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POROSITY AND CONVERGENCE RESULTS FOR SEQUENCES OF NONEXPANSIVE MAPPINGS ON UNBOUNDED SETS

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Abstract. We study sequences of nonexpansive self-mappings of closed and convex subsets of a complete hyperbolic space which are not necessarily bounded. Using the notion of porosity, we show that most of these sequences are, in fact, contractive. We also study the convergence of infinite products generated by contractive sequences of mappings.

1. Introduction and preliminaries

It is well known that infinite products of operators find applications in many areas of mathematics (see, for instance, [1–5,11,14] and the references mentioned therein). This naturally leads to the study of sequences of operators. In the present paper we consider sequences of nonexpansive (that is, 1-Lipschitz) self-mappings of closed and convex subsets of a complete hyperbolic space which are not necessarily bounded. Using the notion of porosity, we show (see Theorem 2.1 below) that most of these sequences are, as a matter of fact, contractive in the sense of Rakotch [8]. Theorem 2.1 follows from Propositions 3.1 and 4.1 which are stated and proved in Sections 3 and 4, respectively. We also study the convergence of infinite products generated by contractive sequences of mappings. More precisely, two convergence results for Rakotch contractive (sub)sequences (namely, Theorems 5.1 and 6.1) are established in Sections 5 and 6, respectively. Analogous results for sequences of mappings which are contractive in the sense of Matkowski (Theorems 7.1 and 7.2) are proved in Section 7. *Unrestricted* infinite products of Rakotch contractive mappings are studied in Section 8 (see Theorem 8.1), while Section 9 is devoted to *inexact* infinite products (see Theorem 9.1).

Hyperbolic spaces constitute an important class of metric spaces the definition of which we now recall for the reader's convenience.

Let (X, ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a *metric embedding* of R^1 into X if

$$\rho(c(s), c(t)) = |s - t|$$

for all real s and t. The image of R^1 under a metric embedding is called a *metric line*. The image of a real interval

$$[a, b] = \{t \in R^1 : a \le t \le b\}$$

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under such a mapping is called a *metric segment*.

Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X, there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x and y. We denote this segment by $[x, y]$. For each $0 \le t \le 1$, there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1 - t)\rho(x, y).$$

This point is denoted by $(1 - t)x \oplus ty$. We say that X, or more precisely (X, ρ, M), is a *hyperbolic space* if

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\Big) \le \frac{1}{2}\rho(y, z)$$

for all x, y and z in X. An equivalent requirement is that

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z\Big) \le \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all x, y, z and w in X. This inequality, in its turn, implies that

$$\rho((1 - t)x \oplus ty, (1 - t)w \oplus tz) \le (1 - t)\rho(x, w) + t\rho(y, z)$$

for all points x, y, z and w in X, and all numbers $0 \le t \le 1$.

It is clear that all normed linear spaces are hyperbolic in this sense. A discussion of more examples of hyperbolic spaces and, in particular, of the Hilbert ball with the hyperbolic metric can be found, for example, in [6, 9, 10].

We call a set $K \subset X$ *ρ-convex* if $[x, y] \subset K$ for all x and y in K.

A property of elements of a complete metric space Z is said to be *generic* (typical) in Z if the set of all elements of Z which have this property contains an everywhere dense G_δ subset of Z. In this case we also say that the property holds for a generic (typical) element of Z or that a generic (typical) element of Z has this property.

It is known that a typical nonexpansive self-mapping of a bounded, closed and ρ-convex subset of a complete hyperbolic metric space has a unique fixed point which is the uniform limit of all its iterates [12, 14]. As a matter of fact, the subset of all those nonexpansive mappings which lack this property is small not only in the sense of Baire category, but also in the sense of porosity, a concept which we now recall.

Let Z be a complete metric space. We denote by $B_Z(y, r)$ the closed ball of center $y \in Z$ and radius $r > 0$. A subset $E \subset Z$ is called *porous* in Z if there exist numbers $\alpha \in (0, 1)$ and $r_0 > 0$ such that for each number $r \in (0, r_0]$ and each point $y \in Z$, there exists a point $z \in Z$ for which

$$B_Z(z, \alpha r) \subset B_Z(y, r) \setminus E.$$

A subset of the space Z is called *σ-porous* in Z if it is a countable union of porous subsets of Z.

Note that in the definition of a porous set we can assume that the point y belongs to E.

Other notions of porosity can be found in the literature. We use this rather strong concept of porosity which has already found applications in, for instance, approximation theory, the calculus of variations and nonlinear analysis. See, for example, [14, 18] and the references mentioned therein.

Since porous sets are nowhere dense, all σ-porous sets are of the first Baire category. If Z is a finite-dimensional Euclidean space, then σ-porous sets are also of Lebesgue measure zero. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have Lebesgue measure zero and are of the first Baire category.

2. A porosity result

Suppose that (X, ρ, M) is a complete hyperbolic space and that K is a nonempty, closed and ρ-convex subset of the space X.

Fix a point $\theta \in K$. Denote by $\mathcal{A}$ the set of all operators $A : K \to K$ such that

$$\rho(A(x), A(y)) \le \rho(x, y) \text{ for all } x, y \in K. \tag{2.1}$$

Such operators are said to be *nonexpansive*.

By (2.1), for every pair of mappings $A, B \in \mathcal{A}$ and every point $x \in K$,

$$\begin{aligned}\rho(A(x), B(x)) &\le \rho(A(x), A(\theta)) + \rho(A(\theta), B(\theta)) + \rho(B(\theta), B(x)) \\ &\le \rho(x, \theta) + \rho(A(\theta), B(\theta)) + \rho(x, \theta)\end{aligned}$$

and

$$\rho(A(x), B(x)) \le 2\rho(x, \theta) + \rho(A(\theta), B(\theta)). \tag{2.2}$$

For every pair of mappings $A, B \in \mathcal{A}$, set [15]

$$d(A, B) := \inf\{\lambda > 0 : \ \rho(A(x), B(x)) \le \lambda(\rho(x, \theta) + 1) \text{ for all } x \in K\}. \tag{2.3}$$

In view of (2.2) and (2.3), for every pair of mappings $A, B \in \mathcal{A}$, $d(A, B)$ is well defined,

$$d(A, B) = \sup\{\rho(A(x), B(x))(\rho(x, \theta) + 1)^{-1} : \ x \in K\}, \tag{2.4}$$

$d : \mathcal{A} \times \mathcal{A} \to [0, \infty)$ is a metric on $\mathcal{A}$ and the metric space $(\mathcal{A}, d)$ is complete. It is clear that the topology induced by the metric d is stronger than the topology of uniform convergence on bounded sets, but weaker than the topology of uniform convergence on all of K. This topology does not depend on the choice of the point θ. More precisely, if we take $\theta_1 \ne \theta$, then we get Lipschitz-equivalent metrics. In order to see this, take $\theta_1 \in K \setminus \{\theta\}$ and for every pair of mappings $A, B \in \mathcal{A}$, define

$$d_1(A, B) := \sup\{\rho(A(x), B(x))(\rho(x, \theta_1) + 1)^{-1} : \ x \in K\}.$$

Let $A, B \in \mathcal{A}$ be given. Then for every point $x \in K$, we have

$$\begin{aligned}\rho(A(x), B(x)) \le d(A, B)(\rho(x, \theta) + 1) &\le d(A, B)(\rho(x, \theta_1) + 1 + \rho(\theta_1, \theta)) \\ &\le d(A, B)(\rho(x, \theta_1) + 1)(\rho(\theta_1, \theta) + 1).\end{aligned}$$

In view of the relation above,

$$d_1(A, B) \le d(A, B)(\rho(\theta, \theta_1) + 1).$$

For every point $x \in K$ and every positive number r, set

$$B(x, r) := \{y \in K : \ \rho(x, y) \le r\}.$$

Denote by $\mathcal{M}$ the collection of all sequences of operators $\{A_t\}_{t=1}^{\infty} \subset \mathcal{A}$ such that

$$\sup\{\rho(\theta, A_t(\theta)) : \ t = 1, 2, \dots\} < \infty. \tag{2.5}$$

Let $\{A_t\}_{t=1}^{\infty},\ \{B_t\}_{t=1}^{\infty} \in \mathcal{M}$. It follows from (2.2) and (2.5) that for each $x \in K$ and each integer $t \geq 1$,

$$\begin{aligned}\rho(A_t(x), B_t(x)) &\leq 2\rho(x,\theta) + \rho(A_t(\theta), B_t(\theta))\\ &\leq 2\rho(x,\theta) + \rho(A_t(\theta),\theta) + \rho(\theta, B_t(\theta))\\ &\leq (\rho(x,\theta)+1)(2 + \sup\{\rho(A_i(\theta),\theta) :\ i = 1,2,\dots\}\\ &\quad + \sup\{\rho(B_i(\theta),\theta) :\ i = 1,2,\dots\}).\end{aligned} \tag{2.6}$$

Define

$$\begin{aligned}&d_{\mathcal{M}}(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty})\\ &\quad := \sup\{\rho(A_t(x), B_t(x))(\rho(x,\theta)+1)^{-1} :\ x \in K,\ t = 1,2,\dots\}.\end{aligned} \tag{2.7}$$

By (2.6) and (2.7), $d_{\mathcal{M}}(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) < \infty$. In view of (2.4) and (2.7),

$$d_{\mathcal{M}}(\{A_t\}_{t=1}^{\infty}, \{B_t\}_{t=1}^{\infty}) = \sup\{d(A_t, B_t) :\ t = 1,2,\dots\}. \tag{2.8}$$

It is not difficult to see that $d_{\mathcal{M}}$ is a metric on $\mathcal{M}$ and that the metric space $(\mathcal{M}, d_{\mathcal{M}})$ is complete.

Let $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ and let D be a nonempty subset of K. We say that the sequence $\{A_t\}_{t=1}^{\infty}$ is *contractive* in the sense of Rakotch [8] on D if there exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that

$$\phi(t) < 1 \text{ for all } t > 0 \tag{2.9}$$

and

$$\rho(A_t(x), A_t(y)) \leq \phi(\rho(x,y))\rho(x,y) \text{ for all } x, y \in D \text{ and all } t = 1,2\dots. \tag{2.10}$$

It is not difficult to see that $\{A_t\}_{t=1}^{\infty}$ is Rakotch contractive on D if and only if for every $\beta > 0$,

$$\sup\{\rho(A_t(x), A_t(y))\rho(x,y)^{-1} :\ t \in \{1,2,\dots\},\ x,y \in D \text{ and } \rho(x,y) \geq \beta\} < 1.$$

One of our goals in this paper is to establish the following result.

Theorem 2.1. *There exists a set $\mathcal{F}_* \subset \mathcal{M}$ such that its complement $\mathcal{M} \setminus \mathcal{F}_*$ is σ-porous in $(\mathcal{M}, d_{\mathcal{M}})$ and for each sequence $\{A_t\}_{t=1}^{\infty} \in \mathcal{F}_*$, there exists $M_A > 0$ such that for each $M \geq M_A$, we have*

$$A_t(B(\theta, M)) \subset B(\theta, M),\ t = 1,2,\dots,$$

and the sequence $\{A_t\}_{t=1}^{\infty}$ is Rakotch contractive on $B(\theta, M)$.

At this point we note that several results regarding the asymptotic behavior of contractive sequences can be found in [13, 14, 16, 17].

3. An auxiliary result

Proposition 3.1. *There exists a set $\mathcal{F} \subset \mathcal{M}$ such that its complement $\mathcal{M} \setminus \mathcal{F}$ is porous in $(\mathcal{M}, d_{\mathcal{M}})$ and each sequence $\{A_t\}_{t=1}^{\infty} \in \mathcal{F}$ has the following property:*

(P1) there exists a positive number M_* such that for every number $M \geq M_*$,

$$\cup\{B(A_t(z), 1) :\ z \in B(\theta, M),\ t \in \{1,2,\dots\}\} \subset B(\theta, M).$$

Proof. Denote by $\mathcal{F}$ the collection of all sequences of mappings $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ for which property (P1) holds. We claim that $\mathcal{M} \setminus \mathcal{F}$ is porous in $(\mathcal{M}, d_{\mathcal{M}})$. To see this, we first set

$$\alpha = 1/8. \tag{3.1}$$

Let

$$\{A_t\}_{t=1}^{\infty} \in \mathcal{M} \setminus \mathcal{F} \text{ and } r \in (0,1]. \tag{3.2}$$

Fix a natural number n for which

$$\rho(\theta, A_t(\theta)) \leq n, \ t = 1, 2, \ldots. \tag{3.3}$$

Set

$$\gamma = 4^{-1} r \tag{3.4}$$

and

$$M_* = 8(n+2)r^{-1}. \tag{3.5}$$

For every point $x \in K$ and every $t \in \{1, 2, \ldots\}$, define

$$A_t^{(\gamma)}(x) := (1-\gamma)A_t(x) \oplus \gamma A_t(\theta). \tag{3.6}$$

It follows from (2.1) and (3.6) that for every integer $t \geq 1$ and every pair of points $x, y \in K$,

$$\begin{aligned} \rho(A_t^{(\gamma)}(x), A_t^{(\gamma)}(y)) &= \rho((1-\gamma)A_t(x) \oplus \gamma A_t(\theta), (1-\gamma)A_t(y) \oplus \gamma A_t(\theta)) \\ &\leq (1-\gamma)\rho(A_t(x), A_t(y)) \leq (1-\gamma)\rho(x,y). \end{aligned} \tag{3.7}$$

Relations (2.1) and (3.6) imply that

$$\begin{aligned} &d(\{A_t^{(\gamma)}\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) \\ &= \sup\{\rho(A_t^{(\gamma)}(x), A_t(x))(\rho(x,\theta)+1)^{-1} : \ x \in K, \ t \in \{1,2,\ldots\}\} \\ &\leq \sup\{\gamma\rho(A_t(x), A_t(\theta))(\rho(x,\theta)+1)^{-1} : \ x \in K, \ t \in \{1,2,\ldots\}\} \leq \gamma. \end{aligned} \tag{3.8}$$

Now assume that a sequence of mappings $\{C_t\}_{t=1}^{\infty} \in \mathcal{M}$ satisfies

$$d_{\mathcal{M}}(\{C_t\}_{t=1}^{\infty}, \{A_t^{(\gamma)}\}_{t=1}^{\infty}) \leq \alpha r. \tag{3.9}$$

By (3.1), (3.4), (3.8) and (3.9), we have

$$\begin{aligned} d_{\mathcal{M}}(\{C_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) &\leq d_{\mathcal{M}}(\{C_t\}_{t=1}^{\infty}, \{A_t^{(\gamma)}\}_{t=1}^{\infty}) + d_{\mathcal{M}}(\{A_t^{(\gamma)}\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) \\ &\leq \alpha r + \gamma \leq r/2. \end{aligned} \tag{3.10}$$

Assume that

$$M \geq M_*, \ z \in B(\theta, M), \ t \in \{1,2,\ldots\}, \ u \in B(C_t(z), 1). \tag{3.11}$$

In view of (3.11),

$$\rho(\theta, u) \leq \rho(\theta, C_t(z)) + \rho(C_t(z), u) \leq \rho(\theta, C_t(z)) + 1. \tag{3.12}$$

Relations (2.7), (3.9) and (3.11) imply that

$$\begin{aligned}\rho(\theta, C_t(z)) &\le \rho(\theta, A_t^{(\gamma)}(z)) + \rho(A_t^{(\gamma)}(z), C_t(z)) \\ &\le \rho(\theta, A_t^{(\gamma)}(z)) + d_{\mathcal{M}}(\{C_t\}_{t=1}^{\infty}, \{A_t^{(\gamma)}\}_{t=1}^{\infty})(\rho(z,\theta)+1) \\ &\le \rho(\theta, A_t^{(\gamma)}(z)) + \alpha r(\rho(z,\theta)+1) \\ &\le \rho(\theta, A_t^{(\gamma)}(z)) + \alpha r(M+1).\end{aligned} \tag{3.13}$$

It follows from (2.1), (3.3) and (3.6) that

$$\begin{aligned}\rho(\theta, A_t^{(\gamma)}(z)) &\le \rho(\theta, A_t(\theta)) + \rho(A_t(\theta), A_t^{(\gamma)}(z)) \\ &\le n + \rho(A_t(\theta), (1-\gamma)A_t(z) \oplus \gamma A_t(\theta)) \\ &\le n + (1-\gamma)\rho(A_t(\theta), A_t(z)) \\ &\le n + (1-\gamma)\rho(\theta, z).\end{aligned} \tag{3.14}$$

By (3.1), (3.2), (3.4), (3.5) and (3.11)–(3.14), we have

$$\begin{aligned}\rho(\theta, u) &\le 1 + \alpha r(M+1) + n + (1-\gamma)\rho(\theta, z) \\ &\le 1 + \alpha r + n + M(1-\gamma+\alpha r) \\ &\le 2 + n + M(1-\gamma/2) \le M + 2 + n - M\gamma/2 \\ &\le M + 2 + n - (r/8)8(n+2)r^{-1} = M\end{aligned}$$

and

$$\rho(\theta, u) \le M.$$

Thus we have shown that (3.11) implies that $\rho(\theta, u) \le M$ and

$$B(C_t(z), 1) \subset B(\theta, M)$$

for all $z \in B(\theta, M)$ and all $t \in \{1, 2, \dots\}$. Hence

$$\{C_t\}_{t=1}^{\infty} \in \mathcal{F}.$$

Since (3.9) implies (3.10), it follows that

$$\begin{aligned}&\{\{C_t\}_{t=1}^{\infty} \in \mathcal{M} :\ d_{\mathcal{M}}(\{C_t\}_{t=1}^{\infty}, \{A_t^{(\gamma)}\}_{t=1}^{\infty}) \le \alpha r\} \\ &\subset \mathcal{F} \cap \{\{C_t\}_{t=1}^{\infty} \in \mathcal{M} :\ d_{\mathcal{M}}(\{C_t\}_{t=1}^{\infty}, \{A_t\}_{t=1}^{\infty}) \le r\}.\end{aligned}$$

Therefore $\mathcal{M} \setminus \mathcal{F}$ is porous in $(\mathcal{M}, d_{\mathcal{M}})$, as claimed. Proposition 3.1 is proved. □

4. Proof of Theorem 2.1

We may assume without loss of generality that the set K is not a singleton. Hence we can fix a number $\kappa \in (0,1)$ for which there exist two points $u, v \in K$ such that

$$\kappa \le \rho(u, v). \tag{4.1}$$

We also fix a natural number n_0 which satisfies

$$n_0 > \max\{\rho(\theta, u),\ \rho(\theta, v)\}. \tag{4.2}$$

For every natural number $n \ge n_0$, denote by $\mathcal{F}_n$ the collection of all sequences of mappings $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ such that

$$\sup\{\rho(A_t(x), A_t(y))\rho(x,y)^{-1} :$$

$$x, y \in B(\theta, n) \text{ and } \rho(x, y) \geq \kappa n^{-1}, \ t = 1, 2, \dots\} < 1. \tag{4.3}$$

Theorem 2.1 follows from Proposition 3.1 and our next result.

Proposition 4.1. *For each integer $n \geq n_0$, the set $\mathcal{M} \setminus \mathcal{F}_n$ is σ-porous in $(\mathcal{M}, d_{\mathcal{M}})$.*

Proof. For every natural number p, define

$$\mathcal{M}_p := \{\{A_t\}_{t=1}^{\infty} \in \mathcal{M} : \ \rho(\theta, A_t(\theta)) \leq p, \ t = 1, 2, \dots\}. \tag{4.4}$$

Let $n \geq n_0$ be an integer. It is clear that

$$\mathcal{M} \setminus \mathcal{F}_n = \cup_{p=1}^{\infty} (\mathcal{M}_p \setminus \mathcal{F}_n). \tag{4.5}$$

In order to complete the proof of the proposition, it is sufficient to show that for any natural number p, the set $\mathcal{M}_p \setminus \mathcal{F}_n$ is porous in $(\mathcal{M}, d_{\mathcal{M}})$.

To this end, let $p \geq 1$ be an integer. Define

$$\alpha = (64 n^2 p)^{-1} \kappa. \tag{4.6}$$

Assume that

$$\{A_t\}_{t=1}^{\infty} \in \mathcal{M}_p \setminus \mathcal{F}_n \ \text{ and } \ r \in (0, 1]. \tag{4.7}$$

Let

$$\gamma = (4p)^{-1} r. \tag{4.8}$$

For every point $x \in K$ and every integer $t \geq 1$, set

$$A_t^{(\gamma)}(x) := (1 - \gamma) A_t(x) \oplus \gamma \theta. \tag{4.9}$$

In view of (2.1) and (4.9), for all points $x, y \in K$ and all natural numbers $t \geq 1$, we have

$$\begin{aligned} \rho(A_t^{(\gamma)}(x), A_t^{(\gamma)}(y)) &= \rho((1 - \gamma) A_t(x) \oplus \gamma \theta, (1 - \gamma) A_t(y) \oplus \gamma \theta) \\ &\leq (1 - \gamma) \rho(A_t(x), A_t(y)) \leq (1 - \gamma) \rho(x, y). \end{aligned} \tag{4.10}$$

Relations (2.1), (4.4) and (4.9) imply that for all points $x \in K$ and all natural numbers t, we have

$$\begin{aligned} \rho(A_t(x), A_t^{(\gamma)}(x)) &= \rho((1 - \gamma) A_t(x) \oplus \gamma \theta, A_t(x)) \\ &\leq \gamma \rho(A_t(x), \theta) \leq \gamma(\rho(A_t(x), A_t(\theta)) + \rho(A_t(\theta), \theta)) \\ &\leq \gamma(\rho(x, \theta) + \rho(A_t(\theta), \theta)) \\ &\leq \gamma(\rho(x, \theta) + p) \\ &\leq p\gamma(\rho(x, \theta) + 1). \end{aligned} \tag{4.11}$$

It follows from (2.7), (4.8) and (4.11) that

$$d_{\mathcal{M}}(\{A_t\}_{t=1}^{\infty}, \{A_t^{(\gamma)}\}_{t=1}^{\infty}) \leq p\gamma = r/4. \tag{4.12}$$

Assume that the sequence $\{B_t\}_{t=1}^{\infty} \in \mathcal{M}$ satisfies

$$d_{\mathcal{M}}(\{B_t\}_{t=1}^{\infty}, \{A_t^{(\gamma)}\}_{t=1}^{\infty}) \leq \alpha r. \tag{4.13}$$

By (4.6), (4.12) and (4.13), we have

$$
\begin{aligned}
d_{\mathcal{M}}(\{B_t\}_{t=1}^{\infty},\{A_t\}_{t=1}^{\infty}) &\le d_{\mathcal{M}}(\{B_t\}_{t=1}^{\infty},\{A_t^{(\gamma)}\}_{t=1}^{\infty}) \\
&\quad + d_{\mathcal{M}}(\{A_t^{(\gamma)}\}_{t=1}^{\infty},\{A_t\}_{t=1}^{\infty}) \\
&\le \alpha r + r/4 < r.
\end{aligned} \tag{4.14}
$$

Assume that

$$
x,\ y \in B(\theta,n),\ \rho(x,y)\ge \kappa n^{-1},\ t\in\{1,2,\dots\}. \tag{4.15}
$$

In view of (4.10), (4.13) and (4.15),

$$
\begin{aligned}
\rho(B_t(x),B_t(y)) &\le \rho(B_t(x),A_t^{(\gamma)}(x)) + \rho(A_t^{(\gamma)}(x),A_t^{(\gamma)}(y)) + \rho(A_t^{(\gamma)}(y),B_t(y)) \\
&\le \alpha r(\rho(x,\theta)+1) + (1-\gamma)\rho(x,y) + \alpha r(\rho(x,\theta)+1) \\
&\le 2\alpha r(n+1) + (1-\gamma)\rho(x,y).
\end{aligned}
$$

When combined with (4.6), (4.8) and (4.15), the above inequality implies that

$$
\begin{aligned}
\rho(B_t(x),B_t(y))\rho(x,y)^{-1} &\le 1-\gamma+2\alpha r(n+1)\kappa^{-1}n \\
&\le 1-(4p)^{-1}r + r(8p)^{-1}.
\end{aligned} \tag{4.16}
$$

Thus (4.15) implies (4.16). Therefore

$$
\begin{aligned}
\sup\{\rho(B_t(x),B_t(y))\rho(x,y)^{-1} : x,\ y\in B(\theta,n) \text{ and } \rho(x,y)\ge \kappa n^{-1},\ t\in\{1,2,\dots\}\} \\
\le 1-(8p)^{-1}r
\end{aligned}
$$

and

$$
\{B_t\}_{t=1}^{\infty}\in \mathcal{F}_n.
$$

Together with (4.14) this implies that

$$
\begin{aligned}
&\{\{B_t\}_{t=1}^{\infty}\in\mathcal{M}:\ d_{\mathcal{M}}(\{B_t\}_{t=1}^{\infty},\{A_t^{(\gamma)}\}_{t=1}^{\infty})\le \alpha r\} \\
&\subset \mathcal{F}_n\cap\{\{B_t\}_{t=1}^{\infty}\in\mathcal{M}:\ d(\{B_t\}_{t=1}^{\infty},\{A_t\}_{t=1}^{\infty})\le r\}.
\end{aligned}
$$

Thus $\mathcal{M}_p\backslash\mathcal{F}_n$ is indeed porous in $(\mathcal{M},d_{\mathcal{M}})$. This completes the proof of Proposition 4.1. □

5. First convergence result

Using the notations, definitions and assumptions introduced in Section 2, we prove in this section the following result.

Theorem 5.1. *Assume that $\{A_t\}_{t=1}^{\infty}\in\mathcal{M}$, there exists a positive number M_A such that for each $M\ge M_A$,*

$$
A_t(B(\theta,M))\subset B(\theta,M),\ t=1,2,\dots, \tag{5.1}
$$

and that the sequence of mappings $\{A_t\}_{t=1}^{\infty}$ is Rakotch contractive on $B(\theta,M)$.

Let $M,\epsilon>0$. Then there exists a natural number n_ϵ such that for each mapping

$$
r:\{1,2,\dots\}\to\{1,2,\dots\},
$$

each pair of points $x,y\in B(\theta,M)$ and each integer $n\ge n_\epsilon$,

$$
\rho(A_{r(n)}\cdots A_{r(1)}(x),A_{r(n)}\cdots A_{r(1)}(y))\le\epsilon.
$$

Proof. We may assume without loss of generality that $M \geq M_A$. Since the sequence of mappings $\{A_t\}_{t=1}^{\infty}$ is Rakotch contractive on $B(\theta, M)$, there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ such that

$$\phi(t) < 1 \text{ for all } t > 0 \tag{5.2}$$

and

$$\rho(A_t(x), A_t(y)) \leq \phi(\rho(x, y))\rho(x, y) \text{ for all } x, y \in B(\theta, M) \tag{5.3}$$

and all integers $t \geq 1$.

Denote by A_0 the identity operator $I : K \to K$: $I(x) = x$ for all $x \in K$.

Choose a natural number

$$n_\epsilon > 2M(\epsilon(1 - \phi(\epsilon))^{-1}. \tag{5.4}$$

Assume that

$$r : \{1, 2, \dots\} \to \{1, 2, \dots\}$$

and that

$$x, y \in B(\theta, M). \tag{5.5}$$

In order to complete the proof of the theorem, we should show that for all integers $n \geq n_\epsilon$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \leq \epsilon.$$

Since all the mappings A_t, $t = 1, 2, \dots$, are nonexpansive, it is sufficient to show that there exists an integer $m \in [1, n_\epsilon]$ such that

$$\rho(A_{r(m)} \cdots A_{r(1)}(x), A_{r(m)} \cdots A_{r(1)}(y)) \leq \epsilon.$$

Suppose to the contrary that this is not true. Then

$$\rho(A_{r(m)} \cdots A_{r(1)}(x), A_{r(m)} \cdots A_{r(1)}(y)) > \epsilon \tag{5.6}$$

for all $m = 1, \dots, n_\epsilon$. Since all the mappings A_t, $t = 1, 2, \dots$, are nonexpansive, we have

$$\rho(x, y) > \epsilon. \tag{5.7}$$

Set

$$r(0) = 0.$$

Assume that an integer

$$t \in [0, n_\epsilon - 1].$$

By (5.1), (5.3) and (5.5),

$$\rho(A_{r(t+1)} \cdots A_{r(0)}(x), A_{r(t+1)} \cdots A_{r(0)}(y)) \leq \phi(\rho(A_{r(t)} \cdots A_{r(0)}(x), A_{r(t)} \cdots A_{r(0)}(y)))$$

$$\times \rho(A_{r(t)} \cdots A_{r(0)}(x), A_{r(t)} \cdots A_{r(0)}(y)).$$

Since the function ϕ is decreasing, it follows from (5.6), (5.7) and the above relation that

$$\begin{aligned}\rho(A_{r(t)}\cdots A_{r(0)}(x), A_{r(t)}\cdots A_{r(0)}(y))\\ -\rho(A_{r(t+1)}\cdots A_{r(0)}(x), A_{r(t+1)}\cdots A_{r(0)}(y))\\ \geq \rho(A_{r(t)}\cdots A_{r(0)}(x), A_{r(t)}\cdots A_{r(0)}(y))\\ \times(1-\phi(\rho(A_{r(t)}\cdots A_{r(0)}(x), A_{r(t)}\cdots A_{r(0)}(y))))\\ \geq \epsilon(1-\phi(\epsilon)).\end{aligned} \tag{5.8}$$

By (5.5) and (5.8),

$$\begin{aligned}2M \geq \rho(x,y) &= \rho(A_{r(0)}(x), A_{r(0)}(y))\\ &\geq \rho(A_{r(0)}(x), A_{r(0)}(y)) - \rho(A_{r(n_\epsilon)}\cdots A_{r(0)}(x), A_{r(n_\epsilon)}\cdots A_{r(0)}(y))\\ &= \sum_{t=0}^{n_\epsilon-1}[\rho(A_{r(t)}\cdots A_{r(0)}(x), A_{r(t)}\cdots A_{r(0)}(y))\\ &\quad -\rho(A_{r(t+1)}\cdots A_{r(0)}(x), A_{r(t+1)}\cdots A_{r(0)}(y))]\\ &\geq n_\epsilon\epsilon(1-\phi(\epsilon))\end{aligned}$$

and

$$n_\epsilon \leq 2M(\epsilon(1-\phi(\epsilon))^{-1}.$$

This, however, contradicts (5.4). The contradiction we have reached completes the proof of Theorem 5.1. □

Proposition 5.2. *Let* $\{A_t\}_{t=1}^\infty \in \mathcal{M}$, $M_A > 0$ *and let*

$$A_t(B(\theta, M_A)) \subset B(\theta, M_A),\ t = 1, 2, \ldots. \tag{5.9}$$

Then for each $M \geq M_A$,

$$A_t(B(\theta, M)) \subset B(\theta, M),\ t = 1, 2, \ldots.$$

Proof. Let

$$M > M_A,\ x \in B(\theta, M). \tag{5.10}$$

In view of (5.10) and the properties of hyperbolic spaces, there exists

$$z \in [x, \theta] \tag{5.11}$$

such that

$$\rho(\theta, z) = M_A \tag{5.12}$$

and

$$\rho(z, x) = M - M_A. \tag{5.13}$$

By (2.1), (5.9), (5.12) and (5.13), for any natural number t,

$$\begin{aligned}\rho(\theta, A_t(x)) &\leq \rho(\theta, A_t(z)) + \rho(A_t(z), A_t(x))\\ &\leq M_A + \rho(z, x) \leq M.\end{aligned}$$

Proposition 5.2 is proved. □

6. Second convergence result

We again use the notations, definitions and assumptions introduced in Section 2. This section is devoted to the proof of the following result.

Theorem 6.1. *Assume that* $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$, $M_A > 0$,

$$A_t(B(\theta, M_A)) \subset B(\theta, M_A),\ t = 1, 2, \ldots, \tag{6.1}$$

and that a subsequence of mappings $\{A_{t_k}\}_{k=1}^{\infty}$ *is Rakotch contractive on* $B(\theta, M)$ *for every* $M \geq M_A$.

Let $M, \epsilon > 0$. *Then there exists a natural number* n_ϵ *such that for each pair of points* $x, y \in B(\theta, M)$ *and each integer* $n \geq n_\epsilon$,

$$\rho(A_n \cdots A_1(x), A_n \cdots A_1(y)) \leq \epsilon.$$

Proof. We may assume without loss of generality that $M > M_A$. Proposition 5.2 and (6.1) imply that

$$A_t(B(\theta, M)) \subset B(\theta, M),\ t = 1, 2, \ldots, \tag{6.2}$$

and there exists a decreasing function $\phi : [0, \infty) \to [0, 1]$ such that

$$\phi(t) < 1 \text{ for all } t > 0 \tag{6.3}$$

and

$$\rho(A_{t_k}(x), A_{t_k}(y)) \leq \phi(\rho(x, y))\rho(x, y) \text{ for all } x, y \in B(\theta, M) \tag{6.4}$$

and all integers $k \geq 1$.

Denote by A_0 the identity operator $I : K \to K$: $I(x) = x$ for all $x \in K$.

Choose a natural number

$$k_\epsilon > 2M(\epsilon(1 - \phi(\epsilon))^{-1} + 1, \tag{6.5}$$

set

$$n_\epsilon = t_{k_\epsilon} \tag{6.6}$$

and

$$t_0 = 0.$$

Assume that

$$x, y \in B(\theta, M). \tag{6.7}$$

In order to complete the proof of the theorem, we need to show that for all integers $n \geq n_\epsilon$,

$$\rho(A_n \cdots A_1(x), A_n \cdots A_1(y)) \leq \epsilon.$$

Since all the mappings A_t, $t = 1, 2, \ldots$, are nonexpansive, it is sufficient to show that there exists an integer $m \in [1, n_\epsilon]$ such that

$$\rho(A_m \cdots A_1(x), A_m \cdots A_1(y)) \leq \epsilon.$$

Suppose to the contrary that this does not hold. Then

$$\rho(A_m \cdots A_1(x), A_m \cdots A_1(y)) > \epsilon \tag{6.8}$$

for all $m = 1, \dots, n_\epsilon$. Since all mappings A_t, $t = 1, 2, \dots$, are nonexpansive, in view of (6.8), we have

$$\rho(x, y) > \epsilon. \tag{6.9}$$

Assume that an integer

$$k \in [0, k_\epsilon - 1].$$

By (6.2) and (6.7),

$$A_{t_k} \cdots A_0(x) \in B(\theta, M)$$

and

$$A_{t_k} \cdots A_0(y) \in B(\theta, M). \tag{6.10}$$

By (6.7)–(6.9),

$$\rho(A_{t_k} \cdots A_0(x), A_{t_k} \cdots A_0(y)) > \epsilon. \tag{6.11}$$

It follows from (6.4) that

$$\begin{aligned} \rho(A_{t_{k+1}} \cdots A_0(x), A_{t_{k+1}} \cdots A_0(y)) \le \phi(\rho(A_{t_k} \cdots A_0(x), A_{t_k} \cdots A_0(y))) \\ \times \rho(A_{t_k} \cdots A_0(x), A_{t_k} \cdots A_0(y)). \end{aligned} \tag{6.12}$$

By (6.11) and (6.12),

$$\begin{aligned} &\rho(A_{t_k} \cdots A_0(x), A_{t_k} \cdots A_0(y)) - \rho(A_{t_{k+1}} \cdots A_0(x), A_{t_{k+1}} \cdots A_0(y)) \\ &\ge \rho(A_{t_k} \cdots A_0(x), A_{t_k} \cdots A_0(y)) \\ &\quad \times (1 - \phi(\rho(A_{t_k} \cdots A_0(x), A_{t_k} \cdots A_0(y)))) \\ &\ge \epsilon(1 - \phi(\epsilon)) \end{aligned}$$

and

$$\rho(A_{t_{k+1}} \cdots A_0(x), A_{t_{k+1}} \cdots A_0(y)) \le \rho(A_{t_k} \cdots A_0(x), A_{t_k} \cdots A_0(y)) - \epsilon(1 - \phi(\epsilon)). \tag{6.13}$$

By (6.7),

$$\begin{aligned} 2M \ge \rho(x, y) &\ge \rho(x, y) - \rho(A_{n_\epsilon} \cdots A_0(x), A_{n_\epsilon} \cdots A_0(y)) \\ &= \sum_{t=0}^{n_\epsilon - 1} [\rho(A_{t_k} \cdots A_{t_0}(x), A_{t_k} \cdots A_{t_0}(y)) \\ &\quad - \rho(A_{t_{k+1}} \cdots A_{t_0}(x), A_{t_{k+1}} \cdots A_{t_0}(y))] \\ &\ge k_\epsilon \epsilon(1 - \phi(\epsilon)) \end{aligned}$$

and

$$k_\epsilon \le 2M(\epsilon(1 - \phi(\epsilon))^{-1}.$$

This contradicts (6.5). The contradiction we have reached completes the proof of Theorem 6.1. □

7. Matkowski contractions

We continue to use the notations, definitions and assumptions introduced in Section 2.

Let $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$ and let D be a nonempty subset of K. We say that the sequence $\{A_t\}_{t=1}^{\infty}$ is *contractive* in the sense of Matkowski [7,14] on D if there exists an increasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{n\to\infty} \psi^n(s) = 0 \text{ for all } s > 0 \tag{7.1}$$

and

$$\rho(A_t(x), A_t(y)) \le \psi(\rho(x,y)) \text{ for all } x, y \in D \text{ and all } t = 1, 2 \dots.$$

In this section we prove the following two theorems (Theorems 7.1 and 7.2).

Theorem 7.1. *Assume that* $\{A_t\}_{t=1}^{\infty} \in \mathcal{M}$, $M_A > 0$, *for each* $M \ge M_A$,

$$A_t(B(\theta, M)) \subset B(\theta, M),\ t = 1, 2, \dots, \tag{7.1}$$

and that the sequence of mappings $\{A_t\}_{t=1}^{\infty}$ *is Matkowski contractive on* $B(\theta, M)$.

Let $M, \epsilon > 0$. *Then there exists a natural number* n_ϵ *such that for each mapping*

$$r : \{1, 2, \dots\} \to \{1, 2, \dots\},$$

each pair of points $x, y \in B(\theta, M)$ *and each integer* $n \ge n_\epsilon$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \le \epsilon.$$

Proof. We may assume without loss of generality that $M \ge M_A$. Since the sequence of mappings $\{A_t\}_{t=1}^{\infty}$ is Matkowski contractive on $B(\theta, M)$, there exists an increasing function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\lim_{n\to\infty} \psi^n(s) = 0 \text{ for all } s > 0 \tag{7.3}$$

and

$$\rho(A_t(x), A_t(y)) \le \psi(\rho(x,y)) \text{ for all } x, y \in B(\theta, M) \text{ and all } t = 1, 2 \dots. \tag{7.4}$$

Denote by A_0 the identity operator $I : K \to K$: $I(x) = x$ for all $x \in K$. By (7.3), there exists a natural number n_ϵ such that

$$\psi^{n_\epsilon}(2M) < \epsilon. \tag{7.5}$$

Assume that

$$r : \{1, 2, \dots\} \to \{1, 2, \dots\}$$

and that

$$x, y \in B(\theta, M). \tag{7.6}$$

Set

$$r(0) = 0.$$

In order to complete the proof of the theorem, we should show that for all integers $n \ge n_\epsilon$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \le \epsilon.$$

Since all the mappings A_t, $t = 1, 2, \dots$, are nonexpansive, it suffices to show that

$$\rho(A_{r(n_\epsilon)} \cdots A_{r(1)}(x), A_{r(n_\epsilon)} \cdots A_{r(1)}(y)) \le \epsilon.$$

It follows from (7.6) that for all integers $n \ge 1$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \le 2M.$$

By (7.2) and (7.4), for all integers $n \in [0, n_\epsilon - 1]$,

$$\rho(A_{r(n+1)} \cdots A_{r(0)}(x), A_{r(n+1)} \cdots A_{r(0)}(y))$$
$$\le \psi(\rho(A_{r(n)} \cdots A_{r(0)}(x), A_{r(n)} \cdots A_{r(0)}(y))).$$

Together with (7.5) and (7.6) this implies that

$$\rho(A_{r(n_\epsilon)} \cdots A_{r(0)}(x), A_{r(n_\epsilon)} \cdots A_{r(0)}(y))$$
$$\le \psi^{n_\epsilon}(\rho(x,y)) \le \psi^{n_\epsilon}(2M) < \epsilon.$$

This completes the proof of Theorem 7.1. □

Theorem 7.2. *Assume that* $\{A_t\}_{t=1}^\infty \in \mathcal{M}$, $M_A > 0$, *that for every* $M \ge M_A$,

$$A_t(B(\theta, M)) \subset B(\theta, M), \ t = 1, 2, \ldots, \tag{7.7}$$

and that a subsequence of mappings $\{A_{t_k}\}_{k=1}^\infty$ *is Matkowski contractive on* $B(\theta, M)$ *for every* $M \ge M_A$.

Let $M, \epsilon > 0$. *Then there exists a natural number* n_ϵ *such that for each pair of points* $x, y \in B(\theta, M)$ *and each integer* $n \ge n_\epsilon$,

$$\rho(A_n \cdots A_1(x), A_n \cdots A_1(y)) \le \epsilon.$$

Proof. We may assume without loss of generality that $M > M_A$. There exists an increasing function $\psi : [0, \infty) \to [0, \infty)$ such that

$$\lim_{n\to\infty} \psi^n(s) = 0 \text{ for all } s > 0 \tag{7.8}$$

and

$$\rho(A_{t_k}(x), A_{t_k}(y)) \le \psi(\rho(x,y)) \text{ for all } x, y \in B(\theta, M) \text{ and all } k = 1, 2 \ldots. \tag{7.9}$$

By (7.8), there exists a natural number k_ϵ such that

$$\psi^k(2M) < \epsilon \ \text{ for all integers } k \ge k_\epsilon. \tag{7.10}$$

Set

$$n_\epsilon = t_{k_\epsilon}$$

and

$$t_0 = 0.$$

Denote by A_0 the identity operator $I : K \to K$: $I(x) = x$ for all $x \in K$.

Assume that

$$x, y \in B(\theta, M). \tag{7.11}$$

Since all the mappings A_t, $t = 1, 2, \ldots$, are nonexpansive, for all integers $n \ge 1$, we have

$$\rho(A_n \cdots A_1(x), A_n \cdots A_1(y)) \le 2M.$$

By (7.7) and (7.9), for all integers $k \ge 0$,

$$\rho(A_{t_{k+1}} \cdots A_{t_0}(x), A_{t_{k+1}} \cdots A_{t_0}(y)) \le \psi(\rho(A_{t_{k+1}-1} \cdots A_{t_0}(x), A_{t_{k+1}-1} \cdots A_{t_0}(y)))$$
$$\le \psi(\rho(A_{t_k} \cdots A_{t_0}(x), A_{t_k} \cdots A_{t_0}(y))).$$

When combined with (7.10) and (7.11), this implies that for all integers $k \geq k_\epsilon$,

$$\rho(A_{t_k} \cdots A_{t_0}(x), A_{t_k} \cdots A_{t_0}(y)) \leq \psi^k(\rho(x,y)) \leq \psi^k(2M) \leq \epsilon.$$

This completes the proof of Theorem 7.2. □

8. Unrestricted infinite products

We again use the notation, definitions and assumptions introduced in Section 2.

Let $E \subset \mathcal{A}$ and let D be a nonempty subset of K. We say that the set E is *contractive* in the sense of Rakotch [8] on D if there exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that

$$\phi(t) < 1 \text{ for all } t > 0 \tag{8.1}$$

and

$$\rho(A(x), A(y)) \leq \phi(\rho(x,y))\rho(x,y) \text{ for all } x, y \in D \text{ and all } A \in E. \tag{8.2}$$

In this section we prove the following result.

Theorem 8.1. *Assume that* $\{A_t\}_{t=1}^\infty \in \mathcal{M}$,

$$E \subset \{A_t : \ t = 1, 2, \dots\}$$

is nonempty and there exists a positive number M_A *such that for each* $M \geq M_A$,

$$A_t(B(\theta, M)) \subset B(\theta, M), \ t = 1, 2, \dots, \tag{8.3}$$

and that the set E *is Rakotch contractive on* $B(\theta, M)$.

Let $M, \epsilon > 0$ *and* $\bar{p}$ *be a natural number. Then there exists a natural number* n_* *such that for each mapping*

$$r : \{1, 2, \dots\} \to \{1, 2, \dots\}$$

which satisfies

$$\{A_{r(i)} : \ i = k\bar{p}+1, \dots, (k+1)\bar{p}\} \cap E \neq \emptyset, \ k = 0, 1, \dots, \tag{8.4}$$

each pair of points $x, y \in B(\theta, M)$ *and for each integer* $n \geq n_*$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \leq \epsilon.$$

Proof. We may assume without any loss of generality that $M \geq M_A$. Since the set E is Rakotch contractive on $B(\theta, M)$, there exists a decreasing function $\phi : [0,\infty) \to [0,1]$ such that

$$\phi(t) < 1 \text{ for all } t > 0 \tag{8.5}$$

and

$$\rho(A(x), A(y)) \leq \phi(\rho(x,y))\rho(x,y) \text{ for all } x, y \in B(\theta, M) \tag{8.6}$$

and all $A \in E$.

Denote by A_0 the identity operator $I : K \to K$: $I(x) = x$ for all $x \in K$.

Choose a natural number

$$k_* > 2M(\epsilon(1 - \phi(\epsilon))^{-1} \tag{8.7}$$

and set

$$n_* = k_*\bar{p} + 1. \tag{8.8}$$

Assume that the mapping

$$r : \{1, 2, \dots\} \to \{1, 2, \dots\}$$

satisfies (8.4) and that

$$x, y \in B(\theta, M). \tag{8.9}$$

In order to complete the proof of the theorem, we need to show that for all integers $n \geq n_*$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \leq \epsilon.$$

Since all the mappings A_t, $t = 1, 2, \dots$, are nonexpansive, it suffices to show that

$$\rho(A_{r(n_*)} \cdots A_{r(1)}(x), A_{r(n_*)} \cdots A_{r(1)}(y)) \leq \epsilon.$$

Suppose to the contrary that this inequality does not hold. Then

$$\rho(A_{r(n_*)} \cdots A_{r(1)}(x), A_{r(n_*)} \cdots A_{r(1)}(y)) > \epsilon. \tag{8.10}$$

Since all the mappings A_t, $t = 1, 2, \dots$, are nonexpansive, it follows from (8.10) that

$$\rho(x, y) > \epsilon \tag{8.11}$$

and

$$\rho(A_{r(k)} \cdots A_{r(1)}(x), A_{r(k)} \cdots A_{r(1)}(y)) > \epsilon \tag{8.12}$$

for all $k = 1, \dots, n_*$.

Set

$$r(0) = 0.$$

Assume that an integer

$$t \in [0, n_* - 1]. \tag{8.13}$$

It is clear that

$$\rho(A_{r(t+1)} \cdots A_{r(0)}(x), A_{r(t+1)} \cdots A_{r(0)}(y)) \leq \rho(A_{r(t)} \cdots A_{r(0)}(x), A_{r(t)} \cdots A_{r(0)}(y))$$

and if

$$A_{r(t+1)} \in E,$$

then in view of (8.6) and (8.11)–(8.13),

$$\begin{aligned} &\rho(A_{r(t+1)} \cdots A_{r(0)}(x), A_{r(t+1)} \cdots A_{r(0)}(y)) \\ &\leq \phi(\rho(A_{r(t)} \cdots A_{r(0)}(x), A_{r(t)} \cdots A_{r(0)}(y))) \\ &\quad \times \rho(A_{r(t)} \cdots A_{r(0)}(x), A_{r(t)} \cdots A_{r(0)}(y)) \\ &\leq \phi(\epsilon)\rho(A_{r(t)} \cdots A_{r(0)}(x), A_{r(t)} \cdots A_{r(0)}(y)). \end{aligned} \tag{8.14}$$

By (8.9) and (8.11)–(8.14),

$$\begin{aligned}
2M &\geq \rho(x,y) \\
&= \rho(A_{r(0)}(x), A_{r(0)}(y)) \\
&\geq \rho(A_{r(0)}(x), A_{r(0)}(y)) - \rho(A_{r(n_*)}\cdots A_{r(0)}(x), A_{r(n_*)}\cdots A_{r(0)}(y)) \\
&= \sum_{i=0}^{n_*-1} [\rho(A_{r(i)}\cdots A_{r(0)}(x), A_{r(i)}\cdots A_{r(0)}(y)) \\
&\quad - \rho(A_{r(i+1)}\cdots A_{r(0)}(x), A_{r(i+1)}\cdots A_{r(0)}(y))] \\
&\geq \sum \{\rho(A_{r(i)}\cdots A_{r(0)}(x), A_{r(i)}\cdots A_{r(0)}(y)) \\
&\quad - \rho(A_{r(i+1)}\cdots A_{r(0)}(x), A_{r(i+1)}\cdots A_{r(0)}(y)) : \\
&\quad i \in \{0,\dots,n_*-1\},\ A_{r(i+1)} \in E\} \\
&\geq \sum \{(1-\phi(\epsilon))\rho(A_{r(i)}\cdots A_{r(0)}(x), A_{r(i)}\cdots A_{r(0)}(y)) : \\
&\quad i \in \{0,\dots,n_*-1\},\ A_{r(i+1)} \in E\} \\
&\geq \epsilon(1-\phi(\epsilon))\mathrm{Card}\{i \in \{0,\dots,n_\epsilon - 1\} :\ A_{r(i+1)} \in E\}.
\end{aligned}$$

By the above relation, (8.4) and (8.8),

$$2M \geq (\epsilon(1-\phi(\epsilon))k_*.$$

This contradicts (8.7). The contradiction we have reached completes the proof of Theorem 8.1. □

9. Inexact infinite products

We continue to use the notations, definitions and assumptions introduced in Section 2.

In this section we prove the following result.

Theorem 9.1. *Assume that $\{A_t\}_{t=1}^\infty \in \mathcal{M}$, there exists a positive number M_A such that for each $M \geq M_A$,*

$$A_t(B(\theta,M)) \subset B(\theta,M),\ t = 1,2,\dots, \tag{9.1}$$

and that the sequence of mappings $\{A_t\}_{t=1}^\infty$ is Rakotch contractive on $B(\theta,M)$.

Let $M, \epsilon > 0$, $\{\epsilon_i\}_{i=1}^\infty \subset [0,\infty)$ satisfy

$$\sum_{i=0}^{\infty} \epsilon_i < \infty. \tag{9.2}$$

Then there exists a natural number n_* such that for each mapping

$$r : \{1,2,\dots\} \to \{1,2,\dots\}$$

and each pair of sequences $\{x_i\}_{i=0}^\infty$, $\{y_i\}_{i=0}^\infty \subset K$ satisfying

$$x_0,\ y_0 \in B(\theta,M) \tag{9.3}$$

and

$$\rho(x_{i+1}, A_{r(i+1)}(x_i)), \rho(y_{i+1}, A_{r(i+1)}(y_i)) \leq \epsilon_i \tag{9.4}$$

for each integer $i \geq 0$, the inequality

$$\rho(x_n, y_n) \leq \epsilon$$

holds for all integers $n \geq n_*$.

Proof. We may assume without any loss of generality that $M \geq M_A$. Set

$$\Lambda = \sum_{i=0}^{\infty} \epsilon_i. \tag{9.5}$$

Let a sequence $\{z_i\}_{i=0}^{\infty} \subset K$ satisfy for each integer $i \geq 0$,

$$\rho(z_{i+1}, A_{r(i+1)}(z_i)) \leq \epsilon_i. \tag{9.6}$$

Let $k \geq 0$ be an integer. Define

$$z_k^{(k)} = z_k \tag{9.7}$$

and

$$z_{i+1}^{(k)} = A_{r(i+1)}(z_i^{(k)}) \text{ for all integers } i \geq k. \tag{9.8}$$

We claim that for all integers $i \geq k+1$,

$$\rho(z_i, z_i^{(k)}) \leq \sum_{j=k}^{i-1} \epsilon_j. \tag{9.9}$$

By (9.6), (9.7) and (9.8),

$$\rho(z_{k+1}, z_{k+1}^{(k)}) = \rho(z_{k+1}, A_{r(k+1)}(z_k^{(k)})) = \rho(z_{k+1}, A_{r(k+1)}(z_k)) \leq \epsilon_k.$$

Thus (9.9) holds for $i = k+1$.

Assume now that $i \geq k+1$ is an integer and that (9.9) holds. In view of (9.6), (9.8) and (9.9),

$$\begin{aligned}\rho(z_{i+1}, z_{i+1}^{(k)}) &\leq \rho(z_{i+1}, A_{r(i+1)}(z_i)) + \rho(A_{r(i+1)}(z_i), A_{r(i+1)}(z_i^{(k)})) \\ &\leq \epsilon_i + \rho(z_i, z_i^{(k)}) \leq \sum_{j=k}^{i} \epsilon_j.\end{aligned}$$

Thus we have shown by induction that (9.9) indeed holds for all integers $i \geq k+1$.

By (9.1), (9.7) and (9.8),

$$z_i^{(0)} \in B(\theta, M),\ i = 0, 1, \ldots. \tag{9.10}$$

Relations (9.5), (9.9) and (9.10) imply that

$$z_i \in B(\theta, M + \Lambda),\ i = 0, 1, \ldots. \tag{9.11}$$

It follows from (9.1), (9.7), (9.8) and (9.11) that

$$z_i^{(k)} \in B(\theta, M + \Lambda),\ k \in \{0, 1, \ldots\},\ i \in \{k, k+1, \ldots\}. \tag{9.12}$$

By Theorem 5.1, there exists a natural number $\tilde{n}$ such that the following property holds:

(P2) for each mapping

$$r : \{1, 2, \ldots\} \to \{1, 2, \ldots\},$$

each pair of points $x, y \in B(\theta, M+\Lambda)$ and for each integer $n \geq \tilde{n}$,

$$\rho(A_{r(n)} \cdots A_{r(1)}(x), A_{r(n)} \cdots A_{r(1)}(y)) \leq \epsilon/2.$$

In view of (9.2), there exists a natural number n_0 such that

$$\sum_{i=n_0}^{\infty} \epsilon_i < \epsilon/4. \tag{9.13}$$

Set

$$n_* = \tilde{n} + n_0. \tag{9.14}$$

Assume that the mapping

$$r : \{1, 2, \dots\} \to \{1, 2, \dots\},$$

and that sequences $\{x_i\}_{i=0}^{\infty}$, $\{y_i\}_{i=0}^{\infty} \subset K$ satisfy

$$x_0,\ y_0 \in B(\theta, M) \tag{9.15}$$

and

$$\rho(x_{i+1}, A_{r(i+1)}(x_i)), \rho(y_{i+1}, A_{r(i+1)}(y_i)) \leq \epsilon_i \tag{9.16}$$

for each integer $i \geq 0$.

For each integer $k \geq 0$ and each integer $i \geq k$, define $x_i^{(k)}$, $y_i^{(k)}$ as in (9.7) and (9.8). As it was shown in (9.9), (9.11) and (9.12),

$$x_i,\ y_i \in B(\theta, M+\Lambda),\ i \in \{0, 1, \dots\}, \tag{9.17}$$

$$x_i^{(k)},\ y_i^{(k)} \in B(\theta, M+\Lambda),\ k \in \{0, 1, \dots\},\ i \in \{k, k+1, \dots\} \tag{9.18}$$

and for each integer $k \geq 0$ and each integer $i \geq k$,

$$\rho(x_i, x_i^{(k)}),\ \rho(y_i, y_i^{(k)}) \leq \sum_{j=k}^{i-1} \epsilon_j. \tag{9.19}$$

Property (P2), (9.7), (9.8) and (9.17) imply that

$$\rho(x_i^{(n_0)}, y_i^{(n_0)}) \leq \epsilon/2 \text{ for all integers } i \geq \tilde{n} + n_0. \tag{9.20}$$

By (9.13) and (9.19), for all integers $i \geq \tilde{n} + n_0$,

$$\rho(x_i, x_i^{(n_0)}),\ \rho(y_i, y_i^{(n_0)}) \leq \sum_{j=n_0}^{\infty} \epsilon_j < \epsilon/4.$$

When combined with (9.20), this implies that for each integer $i \geq n_* = \tilde{n} + n_0$,

$$\rho(x_i, y_i) < \epsilon.$$

The proof of Theorem 9.1 is complete. □

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