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SECOND-ORDER SUBDIFFERENTIALS AND OPTIMALITY CONDITIONS FOR C^1-SMOOTH OPTIMIZATION PROBLEMS

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Abstract. This paper investigates the possibility of using the Fréchet and limiting second-order subdifferentials to characterize locally optimal solutions of C^1-smooth unconstrained minimization problems. We prove that, for a C^1-smooth function of one real variable or a C^1-smooth function on a Banach space with its derivative being calm at the reference point, the positive semi-definiteness of its Fréchet second-order subdifferential at the point in question is a necessary optimality condition, while this is not true for the limiting counterpart. However, the limiting second-order subdifferential of a $C^{1,1}$-smooth function on $\mathbb{R}^n$ at a local minimizer is positively semi-definite along some of its selection. We also show that, for a C^1-smooth function on an Asplund space, the positive semi-definiteness of its Fréchet second-order subdifferential around a stationary point is sufficient for this point to be a local minimizer of the function. Besides, a sufficient condition via the Fréchet second-order subdifferential for a point to be a tilt stable minimizer is given.

1. Introduction

As suggested by Mordukhovich [14], one can define a second-order subdifferential as the coderivative of a subdifferential mapping. This approach has been used widely in variational analysis and its applications [15, 24]. One can apply second-order subdifferentials to examine sensitivity and stability for variational systems [14, 15]. Especially, Poliquin and Rockafellar [23] proved that the positive definiteness of the limiting second-order subdifferential mapping $\partial^2\varphi(\bar{x}, 0) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ characterizes the tilt stability of a stationary point $\bar{x}$ of a subdifferentially continuous prox-regular function $\varphi : \mathbb{R}^n \to \bar{\mathbb{R}}$, and Levy, Poliquin and Rockafellar [13] showed that the positive definiteness of a parametric limiting second-order subdifferential mapping can be used to study the full stability of the locally optimal points. For the class of prox-bounded functions, some sufficient conditions for the so-called strict local minimizer of order two, which are expressed via second-order subdiffential, were established in [9–11]. Recently, it has been shown in [2–7] that the convexity of a real-valued function can be characterized via the Fréchet and/or limiting second-order subdifferentials.

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Despite a great progress in the theory of second-order subdifferentials, to the best of our knowledge, several basic questions about the use of second-order subdifferentials for optimality conditions have been open so far. For example, to which extent the positive semi-definiteness of second-order subdifferentials is a necessary optimality condition, or what about second-order subdifferential sufficient optimality conditions for minimization problems without prox-regularity or prox-boundedness assumptions.

Our aim in this paper is to investigate the aforementioned questions. More precisely, we examine the possibility of using the Fréchet and limiting second-order subdifferentials to characterize local minimizers of C^1 -smooth unconstrained minimization problems. We prove, for a C^1-smooth function of one real variable or a C^1-smooth function on a Banach space with its derivative being calm at the reference point, the positive semi-definiteness of its Fréchet second-order subdifferential at the reference point is a necessary optimality condition, while it is not true for the limiting counterpart; however, the limiting second order subdifferential of a $C^{1,1}$-smooth function on $\mathbb{R}^n$ at a local minimizer is positively semi-definite along certain selection. We also show that, for a C^1-smooth function on an Asplund space, the positive semi-definiteness of its Fréchet second-order subdifferential around a stationary point is sufficient for this point to be a local minimizer of the function. Besides, a sufficient condition via the Fréchet second-order subdifferential for a point to be a tilt stable minimizer is given.

The paper is organized as follows. After recalling some background material from variational analysis in Section 2, we obtain necessary optimality conditions in Section 3. Finally, Section 4 provides sufficient optimality conditions.

2. Preliminaries

This section recalls some background material from variational analysis, which are needed in the sequel. For more details, we refer the reader to the monographs [15,24].

Let $F : X \rightrightarrows Y$ be a set-valued mapping between topological spaces X and Y. As usual, the effective domain and the graph of F are given, respectively, by

$$\text{dom}F := \big\{x \in X \mid F(x) \neq \emptyset\big\} \ \text{ and } \ \text{gph}F := \big\{(x, y) \in X \times Y \mid y \in F(x)\big\}.$$

The *sequential Painlevé-Kuratowski upper limit* of F at a point $\bar{x}$ in the topology of Y is defined by

$$\operatorname*{Lim\,sup}_{x \to \bar{x}} F(x) := \Big\{ y \in Y \Big| \ \exists \text{ sequences } x_k \to \bar{x} \text{ and } y_k \to y \text{ with } y_k \in F(x_k) \text{ for all } k = 1, 2, \ldots \Big\}.$$

In the sequel, unless otherwise stated, X and Y are Banach spaces, and Ω is a nonempty subset of X. Given a Banach space X, we denote by X^* its topological dual space and by X^{**} its second topological dual space, that is, $X^{**} = (X^*)^*$.

Given $\varepsilon \geq 0$, define the collection of *ε-normals* to Ω at $\bar{x} \in \Omega$ by

$$\widehat{N}_\varepsilon(\bar{x}; \Omega) := \Big\{ x^* \in X^* \Big| \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \leq \varepsilon \Big\}, \tag{2.1}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. If $\bar{x} \notin \Omega$, one puts $\widehat{N}_\varepsilon(\bar{x};\Omega) := \emptyset$ for all $\varepsilon \geq 0$.

When $\varepsilon = 0$, the set $\widehat{N}(\bar{x};\Omega) := \widehat{N}_0(\bar{x};\Omega)$ in (2.1) is a cone, which is called the *Fréchet normal cone* to Ω at $\bar{x}$.

The *limiting normal cone* $N(\bar{x};\Omega)$ is obtained from $\widehat{N}_\varepsilon(x;\Omega)$ by taking the sequential Painlevé-Kuratowski upper limit in the weak* topology of X^* as

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\Omega} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega),$$

where one can put $\varepsilon = 0$ when Ω is closed around $\bar{x}$ and X is an *Asplund space*, i.e., a Banach space whose separable subspaces have separable duals. If $\bar{x} \notin \Omega$, one puts $N(\bar{x};\Omega) := \emptyset$.

The *Fréchet coderivative* of F at $(\bar{x},\bar{y}) \in X \times Y$ is defined by

$$\widehat{D}^*F(\bar{x},\bar{y})(y^*) := \Big\{x^* \in X^* \mid (x^*,-y^*) \in \widehat{N}\big((\bar{x},\bar{y});\operatorname{gph} F\big)\Big\} \quad \forall y^* \in Y^*.$$

The *limiting normal coderivative* of F at $(\bar{x},\bar{y}) \in X \times Y$ is defined by

$$D^*F(\bar{x},\bar{y})(y^*) := \Big\{x^* \in X^* \Big| \ (x^*,-y^*) \in N\big((\bar{x},\bar{y});\operatorname{gph} F\big)\Big\} \quad \forall y^* \in Y^*.$$

If $F(\bar{x}) = \{\bar{y}\}$, then we will omit $\bar{y}$ in the corderivative notation.

A single-valued mapping $\varphi\colon X \to Y$ is said to be *strictly differentiable* at $\bar{x}$ if there is a linear continuous operator $\nabla\varphi(\bar{x})\colon X \to Y$ such that

$$\lim_{x,u \to \bar{x}} \frac{\varphi(x) - \varphi(u) - \langle \nabla\varphi(\bar{x}), x - u\rangle}{\|x - u\|} = 0.$$

It is known that for such mappings one has

$$D^*\varphi(\bar{x})(y^*) = \widehat{D}^*\varphi(\bar{x})(y^*) = \Big\{(\nabla\varphi(\bar{x}))^*y^*\Big\} \quad \forall y^* \in Y^*,$$

where the second equality is still valid if φ is merely Fréchet differentiable at $\bar{x}$; see [15, Theorem 1.38].

Let $\varphi : X \to \bar{\mathbb{R}} := [-\infty,\infty]$ be an extended real-valued function. We define

$$\operatorname{dom}\varphi = \big\{x \in X \mid |\varphi(x)| < \infty\big\}, \quad \operatorname{epi}\varphi = \big\{(x,\mu) \in X \times \mathbb{R} \mid \mu \geq \varphi(x)\big\}.$$

The *Fréchet subdifferential* $\widehat{\partial}\varphi(\bar{x})$ of φ at $\bar{x} \in \operatorname{dom}\varphi$ is defined by

$$\widehat{\partial}\varphi(\bar{x}) := \Big\{x^* \in X^* \mid (x^*,-1) \in \widehat{N}\big((\bar{x},\varphi(\bar{x}));\operatorname{epi}\varphi\big)\Big\}.$$

The *limiting subdifferential* $\partial\varphi(\bar{x})$ of φ at $\bar{x} \in \operatorname{dom}\varphi$ is defined by

$$\partial\varphi(\bar{x}) := \Big\{x^* \in X^* \mid (x^*,-1) \in N\big((\bar{x},\varphi(\bar{x}));\operatorname{epi}\varphi\big)\Big\}.$$

If $\bar{x} \notin \operatorname{dom}\varphi$, then one puts $\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x}) = \emptyset$.

Obviously, $\widehat{\partial}\varphi(\bar{x}) \subset \partial\varphi(\bar{x})$ and $\widehat{\partial}\varphi(\bar{x})$ is a closed convex set (may be empty). Note that $\partial\varphi(\bar{x})$ is nonconvex in general [15]. If $\varphi : \mathbb{R} \to \mathbb{R}$ is Lipschitz near $\bar{x}$ and $\widehat{\partial}\varphi(\bar{x}) \neq \emptyset$, then $\partial\varphi(\bar{x})$ is convex; see [1, Corollary 3.1].

The function φ is said to be *lower regular* at $\bar{x}$ if $\widehat{\partial}\varphi(\bar{x}) = \partial\varphi(\bar{x})$; see [15, Definition 1.91]. If φ is convex and $\varphi(\bar{x})$ is finite, then

$$\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x}) = \big\{x^* \in X^* \mid \langle x^*, x - \bar{x}\rangle \le \varphi(x) - \varphi(\bar{x}) \ \ \forall x \in X\big\};$$

see [15, Theorem 1.93]. If φ is strictly differentiable at $\bar{x}$, then

$$\partial\varphi(\bar{x}) = \widehat{\partial}\varphi(\bar{x}) = \big\{\nabla\varphi(\bar{x})\big\},$$

where the second equality also holds if φ is merely Fréchet differentiable at $\bar{x}$. Thus, C^1 functions (i.e., continuously differentiable functions) and convex functions are lower regular at any point in their effective domains.

If φ is lower regular at $\bar{x}$, then $\partial\varphi(\bar{x})$ is convex. The converse is invalid even for Fréchet differentiable, Lipschitz functions; see [1, Example 3.3].

Definition 2.1 ([15])**.** Let $\varphi : X \to \bar{\mathbb{R}}$ be a function with a finite value at $\bar{x}$.

(i) For any $\bar{y} \in \partial\varphi(\bar{x})$, the map $\partial^2\varphi(\bar{x}, \bar{y}) : X^{**} \rightrightarrows X^*$ with the values

$$\partial^2\varphi(\bar{x}, \bar{y})(u) = (D^*\partial\varphi)(\bar{x}, \bar{y})(u) \quad (u \in X^{**})$$

is said to be the *limiting second-order subdifferential* of φ at $\bar{x}$ relative to $\bar{y}$.

(ii) For any $\bar{y} \in \widehat{\partial}\varphi(\bar{x})$, the map $\widehat{\partial}^2\varphi(\bar{x}, \bar{y}) : X^{**} \rightrightarrows X^*$ with the values

$$\widehat{\partial}^2\varphi(\bar{x}, \bar{y})(u) = (\widehat{D}^*\widehat{\partial}\varphi)(\bar{x}, \bar{y})(u) \quad (u \in X^{**})$$

is said to be the *Fréchet second-order subdifferential* of φ at $\bar{x}$ relative to $\bar{y}$.

The symbol $\bar{y}$ in the second-order subdifferential notation will be removed when the corresponding subdifferential of φ at $\bar{x}$ is $\{\bar{y}\}$. In general, the limiting second-order subdifferential and the Fréchet second-order subdifferential are incomparable. However, if φ is lower regular around $\bar{x} \in X$ and $\bar{y} \in \widehat{\partial}\varphi(\bar{x})$, then

$$\widehat{\partial}^2\varphi(\bar{x}, \bar{y})(u) \subset \partial^2\varphi(\bar{x}, \bar{y})(u) \quad \forall u \in X^{**}.$$

If φ is Fréchet differentiable around $\bar{x}$ and $\nabla\varphi$ is strictly differentiable at $\bar{x}$, then

$$\partial^2\varphi(\bar{x})(u) = \widehat{\partial}^2\varphi(\bar{x})(u) = \big\{(\nabla^2\varphi(\bar{x}))^* u\big\} \quad \forall u \in X^{**};$$

the second equality is still true if φ is twice Fréchet differentiable at $\bar{x}$; see [15, Proposition 1.119]. We refer the reader to [2, 3, 5, 6, 13, 15, 17–20, 23, 24] for more information on the second-order subdifferentials and their applications.

Definition 2.2 (see [22] and [24, Chap. 12])**.** One says that a set-valued map $T : X \rightrightarrows X^*$ is a *monotone operator* if

$$\langle x^* - y^*, x - y\rangle \ge 0 \quad \text{for all } \ x, y \in X, \ x^* \in T(x), \ y^* \in T(y).$$

We refer the reader to [22, 24] for detailed information on monotone operators and their applications.

By analogy with positive semi-definiteness and positive definiteness of real matrices, one can consider the following concepts.

Definition 2.3. A set-valued map $T : X \rightrightarrows X^*$ is *positive semi-definite* if $\langle z, u\rangle \ge 0$ for any $u \in X$ and $z \in T(u)$. If $\langle z, u\rangle > 0$ whenever $u \in X \setminus \{0\}$ and $z \in T(u)$, then T is said to be *positive definite*.

3. Necessary Optimality Conditions

In this section, we establish some new necessary optimality conditions by using the second-order subdifferentials. We begin by considering the problem

$$\text{(P)} \qquad \min\{\varphi(x) \mid x \in \mathbb{R}\},$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a C^1 function.

Theorem 3.1. *If $\bar{x}$ is a local solution of* (P)*, then $\nabla\varphi(\bar{x}) = 0$ and the Fréchet second-order subdifferential $\widehat{\partial}^2\varphi(\bar{x})$ is positive semi-definite, i.e., for any $u \in \mathbb{R}$ and $z \in \widehat{\partial}^2\varphi(\bar{x})(u)$, it holds that $zu \geq 0$.*

Proof. First, since $\bar{x}$ is a local solution of (P), by the Fermat rule, $\nabla\varphi(\bar{x}) = 0$. To obtain the second assertion of the theorem, suppose to the contrary that $\widehat{\partial}^2\varphi(\bar{x})$ is not positive semi-definite. Then there exist $u \in \mathbb{R}$ and $z \in \widehat{\partial}^2\varphi(\bar{x})(u)$ with $zu < 0$. By definition of the Fréchet second-order subdifferential,

$$\begin{aligned} z \in \widehat{\partial}^2\varphi(\bar{x})(u) \quad &\Leftrightarrow z \in \widehat{D}^*\nabla\varphi(\cdot)(\bar{x})(u) \\ &\Leftrightarrow (z, -u) \in \widehat{N}\big((\bar{x}, 0); \mathrm{gph}\nabla\varphi(\cdot)\big) \\ &\Leftrightarrow \limsup_{x\to\bar{x}} \frac{\big\langle (z,-u), (x, \nabla\varphi(x)) - (\bar{x}, 0)\big\rangle}{|x-\bar{x}| + |\nabla\varphi(x) - \nabla\varphi(\bar{x})|} \leq 0. \end{aligned}$$

Reducing the latter, we obtain

$$\limsup_{x\to\bar{x}} \frac{z(x-\bar{x}) - u\nabla\varphi(x)}{|x-\bar{x}| + |\nabla\varphi(x)|} \leq 0. \tag{3.1}$$

Since $zu < 0$, it must happen one of the following cases.
Case 1: $z > 0$ and $u < 0$. Take a sequence $x_k \downarrow \bar{x}$. Since $\bar{x}$ is a local solution of (P), by invoking the classical mean value theorem, for each k large enough, we find $\xi_k \in (\bar{x}, x_k)$ such that

$$0 \leq \varphi(x_k) - \varphi(\bar{x}) = \nabla\varphi(\xi_k)(x_k - \bar{x}).$$

As $x_k - \bar{x} > 0$, this forces

$$\nabla\varphi(\xi_k) \geq 0. \tag{3.2}$$

Due to (3.1), we have

$$\limsup_{k\to\infty} \frac{z(\xi_k-\bar{x}) - u\nabla\varphi(\xi_k)}{|\xi_k-\bar{x}| + |\nabla\varphi(\xi_k)|} \leq 0. \tag{3.3}$$

Let $r := \min\{z, -u\}$, $r > 0$. By (3.2), we have

$$\begin{aligned} \Delta_k : \ &= \frac{z(\xi_k-\bar{x}) - u\nabla\varphi(\xi_k)}{|\xi_k-\bar{x}| + |\nabla\varphi(\xi_k)|} \\ &= \frac{z(\xi_k-\bar{x}) - u\nabla\varphi(\xi_k)}{\xi_k-\bar{x} + \nabla\varphi(\xi_k)} \\ &\geq \frac{r(\xi_k-\bar{x}) + r\nabla\varphi(\xi_k)}{\xi_k-\bar{x} + \nabla\varphi(\xi_k)} \\ &= r > 0. \end{aligned}$$

This contradicts (3.3), which says that $\limsup\limits_{k\to\infty} \Delta_k \leq 0$.

Case 2: $z < 0$ and $u > 0$. Take a sequence $x_k \uparrow \bar{x}$. For each k large enough, there exists $\xi_k \in (x_k, \bar{x})$ such that

$$0 \le \varphi(x_k) - \varphi(\bar{x}) = \nabla\varphi(\xi_k)(x_k - \bar{x}).$$

As $x_k - \bar{x} < 0$, it holds that

$$\nabla\varphi(\xi_k) \le 0. \tag{3.4}$$

By virtue of (3.1), inequality (3.3) holds. Set $r := \max\{z, -u\}$. Then we have $r < 0$. Therefore, due to (3.4), we obtain

$$\begin{aligned}\Delta_k: &= \frac{z(\xi_k - \bar{x}) - u\nabla\varphi(\xi_k)}{|\xi_k - \bar{x}| + |\nabla\varphi(\xi_k)|}\\ &= -\frac{z(\xi_k - \bar{x}) - u\nabla\varphi(\xi_k)}{\xi_k - \bar{x} + \nabla\varphi(\xi_k)}\\ &\ge \frac{-r(\xi_k - \bar{x}) - r\nabla\varphi(\xi_k)}{\xi_k - \bar{x} + \nabla\varphi(\xi_k)}\\ &= -r > 0,\end{aligned}$$

which contradicts (3.3). The proof is complete. □

In Theorem 1, even for $C^{1,1}$ functions, we cannot replace $\widehat{\partial}^2\varphi(\bar{x})$ by $\partial^2\varphi(\bar{x})$. This can be seen from the next example.

Example 3.2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the function defined by $\varphi(x) := \int_0^x g(t)dt$. Here $g : \mathbb{R} \to \mathbb{R}$ is the function given by the following rules:

(1) $g(0) := 0$;
(2) $g(t) := \frac{1}{2}t$ for $t > \frac{1}{2}$;
(3) if $k \in \{1, 2, ...\}$ and $\frac{1}{2^{k+1}} < t \le \frac{3}{2^{k+2}}$, then $g(t) := 2t - \frac{3}{2^{k+2}}$;
(4) if $k \in \{1, 2, ...\}$ and $\frac{3}{2^{k+2}} < t \le \frac{1}{2^k}$, then $g(t) := -t + \frac{3}{2^{k+1}}$;
(5) $g(t) := -g(-t)$ for each $t < 0$.

Since g is continuous on $\mathbb{R}\backslash\{0\}$, and $|g(t)| \le 3|t|$ for all $t \in \mathbb{R}$, g is continuous on $\mathbb{R}$. Thus φ is well-defined and $\nabla\varphi(x) = g(x)$ for all $x \in \mathbb{R}$. Clearly, $\bar{x} := 0$ is a global minimizer of φ. For each $k = 1, 2, ...$, take any $x_k \in \left(\frac{3}{2^{k+2}}, \frac{1}{2^k}\right)$. By simple computation, we have

$$\widehat{N}\Big((x_k, \nabla\varphi(x_k)); \mathrm{gph}\nabla\varphi\Big) = \mathbb{R}(1,1) \quad \forall k = 1, 2, ...$$

Since $\big(x_k, \nabla\varphi(x_k)\big) \xrightarrow{\mathrm{gph}\nabla\varphi} (0,0)$ as $k \to \infty$, and $\mathrm{gph}\nabla\varphi$ is closed, it holds that

$$\mathbb{R}(1,1) \subset N\Big((0,0); \mathrm{gph}\nabla\varphi\Big).$$

Observe that for $(z, u) := (1, -1)$, we have that $z \in \partial^2\varphi(0)(u)$ and $zu < 0$. Note that g is Lipschitz on $\mathbb{R}$. Therefore, the positive semi-definiteness of the limiting second-order subdiffential is not a necessary optimality condition for $C^{1,1}$ functions.

Let $\varphi : X \to \mathbb{R}$ be a C^1 function defined on a Banach space X. Consider the following problem:

$$(\mathrm{P}_1) \qquad \min\{\varphi(x) \mid x \in X\}.$$

Theorem 3.3. *Suppose that $\bar{x}$ is a local solution of (P_1) and there exists $\ell > 0$ such that*

$$\|\nabla\varphi(x) - \nabla\varphi(\bar{x})\| \leq \ell\|x - \bar{x}\| \tag{3.5}$$

for every x in some neighborhood of $\bar{x}$. Then $\nabla\varphi(\bar{x}) = 0$ and the Fréchet second-order subdifferential $\widehat{\partial}^2\varphi(\bar{x}) : X \rightrightarrows X^*$, where X is canonically embedded in X^{**}, is positive semi-definite.

Proof. Let $\bar{x}$ be a local solution of (P_1). Then, by the Fermat rule, $\nabla\varphi(\bar{x}) = 0$. In order to obtain the positive semi-definiteness of $\widehat{\partial}^2\varphi(\bar{x})$, suppose to the contrary that there exist $u \in X$ and $z \in \widehat{\partial}^2\varphi(\bar{x})(u)$ with

$$\langle z, u\rangle < 0. \tag{3.6}$$

Then, $z \in \widehat{D}^*\nabla\varphi(\cdot)(\bar{x})(u)$ or, equivalently, $(z, -u) \in \widehat{N}\big((\bar{x}, 0); \mathrm{gph}\nabla\varphi(\cdot)\big)$. Recall that X is embedded in X^{**} and $\nabla\varphi(\cdot) : X \to X^*$. Hence

$$\limsup_{x\to\bar{x}} \frac{\langle (z, -u), \big(x, \nabla\varphi(x)\big) - (\bar{x}, 0)\rangle}{\|x - \bar{x}\| + \|\nabla\varphi(x)\|} \leq 0. \tag{3.7}$$

For $x_k := \bar{x} - \frac{1}{k}u$, $k = 1, 2, ...$, one has $x_k \to \bar{x}$ as $k \to \infty$. Since $\bar{x}$ is a local solution of (P_1), for large indexes k, by the classical mean value theorem, there exists $\xi_k \in (\bar{x}, x_k) := \{(1-t)\bar{x} + tx_k \mid t \in (0, 1)\}$ such that $0 \leq \varphi(x_k) - \varphi(\bar{x}) = \langle\nabla\varphi(\xi_k), x_k - \bar{x}\rangle$. Noticing that $x_k - \bar{x} = -\frac{1}{k}u$, this forces

$$\langle\nabla\varphi(\xi_k), u\rangle \leq 0. \tag{3.8}$$

By (3.7) and noting that $\xi_k = \bar{x} - t_k u$ for some $t_k \in (0, \frac{1}{k})$, we have

$$\limsup_{k\to\infty} \frac{\langle z, -t_k u\rangle - \langle u, \nabla\varphi(\xi_k)\rangle}{\|t_k u\| + \|\nabla\varphi(\xi_k)\|} \leq 0. \tag{3.9}$$

By our assumption, for k large enough,

$$\begin{aligned} \|\nabla\varphi(\xi_k)\| &= \|\nabla\varphi(\xi_k) - \nabla\varphi(\bar{x})\| \\ &\leq \ell\|\xi_k - \bar{x}\| = \ell t_k\|u\|. \end{aligned}$$

So, by (3.6) and (3.8) we obtain

$$\begin{aligned} \Delta_k &:= \frac{\langle z, -t_k u\rangle - \langle u, \nabla\varphi(\xi_k)\rangle}{\|t_k u\| + \|\nabla\varphi(\xi_k)\|} \\ &\geq \frac{\langle z, -u\rangle}{\|u\| + t_k^{-1}\|\nabla\varphi(\xi_k)\|} \\ &\geq \frac{\langle z, -u\rangle}{\|u\| + \ell\|u\|} > 0, \end{aligned}$$

which contradicts (3.9). The proof is complete. □

Remark 3.4. Theorem 3.1 does not need the assumption (3.5). It is still unclear to us whether this condition can be dropped in the formulation of Theorem 3.3, or not.

The following example shows that Theorem 3.3 does not cover the class of problems in Theorem 3.1.

Example 3.5. Let us consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \le 0, \\ x^{\frac{3}{2}} & \text{if } x > 0, \end{cases}$$

and $\bar{x} = 0$. Since $\nabla\varphi(x) = \frac{3}{2}\sqrt{x}$ for $x > 0$, condition (3.5) is invalid. Hence Theorem 3.3 fails to apply to this situation. However, due to the continuous differentiability of φ, Theorem 3.1 is applicable for this example.

Example 3.6. Consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \le 0, \\ 3x^2 & \text{if } x > 0, \end{cases}$$

and let $\bar{x} = 0$. Both Theorems 3.1 and 3.3 are applicable to this example.

Observe that if $\operatorname{dom}\widehat{\partial}^2\varphi(\bar{x}) = \{0\}$ (Example 3.2, for instance), the second-order conditions given by Theorems 3.1 and 3.3 is trivial. However, by using the limiting second-order subdiffential we can receive some meaningful second-order information.

Theorem 3.7. *Let $\bar{x}$ be a local solution of (P_1), where $X = \mathbb{R}^n$ and φ is a $C^{1,1}$ function on X. Then $\nabla\varphi(\bar{x}) = 0$ and the limiting second-order subdifferential $\partial^2\varphi(\bar{x}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has a selection $z(\cdot)$ such that*

$$\langle z(u), u\rangle \ge 0 \quad \forall u \in \mathbb{R}^n. \tag{3.10}$$

Here a selection means a mapping $z : \mathbb{R}^n \to \mathbb{R}^n$ with $z(u) \in \partial^2\varphi(\bar{x})(u)$ for all $u \in \mathbb{R}^n$.

Proof. Suppose that $\bar{x}$ is a local solution of (P_1), where $X = \mathbb{R}^n$ and φ is a $C^{1,1}$ function. The equality $\nabla\varphi(\bar{x}) = 0$ is due to the Fermat rule. To prove the second assertion, let

$$J_B\nabla\varphi(\bar{x}) := \Big\{ \lim_{k\to\infty} \nabla^2\varphi(x_k) \;\Big|\; \{x_k\} \subset \Omega_{\nabla\varphi},\ x_k \to \bar{x} \text{ and } \lim_{k\to\infty} \nabla^2\varphi(x_k) \text{ exists in } \mathbb{R}^{n\times n}\Big\}.$$

Here $\Omega_{\nabla\varphi} := \{x \in \mathbb{R}^n \mid \nabla^2\varphi(x) \text{ exists}\}$. It is well-known [12,21] that $J_B\nabla\varphi(\bar{x})$ is an approximate Hessian of φ at $\bar{x}$. Since $\bar{x}$ is a local solution of (P_1), by [12, Corollary 7.4], for each $u \in \mathbb{R}^n$, there exists $M_u \in J_B\nabla\varphi(\bar{x})$ such that $\langle M_u u, u\rangle \ge 0$. Define $z : \mathbb{R}^n \to \mathbb{R}^n$ by $z(u) := M_u^T u$ for all $u \in \mathbb{R}^n$. Then z is an operator with the property that $\langle z(u), u\rangle \ge 0$ for all $u \in \mathbb{R}^n$. We now take any $u \in \mathbb{R}^n$. By definition of $J_B\nabla\varphi(\bar{x})$, there exists $\{x_k\} \in \mathbb{R}^n$ such that

$$M_u = \lim_{k\to\infty} \nabla^2\varphi(x_k). \tag{3.11}$$

By [15, Theorem 1.38] or [16, Proposition 3.5],

$$\widehat{D}^*\nabla\varphi(x_k)(u) = \widehat{\partial}\langle u, \nabla\varphi\rangle(x_k) = \{(\nabla^2\varphi(x_k))^T u\},$$

for all k large enough. Hence

$$\big(\nabla^2\varphi(x_k)^T u, -u\big) \in \widehat{N}\Big(\big(x_k, \nabla\varphi(x_k)\big); \operatorname{gph}\nabla\varphi\Big),$$

for all k large enough. Taking the limits as $k \to \infty$, by (3.11) we have

$$\big(M_u^T u, -u\big) \in N\Big(\big(\bar{x}, \nabla\varphi(\bar{x})\big); \mathrm{gph}\nabla\varphi\Big).$$

Therefore, $z(u) = M_u^T u \in \partial^2\varphi(\bar{x})(u)$ for all $u \in \mathbb{R}^n$. □

We note that the second-order necessary optimality conditions provided by Theorem 3.1 and Theorem 3.3 on one hand, and by Proposition 3.7 on the other hand, are independent. For more details, let us consider the following example.

Example 3.8. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the function defined by $\varphi(x) := \int_0^x g(t)dt$. Here $g : \mathbb{R} \to \mathbb{R}$ is the function given by the following rules:
(1) $g(0) := t$ for $t \leq 0$;
(2) $g(t) := 0$ for $t > 1$;
(3) if $k \in \{0, 1, 2, ...\}$ and $\frac{1}{2^{k+1}} < t \leq \frac{1}{2^k}$, then

$$g(t) := -\frac{3}{2}\Big(\big|t - \frac{3}{2^{k+2}}\big| - \frac{1}{2^{k+2}}\Big).$$

We have that g is a Lipschitz function on $\mathbb{R}$. Hence φ is well-defined, $\nabla\varphi(x) = g(x)$ for all $x \in \mathbb{R}$, and moreover φ is $C^{1,1}$. Let $\bar{x} = 0$. Note that φ is not a convex function (the second-order information becomes more important). We have that

$$\widehat{N}\Big(\big(\bar{x}, \nabla\varphi(\bar{x})\big); \mathrm{gph}\nabla\varphi\Big) = \big\{(z, u) \in \mathbb{R}^2 \mid -z \leq u \leq -2z\big\}.$$

Hence,

$$\widehat{\partial}^2\varphi(\bar{x})(u) = \begin{cases} [u, \frac{1}{2}u] & \text{if } u \geq 0, \\ \emptyset & \text{if } u < 0. \end{cases}$$

Observe that $\bar{x}$ is a global minimizer of φ. Thus, according to Theorem 3.1 or Theorem 3.3, if $u \in \mathbb{R}$ and $z \in \widehat{\partial}^2\varphi(\bar{x})(u)$, then $zu \geq 0$; while Theorem 3.7 says that the limiting second-order subdifferential $\partial^2\varphi(\bar{x}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has a selection $z(\cdot)$ (for instance, $z(u) := u$ for all $u \in \mathbb{R}$) such that $\langle z(u), u\rangle \geq 0 \quad \forall u \in \mathbb{R}^n$. Therefore, in this situation, Theorem 3.7 provides more information than Theorem 3.1 and Theorem 3.3 for $u < 0$, the same at $u = 0$, but less for $u > 0$.

4. Sufficient Optimality Conditions

This section is devoted to the study of sufficient optimality conditions. Our first result in this direction reads as follows.

Theorem 4.1 (Optimality condition I)**.** *Let X be an Asplund space and $\bar{x} \in X$. Suppose that $\varphi : X \to \mathbb{R}$ is a C^1 function, and $\nabla\varphi(\bar{x}) = 0$. If there exists $\delta > 0$ such that, for all $x \in \mathbb{B}_\delta(\bar{x})$,*

$$\langle z, u\rangle \geq 0 \quad \forall u \in X,\ \forall z \in \widehat{\partial}^2\varphi(x)(u), \tag{4.1}$$

that is, $\widehat{\partial}^2\varphi(x)$ is positive semi-definite for all $x \in \mathbb{B}_\delta(\bar{x})$, then $\bar{x}$ is a local solution of (P_1).

Proof. If (4.1) is valid, then by [4, Corollary 3.5] (also, see [2, Theorem 3.1] or [3, Corollary 3.8]) φ is convex on $\mathbb{B}_\delta(\bar{x})$. Since $\nabla\varphi(\bar{x}) = 0$, by the convexity of φ, it holds that

$$\varphi(x) - \varphi(\bar{x}) \geq \langle \nabla\varphi(\bar{x}), x - \bar{x}\rangle = 0 \quad \forall x \in \mathbb{B}_\delta(\bar{x}).$$

Hence $\bar{x}$ is a local solution of (P_1). □

In Theorem 4.1, checking the positive definiteness *around* $\bar{x}$ only to justify that $\bar{x}$ is a local solution is our unexpected thing. It would be more favorable if condition (4.1) was replaced by the positive definiteness of the Fréchet second-order subdifferential $\widehat{\partial}^2\varphi(\bar{x})$. Unfortunately, as shown by the following example, this is impossible.

Example 4.2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the function defined by $\varphi(x) := \int_0^x g(t)dt$, where

$$g(t) := \begin{cases} |t|^\alpha \cos\frac{1}{t} & \text{if } t \neq 0, \\ 0 & \text{if } t = 0, \end{cases}$$

with $0 < \alpha \leq 1$. Let $\bar{x} = 0$. Since g is continuous, φ is C^1 and $\nabla\varphi(x) = g(x)$ for all x. We claim that $\widehat{N}\big((\bar{x}, \nabla\varphi(\bar{x})); \mathrm{gph}\nabla\varphi\big) = \{(0,0)\}$. Indeed, since $\{(0,0)\} \subset \widehat{N}\big((\bar{x}, \nabla\varphi(\bar{x})); \mathrm{gph}\nabla\varphi\big)$, it suffices to prove the reverse inclusion. Take any $(z,u) \in \widehat{N}\big((\bar{x}, \nabla\varphi(\bar{x})); \mathrm{gph}\nabla\varphi\big)$. It holds that

$$\begin{aligned} 0 &\geq \limsup_{x\to\bar{x}} \frac{\langle (z,u), (x, \nabla\varphi(x)) - (\bar{x}, \nabla\varphi(\bar{x}))\rangle}{|x - \bar{x}| + |\nabla\varphi(x) - \nabla\varphi(\bar{x})|} \\ &= \limsup_{x\to 0} \frac{zx + u\nabla\varphi(x)}{|x| + |\nabla\varphi(x)|} \\ &= \limsup_{x\to 0} \frac{zx + ug(x)}{|x| + |g(x)|}. \end{aligned} \tag{4.2}$$

Choosing $x_k := \frac{1}{\frac{\pi}{2}+2k\pi}$, $k \in \mathbb{N}$, we have that $x_k \to 0$ as $k \to \infty$ and $g(x_k) = 0$ for all $k \in \mathbb{N}$. By (4.2), $0 \geq \limsup\limits_{k\to\infty} \frac{zx_k}{x_k} = z$. For $x_k := \frac{-1}{\frac{\pi}{2}+2k\pi}$, $k \in \mathbb{N}$, noting that $x_k \to 0$ as $k \to \infty$ and $g(x_k) = 0$ for all $k \in \mathbb{N}$, by (4.2) one gets $0 \geq \limsup\limits_{k\to\infty} \frac{-zx_k}{x_k} = -z$. Hence $z = 0$. From (4.2), we have

$$\begin{aligned} 0 &\geq \limsup_{x\to 0} \frac{ug(x)}{|x| + |g(x)|} \\ &= \limsup_{x\to 0} \frac{u\cos\frac{1}{x}}{|x|^{1-\alpha} + |\cos\frac{1}{x}|}. \end{aligned} \tag{4.3}$$

For $x_k = \frac{1}{2k\pi}$, $k \in \mathbb{N}$, by (4.3) we get $u \leq 0$. For $x_k = \frac{1}{\pi+2k\pi}$, $k \in \mathbb{N}$, by (4.3) we obtain $u \geq 0$. This is to say that $u = 0$. Hence

$$\widehat{N}\big((\bar{x}, \nabla\varphi(\bar{x})); \mathrm{gph}\nabla\varphi\big) = \{(0,0)\}.$$

Therefore, if $u \in \mathbb{R}$ and $z \in \widehat{\partial}^2\varphi(\bar{x}))$, then $z = 0$ and $u = 0$. So, $\widehat{\partial}^2\varphi(\bar{x})$ is positively definite. We claim that $\bar{x}$ is not a local solution of (P_1). Indeed, suppose on contrary that $\bar{x}$ is a local solution of (P_1). Then there exists $\varepsilon > 0$ such that $\varphi(x) \geq \varphi(\bar{x}) = 0$ for all $x \in \mathbb{B}_\varepsilon(\bar{x}) = (-\varepsilon, \varepsilon)$. Observe that $\varphi(x) = -\varphi(-x)$ for all $x \in \mathbb{R}$. Hence

$\varphi(x) = 0$ for all $x \in (-\varepsilon, \varepsilon)$, and thus $g(x) = 0$ for all $x \in (-\varepsilon, \varepsilon)$, which is a contradiction. This proves that $\bar{x}$ is not a local solution of (P_1), while $\nabla\varphi(\bar{x}) = 0$ and $\widehat{\partial}^2\varphi(\bar{x})$ is positive definite.

Definition 4.3 ([8, Definition 4.2])**.** A function $\varphi : X \to \bar{\mathbb{R}}$ is said to be prox-regular at $\bar{x}$ for $\bar{x}^* \in \partial\varphi(\bar{x})$ if φ is finite at $\bar{x}$ and there exist $r > 0$ and $\varepsilon > 0$ such that for all $x, u \in \mathbb{B}_\varepsilon(\bar{x})$ with $|\varphi(u) - \varphi(\bar{x})| \le \varepsilon$ we have

$$\varphi(x) \ge \varphi(u) + \langle u^*, x - u\rangle - \frac{r}{2}\|x - u\|^2 \quad \forall u^* \in \partial\varphi(u) \cap \mathbb{B}_\varepsilon(\bar{x}^*).$$

Let $\varphi : X \to \bar{\mathbb{R}}$ and $\bar{x} \in \mathrm{dom}\varphi$. Recall [17] that $\bar{x}$ is said to be a tilt-stable local minimizer of φ if there is $\gamma > 0$ such that the mapping

$$M_\gamma : x^* \mapsto \mathrm{argmin}\{\varphi(x) - \langle x^*, x\rangle \mid x \in \mathbb{B}_\gamma(\bar{x})\}$$

is single-valued and Lipschitz continuous on some neighborhood of $0 \in X^*$ with $M_\gamma(0) = \{\bar{x}\}$.

Obviously, if $\bar{x}$ is a tilt-stable local minimizer, then it is a local minimizer. The inverse does not hold (e.g., $\bar{x} = 0$ is a local minimizer of $\varphi := 0$, but not a tilt-stable local minimizer of φ).

The following theorem gives a sufficient condition for a point to be a tilt-stable minimizer of a C^1-function.

Theorem 4.4 (Optimality condition II)**.** *Let $\varphi : X \to \mathbb{R}$ is a C^1 function, where X is a Hilbert space. If $\nabla\varphi(\bar{x}) = 0$, and there exist $\delta, r > 0$ such that, for all $x \in \mathbb{B}_\delta(\bar{x})$,*

$$\langle z, u\rangle \ge r\|u\|^2 \quad \forall u \in X,\ \forall z \in \widehat{\partial}^2\varphi(x)(u), \tag{4.4}$$

then $\bar{x}$ is a tilt-stable local minimizer of (P_1). The inverse is also valid if one assumes further that φ is prox-regular at $\bar{x}$ for $\bar{x}^* = 0$.

Proof. Suppose that $\nabla\varphi(\bar{x}) = 0$ and (4.4) holds. According to the proof of [2, Theorem 5.1], there exists $\gamma > 0$ such that φ is strongly convex on $\mathbb{B}_\gamma(\bar{x})$. Since $\nabla\varphi(\bar{x}) = 0$ and φ is strongly convex on $\mathbb{B}_\gamma(\bar{x})$, we have $M_\gamma(0) = \{\bar{x}\}$ and $M_\gamma(x^*)$ is a singleton set, say $\{\nu(x^*)\}$, for all $x^* \in X^*$. We claim that there exists a neighborhood U^* of $0 \in X^*$ in the norm topology such that $\nu(x^*) \in \mathrm{int}\mathbb{B}_\gamma(\bar{x})$ for all $x^* \in U^*$. Indeed, if this is false, then one can find $x_k^* \to 0$ and $x_k \in X$ with $\|x_k - \bar{x}\| = \gamma$ such that $x_k = \nu(x_k^*)$ for all k. Thus, by the strong convexity of φ,

$$\varphi(\bar{x}) > \varphi(x_k) - \langle x_k^*, x_k - \bar{x}\rangle \quad \forall k. \tag{4.5}$$

Since X is a Hilbert space and $\|x_k - \bar{x}\| = \gamma$ for all k, there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ tending weakly to some $\hat{x} \in \mathbb{B}_\gamma(\bar{x})$. Since φ is convex and continuous on $\mathbb{B}_\gamma(\bar{x})$, it is weakly lower semicontinuous on $\mathbb{B}_\gamma(\bar{x})$. Taking the lim inf both sides of (4.5) on the subsequence $\{x_{k_j}\}$, noting that $\lim_{j\to\infty}\langle x_{k_j}^*, x_{k_j} - \bar{x}\rangle = 0$, we have $\varphi(\bar{x}) > \varphi(\hat{x})$. This contradicts the fact that $M_\gamma(0) = \{\bar{x}\}$. The claim has been proved. For each $x^* \in U^*$, since $\nu(x^*) \in \mathrm{int}\mathbb{B}_\gamma(\bar{x})$, according to the Fermat rule, $\nabla\varphi\big(\nu(x^*)\big) = x^*$. So, by the strong convexity of φ on $\mathbb{B}_\gamma(\bar{x})$, there exists $\eta > 0$ such that

$$\begin{aligned}\|\nu(x_1^*) - \nu(x_2^*)\|.\|x_1^* - x_2^*\| &= \|\nu(x_1^*) - \nu(x_2^*)\|.\|\nabla\varphi\big(\nu(x_1^*)\big) - \nabla\varphi\big(\nu(x_2^*)\big)\| \\ &\geq \langle \nabla\varphi\big(\nu(x_1^*)\big) - \nabla\varphi\big(\nu(x_2^*)\big), \nu(x_1^*) - \nu(x_2^*)\rangle \\ &\geq \eta\|\nu(x_1^*) - \nu(x_2^*)\|^2 \quad \forall x_1^*, x_2^* \in U^*,\end{aligned}$$

which implies $\|\nu(x_1^*) - \nu(x_2^*)\| \leq \kappa\|x_1^* - x_2^*\|$ for all $x_1^*, x_2^* \in U^*$, where $\kappa := \eta^{-1}$. This shows that $\bar{x}$ is a tilt-stable minimizer of φ. Conversely, suppose that $\bar{x}$ is a tilt-stable minimizer of φ and φ is prox-regular at $\bar{x}$ for $\bar{x}^* = 0$. Then $\nabla\varphi(\bar{x}) = 0$ and, according to [8, Corollary 4.11], (4.4) is valid. This finishes the proof. □

Remark 4.5. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function that is prox-regular at $\bar{x}$ for $\bar{v} := 0$, with $\nabla\varphi(\bar{x}) = 0$, (for example, φ is of C^2-functions or even of $C^{1,1}$-functions), then it is well-known (see, e.g., [17]) that (4.4) is equivalent to the positiveness of the limiting second-order subdifferential $\partial^2\varphi(\bar{x})$ in the sense that $\langle z, u\rangle > 0$ for all $z \in \partial^2\varphi(\bar{x})(u)$ with $u \neq 0$. However, there exists C^1 functions that are not prox-regular, e.g., the function φ defined in Example 4.2.

To obtain a point-based second-order sufficient optimality condition, we have to restrict our consideration to the class of $C^{1,1}$ functions on $\mathbb{R}^n$, and invoke the limiting second-order subdifferential instead of the Fréchet counterpart.

Next result will be used in our subsequent analysis.

Proposition 4.6. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be $C^{1,1}$ near $\bar{x} \in \mathbb{R}^n$. Then the following assertions are equivalent:*

(i) *$\partial^2\varphi\big(\bar{x}\big)$ is positive definite, i.e.,*

$$\big[u \neq 0,\ z \in \partial^2\varphi\big(\bar{x}\big)(u) \Rightarrow \langle z, u\rangle > 0\big];$$

(ii) *there exist $r > 0$ and $\delta > 0$ such that*

$$\big[x \in \mathbb{B}_\delta(\bar{x}),\ u \in \mathbb{R}^n,\ z \in \widehat{\partial}^2\varphi\big(x\big)(u) \Rightarrow \langle z, u\rangle \geq r\|u\|^2\big]; \tag{4.6}$$

(iii) *there exist $r > 0$ and $\delta > 0$ such that*

$$\big[x \in \mathbb{B}_\delta(\bar{x}),\ u \in \mathbb{R}^n,\ z \in \partial^2\varphi\big(x\big)(u) \Rightarrow \langle z, u\rangle \geq r\|u\|^2\big];$$

(iv) *there exists $r > 0$ with the property that*

$$\langle z, u\rangle \geq r\|u\|^2 \quad \forall u \in \mathbb{R}^n,\ \forall z \in \partial^2\varphi\big(\bar{x}\big)(u).$$

(v) *there exists $r > 0$ such that φ is strongly convex on $\mathbb{B}_\delta(\bar{x})$.*

Proof. *(i)* $\Rightarrow$ *(ii)*: Suppose that φ is $C^{1,1}$ near $\bar{x}$, $\partial^2\varphi\big(\bar{x}\big)$ is positive definite, but (4.6) is invalid. Then, for any sequences $r_j \downarrow 0$ and $\delta_j \downarrow 0$, there exist $x_j \in \mathbb{B}_{\delta_j}(\bar{x})$, $u_j \in \mathbb{R}^n$, and $z_j \in \widehat{\partial}^2\varphi(x_j)$ with

$$\langle z_j, u_j\rangle < r_j\|u_j\|^2 \quad \forall j \in \mathbb{N}. \tag{4.7}$$

Since $z_j \in \widehat{\partial}^2\varphi(x_j)$, it holds that

$$(z_j, -u_j) \in \widehat{N}\big((x_j, \nabla\varphi(x_j)); \mathrm{gph}\nabla\varphi\big). \tag{4.8}$$

Note that, by (4.7), $u_j \neq 0$. Denote $\tilde{z}_j = \frac{1}{\|u_j\|} z_j$ and $\tilde{u}_j = \frac{1}{\|u_j\|} u_j$. Since $\|\tilde{u}_j\| = 1$ and $\{\tilde{u}_j\} \subset \mathbb{R}^n$, we may assume that $\tilde{u}_j \to \tilde{u}$ as $j \to \infty$, with $\|\tilde{u}\| = 1$. By (4.8),

$$(\tilde{z}_j, -\tilde{u}_j) \in \widehat{N}\big((x_j, \nabla\varphi(x_j)); \mathrm{gph}\nabla\varphi\big) \quad \forall j \in \mathbb{N}. \tag{4.9}$$

Let us now consider the following cases.

Case 1: $\{\tilde{z}_j\}$ is bounded. Using a subsequence if necessary, we may assume that $\tilde{z}_j \to \tilde{z} \in \mathbb{R}^n$. Thus, by (4.9), $(\tilde{z}, -\tilde{u}) \in N\big((\bar{x}, \nabla\varphi(\bar{x})); \mathrm{gph}\nabla\varphi\big)$. Hence $\tilde{z} \in \partial^2\varphi\big(\bar{x}\big)(\tilde{u})$. So, by our assumptions, $\langle \tilde{z}, \tilde{u}\rangle > 0$. Meanwhile, (4.7) implies that $\langle \tilde{z}_j, \tilde{u}_j\rangle < r_j$ for all j. Passing this to the limits as $j \to \infty$ yields $\langle \tilde{z}, \tilde{u}\rangle \leq 0$, which contradicts the fact that $\langle \tilde{z}, \tilde{u}\rangle > 0$.

Case 2: $\{\tilde{z}_j\}$ is unbounded. Then we may assume that $\|\tilde{z}_j\| \to \infty$ as $j \to \infty$ and moreover, $\frac{1}{\|\tilde{z}_j\|}\tilde{z}_j \to \tilde{z} \neq 0$ as $j \to \infty$. According to (4.9),

$$\Big(\frac{1}{\|\tilde{z}_j\|}\tilde{z}_j, -\frac{1}{\|\tilde{z}_j\|}\tilde{u}_j\Big) \in \widehat{N}\big((x_j, \nabla\varphi(x_j)); \mathrm{gph}\nabla\varphi\big),$$

which implies that $(\tilde{z}, 0) \in N\big((\bar{x}, \nabla\varphi(\bar{x})); \mathrm{gph}\nabla\varphi\big)$. In other words,

$$0 \neq \tilde{z} \in \partial^2\varphi\big(\bar{x})\big)(0). \tag{4.10}$$

As $\nabla\varphi(\cdot)$ is locally Lipschitz at $\bar{x}$, it is Lipschitz-like around $\big(\bar{x}, \nabla\varphi(\bar{x})\big)$. By the Mordukhovich criterion, $D^*\nabla\varphi(\bar{x}, \nabla\varphi(\bar{x}))(0) = \{0\}$; hence

$$\partial^2\varphi\big(\bar{x}\big)(0) = \{0\},$$

which contradicts (4.10). Therefore, we get (4.6).

$(ii) \Rightarrow (iii)$: Suppose that (ii) holds. Take any $r > 0$ and $\delta > 0$ such that (4.6) is valid. Let $x \in \mathbb{B}_{\delta/2}(\bar{x})$, $u \in \mathbb{R}^n$ and $z \in \partial^2\varphi\big(x\big)(u)$. Choose $x_j \to x$, $u_j \to u$, and $z_j \to z$ such that $z_j \in \widehat{\partial}^2\varphi(x_j)(u_j)$ for all j. By (4.6), $\langle z_j, u_j\rangle \geq r\|u_j\|^2$ for all j large enough. Taking limits as $j \to \infty$, we obtain $\langle z, u\rangle \geq r\|u\|^2$, as required.

Finally, we observe that the implications $(iii) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$ are obvious, while $(ii) \Leftrightarrow (v)$ is due to Theorems 5.1 and 5.2 in [6]. The proof is complete. □

Theorem 4.7. *Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $C^{1,1}$ near $\bar{x} \in \mathbb{R}^n$ and $\nabla\varphi(\bar{x}) = 0$. Then, the following assertions are equivalent:*

- (i) *$\bar{x}$ is a tilt-stable local minimizer of (P_1);*
- (ii) *$\partial^2\varphi(\bar{x})$ is positive definite;*
- (iii) *there exists $r > 0$ such that φ is strongly convex on $\mathbb{B}_\delta(\bar{x})$.*
- (iv) *there exist $r > 0$ and $\delta > 0$ such that*

$$\big[x \in \mathbb{B}_\delta(\bar{x}),\ u \in \mathbb{R}^n,\ z \in \widehat{\partial}^2\varphi\big(x\big)(u) \Rightarrow \langle z, u\rangle \geq r\|u\|^2\big];$$

- (v) *there exist $r > 0$ and $\delta > 0$ such that*

$$\big[x \in \mathbb{B}_\delta(\bar{x}),\ u \in \mathbb{R}^n,\ z \in \partial^2\varphi\big(x\big)(u) \Rightarrow \langle z, u\rangle \geq r\|u\|^2\big];$$

- (vi) *there exists $r > 0$ with the property that*

$$\langle z, u\rangle \geq r\|u\|^2 \quad \forall u \in \mathbb{R}^n,\ \forall z \in \partial^2\varphi\big(\bar{x}\big)(u).$$

Proof. Since φ is $C^{1,1}$ near $\bar{x}$, by [24, Proposition 13.34], φ is prox-regular and subdifferentially continuous around $\bar{x}$. Thus, by [23, Theorem 1.3], we have $(i) \Leftrightarrow (ii)$. Furthermore, it follows from Proposition 4.6 that $(ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v) \Leftrightarrow (vi)$. The proof is complete. □

Note that if $\bar{x}$ is a tilt-stable local minimizer, then it is a locally unique minimizer. Hence the following result is a direct corollary of Theorem 4.7.

Corollary 4.8. *Suppose $\varphi : \mathbb{R}^n \to \mathbb{R}$ is $C^{1,1}$ near $\bar{x} \in \mathbb{R}^n$ and $\nabla\varphi(\bar{x}) = 0$. If $\partial^2\varphi(\bar{x})$ is positive definite, then $\bar{x}$ is a locally unique optimal solution of (P_1).*

Example 4.9. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\varphi(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \le 0, \\ 3x^2 & \text{if } x > 0. \end{cases}$$

Then

$$\nabla\varphi(x) = \begin{cases} x & \text{if } x \le 0, \\ 6x & \text{if } x > 0 \end{cases}$$

and φ is $C^{1,1}$ on $\mathbb{R}$. Let $\bar{x} = 0$. By simple computation,

$$N\big((\bar{x}, \nabla\varphi(\bar{x})); \mathrm{gph}\nabla\varphi\big) = \big\{(z, u) \in \mathbb{R}^2 |\ -6u \ge z \ge -u\big\} \\ \cup\big\{(z, u) \in \mathbb{R}^2 |\ z = -u\big\} \cup \big\{(z, u) \in \mathbb{R}^2 |\ z = -6u\big\}.$$

Hence, for all $z \in \partial^2\varphi(\bar{x})(u)$, $\langle z, u\rangle \ge r\|u\|^2$, with $r = 1$. By Theorem 4.8, $\bar{x}$ is a locally unique solution of (P_1).

Note that since φ is $C^{1,1}$ near $\bar{x}$, $\widehat{\partial}^2\varphi(\bar{x}) \subset \partial^2\varphi(\bar{x})$. Obviously, the computation of $\widehat{\partial}^2\varphi(\bar{x})$ is much simpler than the computation of $\partial^2\varphi(\bar{x})$. So, it is natural to hope that, in the above corollary, we can replace $\partial^2\varphi(\bar{x})$ by $\widehat{\partial}^2\varphi(\bar{x})$. The following example shows that this is impossible.

Example 4.10. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the function defined by $\varphi(x) := \int_0^x g(t)dt$. Here $g : \mathbb{R} \to \mathbb{R}$ is the function given by the following rules:

(1) $g(0) := t$ for $t \le 0$;

(2) $g(t) := 0$ for $t > 1$;

(3) if $k \in \{0, 1, 2, ...\}$ and $\frac{1}{2^{2k+1}} < t \le \frac{1}{2^{2k}}$, then

$$g(t) := 3\Big(\big|t - \frac{3}{2^{2k+2}}\big| - \frac{1}{2^{2k+2}}\Big);$$

(4) if $k \in \{0, 1, 2, ...\}$ and $\frac{1}{2^{2k+2}} < t \le \frac{1}{2^{2k+1}}$, then

$$g(t) := -3\Big(\big|t - \frac{3}{2^{2k+3}}\big| - \frac{1}{2^{2k+3}}\Big).$$

Observe that g is a Lipschitz function on $\mathbb{R}$. Hence φ is well-defined, $\nabla\varphi(x) = g(x)$ for all $x \in \mathbb{R}$, and moreover φ is $C^{1,1}$. Let $\bar{x} = 0$. We have that

$$\widehat{N}\Big((\bar{x}, \nabla\varphi(\bar{x})); \mathrm{gph}\nabla\varphi\Big) = \big\{(z, u) \in \mathbb{R}^2 \mid z + u = 0,\ u \ge 0\big\}.$$

Hence, if $u \in \mathbb{R}$ and $z \in \widehat{\partial}^2\varphi(\bar{x})(u)$, then $zu = |u|^2$, which shows that $\widehat{\partial}^2\varphi(\bar{x})$ is positive definite. For each $k = 1, 2, ...$, let $x_k := \frac{1}{2^{2k}}$. It is not difficult to see that

$\varphi(x_k) < 0 = \varphi(\bar{x})$ for all k, and $x_k \to \bar{x}$ as $k \to \infty$. Hence $\bar{x}$ is not a local minimizer of φ, while φ is a $C^{1,1}$ function with property that $\nabla\varphi(\bar{x}) = 0$ and $\widehat{\partial}^2\varphi(\bar{x})$ is positive definite.

Remark 4.11. In Example 4.10, φ is of $C^{1,1}$, $\nabla\varphi(\bar{x}) = 0$ and the function $z : \mathbb{R} \to \mathbb{R}$ defined by $z(u) := u$ is a selection of the second-order subdifferential $\partial^2\varphi(\bar{x}) : \mathbb{R} \rightrightarrows \mathbb{R}$ satisfying $\langle z(u), u\rangle = |u|^2$ for all $u \in \mathbb{R}$, but $\bar{x}$ is not a local minimizer of φ. This shows that $\nabla\varphi(\bar{x}) = 0$ together (3.10) in Theorem 3.7 is not a sufficient optimality condition.

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