
\end{aligned}$$

Taking limit as $k \to \infty$ in (4.8) and applying (4.6) and (4.7), we deduce

$$\epsilon \leq f(\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, a), \lim_{k\to\infty} \varphi(M(x_{m(k)-1}, x_{n(k)-1}, a)))$$

where $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)-1}, a) = \epsilon$. Then $\epsilon = 0$ or $\varphi(\epsilon) = 0$. i.e., $\epsilon = 0$ which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. Now, since (X, d) is a complete 2-metric space, then there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. At first we assume (iii) holds. That is, T is continuous or 2-α-η-continuous. Then, we have

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*.$$

That is, x^* is a fixed point of T. Next we assume (iv) holds. That is, $\alpha(x_n, x^*, a) \geq \eta(x_n, x^*, a)$ for all $n \in \mathbb{N}$ and all $a \in X$. Then by (4.1), we get

$$d(x_{n+1}, Tx^*, a) = d(Tx_n, Tx^*, a) \leq f(M(x_n, x^*, a), \varphi(M(x_n, x^*, a)))$$

for all $a \in X$ where $\varphi \in \Psi_\varphi$ and

$$\begin{aligned}
M(x_n, x^*, a) = \max\Big\{ & d(x_n, x^*, a), d(x_n, Tx_n, a), d(x^*, Tx^*, a), \\
& \frac{d(x_n, Tx^*, a) + d(x^*, Tx_n, a)}{2}, \frac{d(x_n, Tx_n, a) d(x^*, Tx^*, a)}{1 + d(Tx_n, Tx^*, a)} \Big\} \\
= \max\Big\{ & d(x_n, x^*, a), d(x_n, x_{n+1}, a), d(x^*, Tx^*, a), \\
& \frac{d(x_n, Tx^*, a) + d(x^*, x_{n+1}, a)}{2}, \frac{d(x_n, x_{n+1}, a) d(x^*, Tx^*, a)}{1 + d(x_{n+1}, Tx^*, a)} \Big\}.
\end{aligned}$$

Taking limit as $n \to \infty$ in the above inequality we deduce that

$$d(x^*, Tx^*, a) \leq f(\lim_{n\to\infty} M(x_n, x^*, a), \lim_{n\to\infty} \varphi(M(x_n, x^*, a))).$$

Since $\lim_{n\to\infty} M(x_n, x^*, a) = d(x^*, Tx^*, a)$.
Then$d(x^*, Tx^*, a) = 0$ or, $\varphi(d(x^*, Tx^*, a)) = 0$. i.e., $d(x^*, Tx^*, a) = 0$ for all $a \in X$. Thus, $x^* = Tx^*$. □

If we take $f(s,t) = s - t$ in Theorem 4.2, we obtain Theorem 3.1 [19] as corollary.

Corollary 4.3. *Let (X, d) be a complete 2-metric space. Assume $T : X \to X$ is a modified rational α-φ-f-contraction mapping satisfying the following assertions:*

(i) *T is a triangular 2-α-admissible mapping;*
(ii) *there exists x_0 in X such that, $\alpha(x_0, Tx_0, a) \geq 1$ for all $a \in X$;*
(iii) *T is continuous or 2-α-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, a) \geq 1$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

Example 4.4. Let $X = [0, \infty)$. We define a 2-metric d on X by

$$d(x, y, a) = \min\{\sigma(x, y), \sigma(y, a), \sigma(x, a)\},$$

where

$$\sigma(u, v) = \begin{cases} \max\{u, v\} & \text{if } u \neq v \\ 0, & \text{if } u = v. \end{cases}$$

Clearly (X, d) is a complete 2-metric space. Define $T : X \to X$, $\varphi : [0, \infty) \to [0, \infty)$, $\alpha : X \times X \times X \to [0, \infty)$, and $f : [0, \infty)^2 \to [0, \infty)$ by $f = log\frac{t+a^s}{1+t}$, $\psi(t) = a^{\frac{t}{2}}$ for all $s, t \in [0, \infty)$

$$Tx = \begin{cases} \frac{1}{4} & \text{if } x \in [0, 1] \\ \cos x + 2, & \text{if } x \in (1, 300] \\ x^3 + x + 1, & \text{if } x \in [300, \infty), \end{cases}$$

and

$$\alpha(x, y, a) = \begin{cases} 1, & \text{if } x, y \in [0, 1] \text{ and } a = 0 \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Now, we prove that all the hypotheses of the Corollary 4.3 (Theorem 4.2) are satisfied and hence T has a fixed point.

Proof. As in the proof of Example 3.12 we can show that T is a 2-α-admissible mapping $\alpha(0, T0, a) \geq 1$ for all $a \in X$ and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $a \in X$ and $x_n \to x$ as $n \to +\infty$, then $\alpha(x_n, x, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and all $a \in X$.

Let $\alpha(x, y, a) \geq 1$. Then $x, y \in [0, 1]$ and hence

$$d(Tx, Ty, a) = 0 \leq \log\Big(\frac{\varphi(M(x, y, a)) + a^{M(x,y,a)}}{1 + \varphi(M(x, y, a))}\Big).$$

That is,

$$\alpha(x, y, a) \geq 1 \Longrightarrow d(Tx, Ty, a) \leq \log\left(\tfrac{\varphi(M(x,y,a)) + a^{M(x,y,a)}}{1+\varphi(M(x,y,a))}\right)$$

for all $a \in X$. Hence, T is a modified rational α-φ-contraction mapping. Then, all the conditions of the Corollary 4.3 (Theorem 4.2) are satisfied and hence T has a fixed point. □

By taking $\alpha(x, y, a) = 1$ in Theorem 4.2 we have the following Corollary.

Corollary 4.5. *Let (X, d) be a complete 2-metric space. Assume that $T : X \to X$ is a modified rational η-φ-f-contraction mapping satisfying the following assertions:*

(i) *T is a triangular 2-η-admissible mapping;*
(ii) *there exists x_0 in X such that, $\eta(x_0, Tx_0, a) \leq 1$ for all $a \in X$;*
(iii) *T is continuous or 2-η-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}, a) \leq 1$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\eta(x_n, x, a) \leq 1$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

Corollary 4.6. *Let (X, d) be a complete 2-metric space. Assume that $T : X \to X$ is a rational α-φ-f-contraction mapping satisfying the following assertions:*

(i) *T is a triangular 2-α-admissible mapping;*
(ii) *there exists x_0 in X such that, $\alpha(x_0, Tx_0, a) \geq 1$ for all $a \in X$;*
(iii) *T is continuous or 2-α-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, a) \geq 1$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

Corollary 4.7. *Let (X, d) be a complete 2-metric space. Assume that $T : X \to X$ is a rational η-φ-f-contraction mapping satisfying the following assertions:*

(i) *T is a triangular 2-η-admissible mapping;*
(ii) *there exists x_0 in X such that, $\eta(x_0, Tx_0, a) \leq 1$ for all $a \in X$;*
(iii) *T is continuous or 2-η-continuous or,*
(iv) *if $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}, a) \leq 1$ for all $a \in X$ and $x_n \to x$ as $n \to \infty$, then $\eta(x_n, x, a) \leq 1$ for all $n \in \mathbb{N}$ and all $a \in X$.*

Then T has a fixed point.

5. Fixed point results in partially ordered 2-metric spaces

Recently, there have been so many exciting developments in the field of existence of fixed points in partially ordered sets. This approach has been initiated by Ran and Reurings [35] and they also provided some applications to matrix equations. Their results are a hybrid of the two classical theorems; Banach's fixed point theorem and Tarski's fixed point theorem. Agarwal, et al. [1], Bhaskar and Lakshmikantham [6], Ciric et al. [14] and Hussain et al. [24] presented some new results for nonlinear contractions in partially ordered metric spaces and noted that their theorems can be used to investigate a large class of problems. In this section, as an application of obtained results we prove some fixed point results in partially ordered 2-metric spaces. We also note that the recent fixed point results in [18] can be deduced as simple corollaries.

Recall that if $(X, \preceq)$ is a partially ordered set and $T : X \to X$ is such that for $x, y \in X$, $x \preceq y$ implies $Tx \preceq Ty$, then mapping T is said to be non-decreasing.

Theorem 5.1. *Let $(X, d, \preceq)$ be a complete partially ordered 2-metric space. Assume that $T : X \to X$ is a mapping satisfying the following assertions:*

(i) *T is non-decreasing;*
(ii) *there exists x_0 in X such that, $x_0 \preceq Tx_0$;*
(iii) *T is continuous or,*
(iv) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*
(v)

$$d(Tx, Ty, a) \leq f(\tfrac{1}{2}[d(x, Ty, a) + d(y, Tx, a)], \psi(d(x, Ty, a), d(y, Tx, a))) \tag{5.1}$$

holds for all $x, y, a \in X$ with $x \preceq y$ or $y \preceq x$ where $\psi \in \Psi$ and $f \in \mathcal{C}$.

Then T has a fixed point.

Proof. Define the mapping $\alpha : X \times X \times X \to \mathbb{R}_+$ by

$$\alpha(x, y, a) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha(x, y, a) \geq 1$, then $x \preceq y$. From (5.2), we get

$$d(Tx, Ty, a) \leq f(\tfrac{1}{2}[d(x, Ty, a) + d(y, Tx, a)], \psi(d(x, Ty, a), d(y, Tx, a))).$$

Again let $x, y, a \in X$ such that $\alpha(x, y, a) \geq 1$. This implies that $x \preceq y$. As the mapping T is non-decreasing, we deduce that $Tx \preceq Ty$ and hence $\alpha(Tx, Ty, a) \geq 1$ for all $a \in X$. Also, let $\alpha(x, z, a) \geq 1$ and $\alpha(z, y, a) \geq 1$, then $x \preceq z$ and $z \preceq y$. So from transitivity we have $x \preceq y$. That is, $\alpha(x, y, a) \geq 1$ for all $a \in X$. Thus T is a triangular 2-α-admissible mapping. The condition (ii) ensures that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. This implies that $\alpha(x_0, Tx_0, a) \geq 1$ for all $a \in X$. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $a \in X$ and all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. So, $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Then from (iv) we have $x_n \preceq x$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n, x, a) \geq 1$ for all $n \in \mathbb{N}$ and all $a \in X$. Therefore, all the conditions of the Corollary 3.6 are satisfied, so T has a fixed point in X. $\square$

If we take $f(s, t) = s - t$ in Theorem 5.1, we obtain following main results in [18] as corollary.

Corollary 5.2 (Theorem 2.3 and 2.4 of [18])**.** *Let $(X, d, \preceq)$ be a complete partially ordered 2-metric space. Assume $T : X \to X$ is a mapping satisfying the following assertions:*

(i) *T is non-decreasing;*
(ii) *there exists x_0 in X such that, $x_0 \preceq Tx_0$;*
(iii) *T is continuous or,*
(iv) *if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*
(v)

$$d(Tx, Ty, a) \leq \frac{1}{2}[d(x, Ty, a) + d(y, Tx, a)] - \psi(d(x, Ty, a), d(y, Tx, a))) \tag{5.2}$$

holds for all $x, y, a \in X$ *with* $x \preceq y$ *or* $y \preceq x$ *where* $\psi \in \Psi$.

Then T *has a fixed point.*

Next, we assume the condition (B) as follow:

(B) for all $x, y \in X$ which are not comparable, there exists $z \in X$ such that comparable to x and y.

Theorem 5.3 (Theorem 2.5 of [18])**.** *Adding condition (B) to the hypotheses of Theorem 5.1 we obtain uniqueness of the fixed point of* T.

Proof. Define the mapping $\alpha : X \times X \times X \to \mathbb{R}_+$ as in the proof of Theorem 5.1. Let $x, y \in X$ where $\alpha(x, y, a) < 1$ and $\alpha(y, x, a) < 1$ for all $a \in X$. That is, x and y are not comparable. Hence, by condition (B) there exists $z \in X$ such that comparable to x and y. i.e., $z \preceq x$ or $x \preceq z$ and $z \preceq y$ or $y \preceq x$. That is, $\alpha(z, x, a) \geq 1$ or $\alpha(x, z, a) \geq 1$ and $\alpha(z, y, a) \geq 1$ or $\alpha(y, z, a) \geq 1$ for all $a \in X$. Then conditions of Theorem 3.11 hold and fixed point of T is unique. □

Theorem 5.4. *Let* $(X, d, \preceq)$ *be a complete partially ordered 2-metric space. Assume that* $T : X \to X$ *is a mapping satisfying the following assertions:*

(i) T *is non-decreasing;*

(ii) *there exists* x_0 *in* X *such that,* $x_0 \preceq Tx_0$*;*

(iii) T *is continuous or,*

(iv) *if* $\{x_n\}$ *is a non-decreasing sequence in* X *such that* $x_n \to x$ *as* $n \to \infty$*, then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$*;*

(v) T *is an ordered modified rational* φ*-contraction mapping, that is,*

$$d(Tx, Ty, a) \leq f(M(x, y, a), \varphi(M(x, y, a)))$$

holds for all $x, y, a \in X$ *with* $x \preceq y$ *or* $y \preceq x$ *where* $\varphi \in \Psi_\varphi$, $f \in \mathcal{C}$ *and*

$$M(x, y, a) = \max\left\{d(x, y, a), d(x, Tx, a), d(y, Ty, a), \frac{d(x, Ty, a) + d(y, Tx, a)}{2}, \frac{d(x, Tx, a)d(y, Ty, a)}{1 + d(Tx, Ty, a)}\right\}.$$

Then T *has a fixed point.*

Proof. Define the mapping $\alpha : X \times X \times X \to \mathbb{R}_+$ by

$$\alpha(x, y, a) = \begin{cases} 1 & \text{if } x \preceq y \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha(x, y, a) \geq 1$, then $x \preceq y$. From (5.3), we get

$$d(Tx, Ty, a) \leq f(M(x, y, a), \varphi(M(x, y, a))).$$

Again let $x, y, a \in X$ such that $\alpha(x, y, a) \geq 1$. This implies that $x \preceq y$. As the mapping T is non-decreasing, we deduce that $Tx \preceq Ty$ and hence $\alpha(Tx, Ty, a) \geq 1$ for all $a \in X$. Also, let $\alpha(x, z, a) \geq 1$ and $\alpha(z, y, a) \geq 1$, then, $x \preceq z$ and $z \preceq y$. So from transitivity we have, $x \preceq y$. That is, $\alpha(x, y, a) \geq 1$ for all $a \in X$. Thus T is a triangular 2-α-admissible mapping. The condition (ii) ensures that there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$. This implies that $\alpha(x_0, Tx_0, a) \geq 1$ for all $a \in X$. Let $\{x_n\}$ be a sequence in X such that $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $a \in X$ and all $n \in \mathbb{N}$

and $x_n \to x$ as $n \to \infty$. So, $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. Then from (iv) we have $x_n \preceq x$ for all $n \in \mathbb{N}$. That is, $\alpha(x_n, x, a) \geq 1$ for all $n \in \mathbb{N}$ and all $a \in X$. Therefore, all the conditions of the Corollary 4.3 are satisfied so T has a fixed point in X. $\square$

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P. Sukprasert
Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT) 126, Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
E-mail address: happy_t_ik@hotmail.com

P. Kumam
Theoretical and Computational Science (TaCS) Center, Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT) and, Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT) 126, Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
E-mail address: poom.kum@kmutt.ac.th and poom.kumam@mail.kmutt.ac.th

A.H. Ansari
Department of Mathematics,College of Basic Sciences, Karaj Branch,Islamic Azad University, Alborz, Iran
E-mail address: analsisamirmath2@gmail.com

S. Chandok
School of Mathematics, Thapar University, Patiala-147004, India
E-mail address: sumit.chandok@thapar.edu

N. Hussain
Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: nhusain@kau.edu.sa