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AN IMPLICIT ALGORITHM FOR THE SPLIT COMMON FIXED POINT PROBLEM IN HILBERT SPACES AND APPLICATIONS

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Abstract. In this paper, using an implicit algorithm of Browder's type, we prove a strong convergence theorem for finding a solution of the general split common fixed point problem with zero points of two monotone operators in Hilbert spaces. This solution is the unique solution of the hierarchical variational inequality problem. Using this result, we obtain new and well-known strong convergence theorems in Hilbert spaces.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Given two mappings $T : H_1 \to H_1$ and $U : H_2 \to H_2$ and a bounded linear operator $A : H_1 \to H_2$, the split common fixed point problem is to find a point $z \in H_1$ such that $z \in F(T) \cap A^{-1}F(U)$, where $F(T)$ and $F(U)$ are fixed point sets of T and U, respectively. Such a problem includes the split feasibility problem and the split common null point problem. In fact, let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the split feasibility problem [7] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Defining $T = P_D$ and $U = P_Q$, where P_D and P_Q are the metric projections of H_1 onto D and H_2 onto Q, respectively, we have that $z \in D \cap A^{-1}Q$ is equivalent to $z \in F(T) \cap A^{-1}F(U)$. Furthermore, given maximal monotone operators $G : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the split common null point problem [6] is to find a point $z \in H_1$ such that $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $G^{-1}0$ and $B^{-1}0$ are null point sets of G and B, respectively. Defining $T = J_\lambda$ and $U = Q_\mu$, where J_λ and Q_μ are the resolvents of G for $\lambda > 0$ and B for $\mu > 0$, respectively, we have that $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$ is equivalent to $z \in F(T) \cap A^{-1}F(U)$. Thus the split common fixed point problem generalizes the split feasibility problem and the split common null point problem. There are many papers for the split feasibility problem, the split common null point problem and the split common fixed point problem; see, for instance, [6, 7, 16, 19, 30].

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Recently, Takahashi [26] introduced a class of new nonlinear mappings which contains strict pseudo-contractions [5] and generalized hybrid mappings [13] in a Hilbert space and the metric projections and the metric resolvents of a maximal monotone operator in a Banach space. Then Takahashi, Wen and Yao [27] proved a strong convergence theorem for finding a solution of the general split common fixed point problem with zero points of two monotone operators in Hilbert spaces by using an explicit algorithm of Halpern's type [11]. The new nonlinear mapping is as follows: Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2,$$

where $F(U)$ is the set of fixed points of U and J is the duality mapping on E.

On the other hand, we know the implicit algorithm defined by Browder [3] for finding a fixed point of a nonexpansive mapping in a Hilbert space.

In this paper, using an implicit algorithm of Browder's type, we prove a strong convergence theorem for finding a solution of the general split common fixed point problem with zero points of two monotone operators in Hilbert spaces. This solution is the unique solution of the hierarchical variational inequality problem. The proof is different from that of the explicit algorithm. Using this result, we obtain new and well-known strong convergence theorems in Hilbert spaces.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. Let H be a real Hilbert space with inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ and norm $\|\cdot\|$, respectively. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [24] that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.1}$$

Furthermore we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}$$

If $x = y + z$, then

$$\|x\|^2 \leq \|y\|^2 + 2\langle z, x\rangle. \tag{2.3}$$

Let C be a nonempty, closed and convex subset of H and let $T\colon C \to H$ be a mapping. We denote by $F(T)$ be the set of fixed points for T. A mapping $T : C \to H$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. If $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see [12]. For a nonempty, closed and convex subset C of H, the nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_Cx\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive;

$$\|P_Cx - P_Cy\|^2 \leq \langle P_Cx - P_Cy, x - y\rangle$$

for all $x, y \in H$. Furthermore $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see [10, 22]. Let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy\rangle \geq \alpha \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. If a mapping $U : C \to H$ is α-inverse strongly monotone and $0 < \lambda \leq 2\alpha$, then $I - \lambda U : C \to H$ is nonexpansive. In fact, we have that for all $x, y \in C$,

$$\begin{aligned} \|(I - \lambda U)x - (I - \lambda U)y\|^2 &= \|x - y - \lambda(Ux - Uy)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ux - Uy\rangle + \lambda^2\|Ux - Uy\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\alpha\|Ux - Uy\|^2 + \lambda^2\|Ux - Uy\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ux - Uy\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \tag{2.4}$$

Thus $I - \lambda U$ is nonexpansive; see [1, 18, 24] for more results of inverse strongly monotone mappings.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1} \colon H \to D(B)$, which is called the resolvent of B for r. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of B for $r > 0$. We know from [23] that

$$A_r x \in BJ_r x, \quad \forall x \in H, \ r > 0. \tag{2.5}$$

Let B be a maximal monotone operator on H and let

$$B^{-1}0 = \{x \in H : 0 \in Bx\}.$$

Then $B^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent J_r is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \leq \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{2.6}$$

We also know the following lemma from [21].

Lemma 2.1 ([21]). *Let H be a Hilbert space and let B be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s - t}{s}\langle J_s x - J_t x, J_s x - x\rangle \geq \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

From Lemma 2.1, we have that

$$\|J_\lambda x - J_\mu x\| \leq (|\lambda - \mu| / \lambda) \|x - J_\lambda x\| \tag{2.7}$$

for all $\lambda, \mu > 0$ and $x \in H$; see also [9, 22].

We also have the following lemma from Alsulami and Takahashi [1].

Lemma 2.2 ([1])**.** *Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $U : H_2 \to H_2$ be an α-inverse strongly monotone mapping. Then a mapping $A^*UA : H_1 \to H_1$ is $\frac{\alpha}{\|A\|^2}$-inverse strongly monotone.*

If T is a nonexpansive mapping, then $I - T$ is $\frac{1}{2}$-inverse strongly monotone. So we have the following result from Lemma 2.2.

Lemma 2.3. *Let H_1 and H_2 be Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping. Then the mapping $A^*(I - T)A : H_1 \to H_1$ is $\frac{1}{2\|A\|^2}$-inverse strongly monotone.*

The following lemma was proved in Takahashi, Xu and Yao [30].

Lemma 2.4 ([30])**.** *Let H_1 and H_2 be Hilbert spaces. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $T : H_2 \to H_2$ be a nonexpansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}F(T) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rA^*(I - T)A)z$;
(ii) $0 \in A^*(I - T)Az + Bz$;
(iii) $z \in B^{-1}0 \cap A^{-1}F(T)$.

Using Lemma 2.4, Plubtieng and Takahashi [19] proved the following lemma. This lemma is crucial for the proof of our main result.

Lemma 2.5 ([19])**.** *Let H_1 and H_2 be Hilbert spaces and let $\alpha > 0$. Let $B : H_1 \to 2^{H_1}$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$. Let $U : H_2 \to H_2$ be an α-inverse strongly monotone mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $B^{-1}0 \cap A^{-1}(U^{-1}0) \neq \emptyset$. Let $\lambda, r > 0$ and $z \in H_1$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rA^*UA)z$;
(ii) $0 \in A^*UAz + Bz$;
(iii) $z \in B^{-1}0 \cap A^{-1}(U^{-1}0)$.

3. Strong Convergence Theorem by Implicit Algorithm

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$. A mapping $T : C \to H$ is called a k-strict pseudo-contraction [5] if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 \tag{3.1}$$

for all $x, y \in C$. Let $\alpha > 0$ and let A be an α-inverse strongly monotone mapping of C into H. If $0 < \lambda \leq 2\alpha$, then $I - \lambda A : C \to H$ is nonexpansive from (2.4). Using this, we have that if $T : C \to H$ is a k-strict pseudo-contraction and $0 < \lambda \leq 1 - k$, then $(1 - \lambda)I + \lambda T$ is nonexpansive. In fact, if $T : C \to H$ is a k-strict pseudo-contraction and $U = I - T$, then we have

$$\|(I - U)x - (I - U)y\|^2 \leq \|x - y\|^2 + k\|Ux - Uy\|^2.$$

Since $\|(I-U)x-(I-U)y\|^2 = \|x-y\|^2 - 2\langle x-y, Ux-Uy\rangle + \|Ux-Uy\|^2$, we have

$$\langle x-y, Ux-Uy\rangle \geq \frac{1-k}{2}\|Ux-Uy\|^2.$$

Thus, if $0 < \lambda \leq 2 \cdot \frac{1-k}{2} = 1-k$, then

$$I - \lambda U = I - \lambda(I-T)U = (1-\lambda)I + \lambda T$$

is nonexpansive. We also know the following lemma for strict pseudo-contractions obtained by Marino and Xu [15].

Lemma 3.1 ([15]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and $T : C \to C$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in F(T)$.*

A mapping $g : H \to H$ is a contraction if there exists $k \in (0,1)$ such that $\|g(x) - g(y)\| \leq k\|x-y\|$ for all $x, y \in H$. We also call such a mapping g a k-contraction. A nonlinear operator $V : H \to H$ is called strongly monotone if there exists $\overline{\gamma} > 0$ such that $\langle x-y, Vx-Vy\rangle \geq \overline{\gamma}\|x-y\|^2$ for all $x, y \in H$. Such V is also called $\overline{\gamma}$-strongly monotone. A nonlinear operator $V : H \to H$ is called Lipschitzian continuous if there exists $L > 0$ such that $\|Vx - Vy\| \leq L\|x-y\|$ for all $x, y \in H$. Such V is also called L-Lipschitzian continuous. We know the following lemmas in a Hilbert space; see Lin and Takahashi [14].

Lemma 3.2 ([14]). *Let H be a Hilbert space and let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\overline{\gamma} > 0$ and $L > 0$. Let $t > 0$ satisfy $2\overline{\gamma} > tL^2$ and $1 > 2t\overline{\gamma}$. Then $0 < 1 - t(2\overline{\gamma} - tL^2) < 1$ and $I - tV : H \to H$ is a contraction, where I is the identity operator on H.*

Lemma 3.3 ([14]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let P_C be the metric projection of H onto C and let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator on H with $\overline{\gamma} > 0$ and $L > 0$. Let $t > 0$ satisfy $2\overline{\gamma} > tL^2$ and $1 > 2t\overline{\gamma}$ and let $z \in C$. Then the following are equivalent:*

(1) *$z = P_C(I - tV)z$;*
(2) *$\langle Vz, y - z\rangle \geq 0, \quad \forall y \in C$;*
(3) *$z = P_C(I - V)z$.*

Such $z \in C$ always exists and is unique.

Now, we prove the following strong convergence theorem by an implicit iteration of Browder's type [3] in a Hilbert space.

Theorem 3.4. *Let H_1 and H_2 be Hilbert spaces. Let B and G be maximal monotone operators on H_1. Let $J_{\lambda_n} = (I + \lambda_n B)^{-1}$ and $T_{r_n} = (I + r_n G)^{-1}$ be the resolvents of B and G for $\lambda_n > 0$ and $r_n > 0$, respectively. Let $\eta \in [0,1)$ and let S be an η-strict pseudo-contraction of H_1 into H_1. Define $S_\lambda = (1-\lambda)I + \lambda S$ for some λ with $0 < \lambda \leq 1 - \eta$. Let $\alpha > 0$ and let U be an α-inverse strongly monotone mapping of H_2 into H_2. Let $0 < k < 1$ and let g be a k-contraction of H_1 into itself. Let V be* *a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator of H_1 into H_1 with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $A : H_1 \to H_2$ be a bounded linear operator such that $\|A\| \neq 0$ and let A^* be the adjoint operator of A. Suppose $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0 \neq \emptyset$. Assume that $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, \infty)$ and $\{r_n\} \subset (0, \infty)$ are real sequences such that

$$\alpha_n \to 0, \quad 0 < a \leq \lambda_n \leq b < \frac{2\alpha}{\|A\|^2} \quad \text{and} \quad \liminf_{n\to\infty} r_n > 0$$

for some $a, b \in \mathbb{R}$. Then, the following hold:

(i) *For any $n \in \mathbb{N}$, define $T_n : H_1 \to H_1$ by*

$$T_n x = \alpha_n \gamma g(x) + (I - \alpha_n V) S_\lambda J_{\lambda_n} (I - \lambda_n A^* U A) T_{r_n} x, \quad \forall x \in H_1.$$

Then, T_n has a unique fixed point x_n in H_1 and $\{x_n\}$ is bounded.

(ii) *For any nonempty, closed and convex subset C of H_1, $P_C(I - V + \gamma g)$ has a unique fixed point z_0 in C. This point $z_0 \in C$ is also a unique solution of the variational inequality*

$$\langle (V - \gamma g) z_0, q - z_0 \rangle \geq 0, \quad \forall q \in C.$$

In particular, $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$ is a nonempty, closed and convex subset of H_1 and $P_{F(S)\cap B^{-1}0\cap A^{-1}(U^{-1}0)\cap G^{-1}0}(I - V + \gamma g)$ has a unique fixed point z_0 in $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$. This point z_0 is also a unique solution of the variational inequality

$$\langle (V - \gamma g) z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0.$$

(iii) *The sequence $\{x_n\}$ in H_1 converges strongly to a unique element z_0 in $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$, where*

$$\langle (V - \gamma g) z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0.$$

Proof. Let us prove (i). For any $n \in \mathbb{N}$, define $T_n, U_n : H_1 \to H_1$ by

$$T_n x = \alpha_n \gamma g(x) + (I - \alpha_n V) S_\lambda J_{\lambda_n} (I - \lambda_n A^* U A) T_{r_n} x$$

and

$$U_n x = S_\lambda J_{\lambda_n} (I - \lambda_n A^* U A) T_{r_n} x, \quad \forall x \in H_1.$$

Since S is an η-strict pseudo-contraction of H_1 into H_1 and $S_\lambda = (1 - \lambda)I + \lambda S$ for some λ with $0 < \lambda \leq 1 - \eta$, S_λ is nonexpansive. Furthermore, from Lemma 2.2, (2.4) and $0 < \lambda_n \leq \frac{2\alpha}{\|A\|^2}$, we have that $I - \lambda_n A^* U A$ is also nonexpansive. Thus, we have that for any $x, y \in H_1$,

$$\begin{aligned}
\|U_n x - U_n y\| &= \|S_\lambda J_{\lambda_n} (I - \lambda_n A^* U A) T_{r_n} x - S_\lambda J_{\lambda_n} (I - \lambda_n A^* U A) T_{r_n} y\| \\
&\leq \|J_{\lambda_n} (I - \lambda_n A^* U A) T_{r_n} x - J_{\lambda_n} (I - \lambda_n A^* U A) T_{r_n} y\| \\
&\leq \|(I - \lambda_n A^* U A) T_{r_n} x - (I - \lambda_n A^* U A) T_{r_n} y\| \\
&\leq \|T_{r_n} x - T_{r_n} y\| \\
&\leq \|x - y\|.
\end{aligned}$$

Put $\tau = \overline{\gamma} - \frac{L^2\mu}{2}$. We have $\tau > 0$. Since $\lim_{n\to\infty} \alpha_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $1 - \alpha_n\tau > 0$ and $\alpha_n < \mu$ for all $n \geq n_0$. Then, we have that for any $x, y \in H_1$ and $n \geq n_0$,

$$\begin{aligned}
\|(I-\alpha_n V)x - (I-\alpha_n V)y\|^2 &= \|x - y - \alpha_n(Vx - Vy)\|^2 \\
&= \|x-y\|^2 - 2\alpha_n\langle x-y, Vx - Vy\rangle + \alpha_n^2\|Vx - Vy\|^2 \\
&\leq \|x-y\|^2 - 2\alpha_n\overline{\gamma}\|x-y\|^2 + \alpha_n^2 L^2\|x-y\|^2 \\
&= (1 - 2\alpha_n\overline{\gamma} + \alpha_n^2 L^2)\|x-y\|^2 \\
&= (1 - 2\alpha_n\tau - \alpha_n L^2\mu + \alpha_n^2 L^2)\|x-y\|^2 \\
&\leq \big(1 - 2\alpha_n\tau - \alpha_n(L^2\mu - \alpha_n L^2) + \alpha_n^2\tau^2)\|x-y\|^2 \\
&\leq (1 - 2\alpha_n\tau + \alpha_n^2\tau^2)\|x-y\|^2 \\
&= (1-\alpha_n\tau)^2\|x-y\|^2.
\end{aligned}$$

Since $1 - \alpha_n\tau > 0$, we obtain that for any $x, y \in H_1$ and $n \geq n_0$,

$$\|(I-\alpha_n V)x - (I-\alpha_n V)y\| \leq (1-\alpha_n\tau)\|x-y\|. \tag{3.2}$$

Furthermore, we have that

$$\begin{aligned}
&\|(I-\alpha_n V)U_n x - (I-\alpha_n V)U_n y\| \\
&\qquad\leq (1-\alpha_n\tau)\|U_n x - U_n y\| \leq (1-\alpha_n\tau)\|x-y\|.
\end{aligned}\tag{3.3}$$

Thus, we have that

$$\begin{aligned}
\|T_n x - T_n y\| &= \|\alpha_n\gamma g(x) + (I-\alpha_n V)U_n x - \{\alpha_n\gamma g(y) + (I-\alpha_n V)U_n y\}\| \\
&\leq \alpha_n\gamma\|g(x) - g(y)\| + \|(I-\alpha_n V)U_n x - (I-\alpha_n V)U_n y\| \\
&\leq \alpha_n\gamma\, k\|x-y\| + (1-\alpha_n\tau)\|x-y\| \\
&= (\alpha_n\gamma\, k + 1 - \alpha_n\tau)\|x-y\| \\
&= (1 - \alpha_n(\tau - \gamma\, k))\|x-y\|.
\end{aligned}$$

Since $\tau > \gamma\, k$ and hence $0 < 1 - \alpha_n\tau < 1 - \alpha_n(\tau - \gamma\, k)$, we have

$$0 < 1 - \alpha_n(\tau - \gamma\, k) < 1.$$

Then, T_n is a $(1 - \alpha_n(\tau - \gamma\, k))$-contraction of H_1 into itself and hence T_n has a unique fixed point x_n in H_1. Next, we show that $\{x_n\}$ is a bounded sequence. Let $u \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$. Using $u = \alpha_n Vu + u - \alpha_n Vu$, we have that for all $n \in \mathbb{N}$,

$$\begin{aligned}
\|x_n - u\| &= \|T_n x_n - \alpha_n Vu - u + \alpha_n Vu\| \\
&= \|\alpha_n(\gamma g(x_n) - Vu) + (I-\alpha_n V)U_n x_n - (I-\alpha_n V)u\| \\
&\leq \alpha_n\|\gamma g(x_n) - Vu\| + (1-\alpha_n\tau)\|x_n - u\| \\
&\leq \alpha_n\gamma\, k\|x_n - u\| + \alpha_n\|\gamma g(u) - Vu\| + (1-\alpha_n\tau)\|x_n - u\|.
\end{aligned}$$

Thus, we have $\alpha_n(\tau - \gamma\, k)\|x_n - u\| \leq \alpha_n\|\gamma g(u) - Vu\|$ and hence

$$(\tau - \gamma\, k)\|x_n - u\| \leq \|\gamma g(u) - Vu\|.$$

So, we have $\|x_n - u\| \leq \frac{\|\gamma g(u) - Vu\|}{\tau - \gamma\, k}$. This implies that $\{x_n\}$ is bounded.

Let us prove (ii). Since $g : H_1 \to H_1$ is a k-contraction and V is a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator of H_1 into H_1 with $\overline{\gamma} > 0$ and $L > 0$, we have that for any $x, y \in H_1$,

$$\begin{aligned}\langle x - y,\ (V - \gamma g)x - (V - \gamma g)y\rangle \\ &= \langle x - y, Vx - Vy\rangle - \langle x - y, \gamma gx - \gamma gy\rangle \\ &\geq \overline{\gamma}\|x - y\|^2 - \gamma\ k\|x - y\|^2 \\ &= (\overline{\gamma} - \gamma\ k)\|x - y\|^2.\end{aligned}$$

Then, $V - \gamma g : H_1 \to H_1$ is a $(\overline{\gamma} - \gamma\ k)$-strongly monotone operator. Furthermore, we have that for any $x, y \in H_1$,

$$\begin{aligned}\|(V - \gamma g)x - (V - \gamma g)y\| &= \|Vx - Vy - \gamma(gx - gy)\| \\ &\leq L\|x - y\| + \gamma k\|x - y\| \\ &= (L + \gamma k)\|x - y\|.\end{aligned}$$

This implies that $V - \gamma g$ is $(L + \gamma k)$-Lipschitzian continuous. Taking a positive number t with $t(L + \gamma\ k)^2 < 2(\overline{\gamma} - \gamma\ k)$ and $2t(\overline{\gamma} - \gamma\ k) < 1$ and using Lemma 3.3, we have that $P_C(I - V + \gamma g)$ has a unique fixed point z_0 in C. We also have from Lemma 3.3 that this point $z_0 \in C$ is a unique solution of the variational inequality

$$\langle (V - \gamma g)z_0, q - z_0\rangle \geq 0, \quad \forall q \in C.$$

In particular, $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$ is a nonempty, closed and convex subset of H_1. Thus, $P_{F(S)\cap B^{-1}0\cap A^{-1}(U^{-1}0)\cap G^{-1}0}(I - V + \gamma g)$ has a unique fixed point z_0 in $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$. This point z_0 is also a unique solution of the variational inequality

$$\langle (V - \gamma g)z_0, q - z_0\rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0.$$

Let us prove (iii). Put $y_n = S_\lambda J_{\lambda_n}(I - \lambda_n A^*UA)T_{r_n}x_n$ and $u_n = T_{r_n}x_n$ for all $n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, $\{y_n\}$ and $\{u_n\}$ are bounded. Furthermore, $\{g(x_n)\}$ is also bounded. Let $z \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$. We note that

$$\begin{aligned}\|y_n - z\| &= \|S_\lambda J_{\lambda_n}(I - \lambda_n A^*UA))u_n - z\| \\ &\leq \|J_{\lambda_n}(I - \lambda_n A^*UA))u_n - z\| \\ &\leq \|u_n - z\|\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\|u_n - y_n\| &\leq \|u_n - x_n\| + \|x_n - y_n\| \\ &= \|u_n - x_n\| + \|\alpha_n\gamma g(x_n) + (I - \alpha_n V)y_n - y_n\| \\ &= \|u_n - x_n\| + \alpha_n\|\gamma g(x_n) - Vy_n\| \\ &= \|u_n - x_n\| + \alpha_n\|\gamma g(x_n) - y_n + (I - V)y_n - (I - V)z + (I - V)z\| \\ &\leq \|u_n - x_n\| + \alpha_n(\|\gamma g(x_n) - y_n\| + (1 - \alpha_n\tau)\|y_n - z\| + \|(I - V)z\|) \\ &\leq \|u_n - x_n\| + \alpha_n(\|\gamma g(x_n) - y_n\| + \|y_n - z\| + \|(I - V)z\|).\end{aligned} \tag{3.5}$$

Using (2.6), we have

$$2\|u_n - z\|^2 = 2\|T_{r_n}x_n - T_{r_n}z\|^2$$

$$\leq 2\langle x_n - z, u_n - z\rangle$$
$$= \|x_n - z\|^2 + \|u_n - z\|^2 - \|u_n - x_n\|^2$$

and hence

$$\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.6}$$

Then, we have from (2.3) and (3.6) that

$$\begin{aligned}
\|x_n - z\|^2 &= \|(I - \alpha_n V)y_n - (I - \alpha_n V)z + \alpha_n(\gamma g(x_n) - Vz)\|^2 \\
&\leq \|(I - \alpha_n V)y_n - (I - \alpha_n V)z\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle \\
&\leq (1 - \alpha_n\tau)^2 \|u_n - z\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle \\
&\leq (1 - \alpha_n\tau)^2(\|x_n - z\|^2 - \|x_n - u_n\|^2) \\
&\quad + 2\alpha_n\gamma\ k\|x_n - z\|^2 + 2\alpha_n\|\gamma g(z) - Vz\|\|x_n - z\| \\
&= \{1 - 2\alpha_n(\tau - \gamma\ k) + \alpha_n^2\tau^2\} \|x_n - z\|^2 \\
&\quad - (1 - \alpha_n\tau)^2\|x_n - u_n\|^2 + 2\alpha_n\|\gamma g(z) - Vz\|\|x_n - z\| \\
&\leq \|x_n - z\|^2 + \alpha_n^2\tau^2 \|x_n - z\|^2 - (1 - \alpha_n\tau)^2\|x_n - u_n\|^2 \\
&\quad + 2\alpha_n\|\gamma g(z) - Vz\|\|x_n - z\|
\end{aligned}$$

and hence

$$(1 - \alpha_n\tau)^2\|x_n - u_n\|^2 \leq \alpha_n^2\tau^2 \|x_n - z\|^2 + 2\alpha_n\|\gamma g(z) - Vz\|\|x_n - z\|.$$

From $\alpha_n \to 0$, we have

$$\|x_n - u_n\| \to 0. \tag{3.7}$$

Then, we have from (3.5) that

$$\|y_n - u_n\| \to 0. \tag{3.8}$$

Put $z_n = J_{\lambda_n}(I - \lambda_n A^*UA)u_n$. Then, for $z \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$,

$$\begin{aligned}
\|z_n - z\|^2 &= \|J_{\lambda_n}(u_n - \lambda_n A^*UAu_n) - J_{\lambda_n}z\|^2 \\
&\leq \|u_n - \lambda_n A^*UAu_n - z\|^2 \\
&= \|u_n - z - \lambda_n A^*UAu_n\|^2 \\
&= \|u_n - z\|^2 - 2\lambda_n \langle u_n - z, A^*UAu_n\rangle + \lambda_n^2 \|A^*UAu_n\|^2 \\
&= \|u_n - z\|^2 - 2\lambda_n \langle Au_n - Az, UAu_n\rangle + \lambda_n^2 \|A^*UAu_n\|^2 \\
&\leq \|u_n - z\|^2 - 2\lambda_n\alpha \|UAu_n\|^2 + \lambda_n^2\|A\|^2 \|UAu_n\|^2 \\
&\leq \|x_n - z\|^2 + \lambda_n(\lambda_n\|A\|^2 - 2\alpha) \|UAu_n\|^2 \\
&\leq \|x_n - z\|^2
\end{aligned} \tag{3.9}$$

and hence

$$\begin{aligned}
\|x_n - z\|^2 &= \|(I - \alpha_n V)y_n - (I - \alpha_n V)z + \alpha_n(\gamma g(x_n) - Vz)\|^2 \\
&\leq \|(I - \alpha_n V)y_n - (I - \alpha_n V)z\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle \\
&\leq (1 - \alpha_n\tau)^2 \|y_n - z\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \|y_n - z\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle \\
&\leq \|z_n - z\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle \\
&\leq \|x_n - z\|^2 + \lambda_n(\lambda_n\|A\|^2 - 2\alpha)\,\|UAu_n\|^2 \\
&\qquad + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle.
\end{aligned} \tag{3.10}
$$

Then, we have that

$$\lambda_n(2\alpha - \lambda_n\|A\|^2)\,\|UAu_n\|^2 \leq 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle.$$

Since $\alpha_n \to 0$ and $0 < a \leq \lambda_n \leq b < \frac{2\alpha}{\|A\|^2}$, we have

$$\|UAu_n\|^2 \to 0. \tag{3.11}$$

Since J_{λ_n} is firmly nonexpansive and $I - \lambda_n A^*UA$ is nonexpansive, we have that

$$
\begin{aligned}
2\|z_n - z\|^2 &= 2\,\|J_{\lambda_n}(u_n - \lambda_n A^*UAu_n) - J_{\lambda_n}z\|^2 \\
&\leq 2\,\langle u_n - \lambda_n A^*UAu_n - z, z_n - z\rangle \\
&= \|u_n - \lambda_n A^*UAu_n - z\|^2 + \|z_n - z\|^2 \\
&\qquad - \|u_n - \lambda_n A^*UAu_n - z_n\|^2 \\
&= \|u_n - \lambda_n A^*UAu_n - (z - \lambda_n A^*UAz)\|^2 + \|z_n - z\|^2 \\
&\qquad - \|u_n - \lambda_n A^*UAu_n - z_n\|^2 \\
&\leq \|u_n - z\|^2 + \|z_n - z\|^2 \\
&\qquad - \|u_n - z_n - \lambda_n A^*UAu_n\|^2 \\
&= \|u_n - z\|^2 + \|z_n - z\|^2 - \|u_n - z_n\|^2 \\
&\qquad + 2\lambda_n\,\langle u_n - z_n, A^*UAu_n\rangle - \lambda_n^2\,\|A^*UAu_n\|^2.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
\|z_n - z\|^2 &\leq \|u_n - z\|^2 - \|u_n - z_n\|^2 \\
&\qquad + 2\lambda_n\,\langle u_n - z_n, A^*UAu_n\rangle - \lambda_n^2\,\|A^*UAu_n\|^2.
\end{aligned} \tag{3.12}
$$

Using (3.10) and (3.12), we obtain

$$
\begin{aligned}
\|x_n - z\|^2 &\leq \|z_n - z\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle \\
&\leq \|u_n - z\|^2 \\
&\qquad - \|u_n - z_n\|^2 + 2\lambda_n\,\langle u_n - z_n, A^*UAu_n\rangle \\
&\qquad - \lambda_n^2\,\|A^*UAu_n\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle \\
&\leq \|x_n - z\|^2 - \|u_n - z_n\|^2 \\
&\qquad + 2\lambda_n\,\langle u_n - z_n, A^*UAu_n\rangle - \lambda_n^2\,\|A^*UAu_n\|^2 \\
&\qquad + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle.
\end{aligned}
$$

Using this, we have that

$$\|u_n - z_n\|^2 \leq 2\lambda_n\,\langle u_n - z_n, A^*UAu_n\rangle$$

$$- \lambda_n^2 \|A^*UAu_n\|^2 + 2\alpha_n\langle \gamma g(x_n) - Vz, x_n - z\rangle).$$

Using (3.11) and $\alpha_n \to 0$, we have that

$$\lim_{n\to\infty} \|u_n - z_n\| = 0. \tag{3.13}$$

Since $\|S_\lambda z_n - z_n\| = \|y_n - z_n\| \leq \|y_n - u_n\| + \|u_n - z_n\|$, we have from (3.8) and (3.13) that

$$\lim_{n\to\infty} \|S_\lambda z_n - z_n\| = 0. \tag{3.14}$$

Take $\lambda_0 \in \mathbb{R}$ with $0 < a \leq \lambda_0 \leq b < \frac{2\alpha}{\|A\|^2}$ arbitrarily. Put $s_n = (I - \lambda_n A^*UA)u_n$. Using $z_n = J_{\lambda_n}(I - \lambda_n A^*UA)u_n$, we have from Lemma 2.1 that

$$\begin{aligned}
\|J_{\lambda_0}(I - \lambda_0 A^*UA)u_n - z_n\| &= \|J_{\lambda_0}(I - \lambda_0 A^*UA)u_n - J_{\lambda_n}(I - \lambda_n A^*UA)u_n\| \\
&= \|J_{\lambda_0}(I - \lambda_0 A^*UA)u_n - J_{\lambda_0}(I - \lambda_n A^*UA)u_n \\
&\quad + J_{\lambda_0}(I - \lambda_n A^*UA)u_n - J_{\lambda_n}(I - \lambda_n A^*UA)u_n\| \\
&\leq \|(I - \lambda_0 A^*UA)u_n - (I - \lambda_n A^*UA)u_n\| + \|J_{\lambda_0}s_n - J_{\lambda_n}s_n\| \\
&\leq |\lambda_0 - \lambda_n|\|A^*UAu_n\| + \frac{|\lambda_0 - \lambda_n|}{\lambda_0}\|J_{\lambda_0}s_n - s_n\|.
\end{aligned} \tag{3.15}$$

We also have from (3.15) that

$$\|u_n - J_{\lambda_0}(I - \lambda_0 A^*UA)u_n\| \leq \|u_n - z_n\| + \|z_n - J_{\lambda_0}(I - \lambda_0 A^*UA)u_n\|. \tag{3.16}$$

We will use (3.15) and (3.16) later.

We know that there exists a unique point $z_0 \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$ such that

$$\langle (V - \gamma g)z_0, q - z_0\rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0.$$

In order to show that $x_n \to z_0$, it suffices to show that if $\{x_{n_i}\}$ is any subsequence of $\{x_n\}$, then we can find a subsequence of $\{x_{n_i}\}$ converging strongly to z_0. Since $\{x_{n_i}\}$ is bounded and $\{\lambda_{n_i}\} \subset [a, b]$, without loss of generality there exist a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ and a subsequence $\{\lambda_{n_{i_j}}\}$ of $\{\lambda_{n_i}\}$ such that $x_{n_{i_j}} \rightharpoonup w$ and $\lambda_{n_{i_j}} \to \lambda_0$ for some $\lambda_0 \in [a, b]$.

We know from (3.14) that $\lim_{n\to\infty} \|S_\lambda z_n - z_n\| = 0$. Since

$$\|S_\lambda z_n - z_n\| = \|(1 - \lambda)z_n + \lambda S z_n - z_n\| = \lambda\|Sz_n - z_n\| \to 0,$$

we have $Sz_n - z_n \to 0$. We also have

$$\|x_n - z_n\| \leq \|x_n - u_n\| + \|u_n - z_n\| \to 0.$$

So, $z_{n_{i_j}} \rightharpoonup w$. Thus, we have $w \in F(S) = F(S_\lambda)$ because S is demiclosed from Lemma 3.1.

From $x_n - u_n \to 0$, we have $u_{n_{i_j}} \rightharpoonup w$. Using $\lambda_{n_{i_j}} \to \lambda_0$ and (3.15), we have that

$$\|J_{\lambda_0}(I - \lambda_0 A^*UA)u_{n_{i_j}} - z_{n_{i_j}}\| \to 0.$$

Furthermore, we have from $\|z_{n_{i_j}} - u_{n_{i_j}}\| \to 0$ and (3.16) that

$$\|J_{\lambda_0}(I - \lambda_0 A^*UA)u_{n_{i_j}} - u_{n_{i_j}}\| \to 0.$$

Since $J_{\lambda_0}(I-\lambda_0 A^*UA)$ is nonexpansive, we have that $w = J_{\lambda_0}(I-\lambda_0 A^*UA)w$ and hence $w \in B^{-1}0 \cap A^{-1}(U^{-1}0)$ from Lemma 2.5. We show $w \in G^{-1}0$. Since G is a maximal monotone operator, we have from (2.5) that $A_{r_{n_{i_j}}} x_{n_{i_j}} \in GT_{r_{n_{i_j}}} x_{n_{i_j}}$, where A_r is the Yosida approximation of G for $r>0$. Furthermore we have that for any $(u,v) \in G$,

$$\left\langle u - u_{n_{i_j}}, v - \frac{x_{n_{i_j}} - u_{n_{i_j}}}{r_{n_{i_j}}} \right\rangle \geq 0.$$

Since $\liminf_{n\to\infty} r_n > 0$, $u_{n_{i_j}} \rightharpoonup w$ and $x_{n_{i_j}} - u_{n_{i_j}} \to 0$, we have

$$\langle u - w, v\rangle \geq 0.$$

Since G is a maximal monotone operator, we have $0 \in Gw$ and hence $w \in G^{-1}0$. Thus we have $w \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$.

From (ii) we can take a unique point $z_0 \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$ such that

$$z_0 = P_{F(S)\cap B^{-1}0\cap A^{-1}(U^{-1}0)\cap G^{-1}0}(I - V + \gamma g)z_0.$$

This point z_0 satisfies

$$\langle (V-\gamma g)z_0, q - z_0\rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0.$$

We also have that

$$x_n - z_0 = \alpha_n(\gamma g(x_n) - Vz_0) + (I-\alpha_n V)y_n - (I-\alpha_n V)z_0.$$

So, we have that

$$\begin{aligned}
&\|x_n - z_0\|^2 \\
&= \alpha_n\langle \gamma g(x_n) - Vz_0, x_n - z_0\rangle + \langle (I-\alpha_n V)y_n - (I-\alpha_n V)z_0, x_n - z_0\rangle \\
&\leq \alpha_n\langle \gamma g(x_n) - Vz_0, x_n - z_0\rangle + (1-\alpha_n\tau)\|y_n - z_0\|\|x_n - z_0\| \\
&\leq \alpha_n\langle \gamma g(x_n) - Vz_0, x_n - z_0\rangle + (1-\alpha_n\tau)\|x_n - z_0\|^2.
\end{aligned}$$

Then we have that

$$\alpha_n\tau\|x_n - z_0\|^2 \leq \alpha_n\langle \gamma g(x_n) - Vz_0, x_n - z_0\rangle$$

and hence

$$\begin{aligned}
\|x_n - z_0\|^2 &\leq \frac{1}{\tau}\langle \gamma g(x_n) - Vz_0, x_n - z_0\rangle \\
&= \frac{1}{\tau}\langle \gamma g(x_n) - \gamma g(z_0) + \gamma g(z_0) - Vz_0, x_n - z_0\rangle \\
&\leq \frac{1}{\tau}\gamma\, k\|x_n - z_0\|^2 + \frac{1}{\tau}\langle \gamma g(z_0) - Vz_0, x_n - z_0\rangle.
\end{aligned}$$

This implies that

$$\|x_n - z_0\|^2 \leq \frac{\langle \gamma g(z_0) - Vz_0, x_n - w\rangle}{\tau - \gamma\, k}.$$

In particular, we have that

$$\|x_{n_{i_j}} - z_0\|^2 \leq \frac{\langle \gamma g(z_0) - Vz_0, x_{n_{i_j}} - z_0\rangle}{\tau - \gamma\, k}.$$

Since $x_{n_{i_j}} \rightharpoonup w$, $w \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$ and

$$\langle (V - \gamma g)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0,$$

we have

$$\begin{aligned} \limsup_{j \to \infty} \|x_{n_{i_j}} - z_0\|^2 &\leq \lim_{j \to \infty} \frac{\langle \gamma g(z_0) - V z_0, x_{n_{i_j}} - z_0 \rangle}{\tau - \gamma\ k} \\ &= \frac{\langle \gamma g(z_0) - V z_0, w - z_0 \rangle}{\tau - \gamma\ k} \leq 0. \end{aligned}$$

Thus, we have $x_{n_{i_j}} \to z_0$. Then, the sequence $\{x_n\}$ converges strongly to a unique point $z_0 \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$ such that

$$\langle (V - \gamma g)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0.$$

This z_0 is also a unique fixed point of $P_{F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0}(I - V + \gamma g)$ in $F(S) \cap B^{-1}0 \cap A^{-1}(U^{-1}0) \cap G^{-1}0$. This completes the proof. □

4. Applications

In this section, using Theorem 3.4, we can obtain well-known and new strong convergence theorems in Hilbert spaces. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to H$ be a strict pseudo-contraction, that is, there exists $k \in \mathbb{R}$ with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

Putting $U = I - T$, we have that

$$\frac{1-k}{2}\|Ux - Uy\|^2 \leq \langle x - y, Ux - Uy \rangle. \tag{4.1}$$

Therefore, $U = I - T$ is $\frac{1-k}{2}$-inverse strongly monotone. In particular, if $T : C \to H$ is nonexpansive, then $U = I - T$ is $\frac{1}{2}$-inverse strongly monotone.

Theorem 4.1. *Let H_1 and H_2 be Hilbert spaces. Let B and G be maximal monotone operators on H_1. Let $J_{\lambda_n} = (I + \lambda_n B)^{-1}$ and $T_{r_n} = (I + r_n G)^{-1}$ be the resolvents of B and G for $\lambda_n > 0$ and $r_n > 0$, respectively. Let S be a nonexpansive mapping of H_1 into H_1. Let $0 \leq \eta < 1$ and let T be an η-strict pseudo-contraction of H_2 into H_2. Let $0 < k < 1$ and let g be a k-contraction of H_1 into itself. Let V be a $\overline{\gamma}$-strongly monotone and L-Lipschitzian continuous operator of H_1 into H_1 with $\overline{\gamma} > 0$ and $L > 0$. Take $\mu, \gamma \in \mathbb{R}$ as follows:*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2 \mu}{2}}{k}.$$

Let $A : H_1 \to H_2$ be a bounded linear operator such that $\|A\| \neq 0$. Suppose $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0 \neq \emptyset$. Assume that $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, \infty)$ and $\{r_n\} \subset (0, \infty)$ are real sequences such that

$$\alpha_n \to 0, \quad 0 < a \leq \lambda_n \leq b < \frac{1 - \eta}{\|A\|^2} \quad \text{and} \quad \liminf_{n \to \infty} r_n > 0$$

for some $a, b \in \mathbb{R}$. Then, the following hold:

(i) *For any* $n \in \mathbb{N}$, *define* $T_n : H_1 \to H_1$ *by*

$$T_n x = \alpha_n \gamma g(x) + (I - \alpha_n V) S J_{\lambda_n} (I - \lambda_n A^*(I - T)A) T_{r_n} x, \quad \forall x \in H_1.$$

Then, T_n *has a unique fixed point* x_n *in* H_1 *and* $\{x_n\}$ *is bounded.*

(ii) *For any nonempty, closed and convex subset* C *of* H_1, $P_C(I - V + \gamma g)$ *has a unique fixed point* z_0 *in* C. *This point* $z_0 \in C$ *is also a unique solution of the variational inequality*

$$\langle (V - \gamma g) z_0, q - z_0 \rangle \geq 0, \quad \forall q \in C.$$

In particular, $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0$ *is a nonempty, closed and convex subset of* H_1 *and* $P_{F(S)\cap B^{-1}0 \cap A^{-1}F(T)\cap G^{-1}0}(I - V + \gamma g)$ *has a unique fixed point* z_0 *in* $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0$. *This point* z_0 *is also a unique solution of the variational inequality*

$$\langle (V - \gamma g) z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0.$$

(iii) *The sequence* $\{x_n\}$ *in* H_1 *converges strongly to a unique element* z_0 *in* $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0$, *where*

$$\langle (V - \gamma g) z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap G^{-1}0.$$

Proof. Taking $S_1 = S$ and $U = I - T$ in Theorem 3.4, we obtain the desired result from Theorem 3.4. □

Let $f : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0, \quad \forall y \in C. \tag{4.2}$$

The set of such solutions $\hat{x}$ is denoted by $EP(f)$, i.e.,

$$EP(f) = \{\hat{x} \in C : f(\hat{x}, y) \geq 0, \ \forall y \in C\}.$$

For solving the equilibrium problem, let us assume that the bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The following lemmas were given in Combettes and Hirstoaga [8] and Takahashi, Takahashi and Toyoda [21]; see also [2].

Lemma 4.2 ([8]). *Let* H *be a real Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Assume that* $f : C \times C \to \mathbb{R}$ *satisfies* (A1) – (A4). *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \ \forall y \in C \right\}.$$

Then, the following hold:

(1) T_r *is single-valued;*

(2) T_r *is a firmly nonexpansive mapping, i.e., for all* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ *is closed and convex.*

We call such T_r the resolvent of f for $r > 0$.

Lemma 4.3 ([21])**.** *Let* H *be a Hilbert space and let* C *be a nonempty, closed convex subset of* H. *Let* $f : C \times C \to \mathbb{R}$ *satisfy* $(A1)-(A4)$. *Let* A_f *be a set-valued mapping of* H *into itself defined by*

$$A_f x = \begin{cases} \{z \in H : f(x, y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \forall x \in C, \\ \emptyset, \quad \forall x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}0$ *and* A_f *is a maximal monotone operator with* $D(A_f) \subset C$. *Furthermore, for any* $x \in H$ *and* $r > 0$, *the resolvent* T_r *of* f *coincides with the resolvent of* A_f, *i.e.,*

$$T_r x = (I + rA_f)^{-1}x.$$

Using Lemma 4.3 and Theorem 3.4, we also obtain the following result with equilibrium problem in Hilbert spaces; see also [17, 20].

Theorem 4.4. *Let* H_1 *and* H_2 *be real Hilbert spaces. Let* $\eta \in [0, 1)$ *and let* S *be an* η*-strict pseudo-contraction of* H_1 *into* H_1. *Define* $S_\lambda = (1 - \lambda)I + \lambda S$ *for some* λ *with* $0 < \lambda \leq 1 - \eta$. *Let* T *be a nonexpansive mapping of* H_2 *into* H_2. *Let* B *be a maximal monotone operator on* H_1 *and let* $J_{\lambda_n} = (I + \lambda_n B)^{-1}$ *be the resolvent of* B *for* $\lambda_n > 0$. *Let* C *be a nonempty, closed and convex subset of* H_1. *Let* $f : C \times C \to \mathbb{R}$ *satisfy the conditions (A1)-(A4) and let* T_{λ_n} *be the resolvent of* A_f *for* $\lambda_n > 0$ *in Lemma 4.3. Let* $k \in (0, 1)$ *and let* g *be a* k*-contraction of* H_1 *into itself. Let* V *be a* $\overline{\gamma}$*-strongly monotone and* L*-Lipschitzian continuous operator of* H_1 *into* H_1 *with* $\overline{\gamma} > 0$ *and* $L > 0$. *Take* $\mu, \gamma \in \mathbb{R}$ *as follows:*

$$0 < \mu < \frac{2\overline{\gamma}}{L^2}, \quad 0 < \gamma < \frac{\overline{\gamma} - \frac{L^2\mu}{2}}{k}.$$

Let $A : H_1 \to H_2$ *be a bounded linear operator such that* $\|A\| \neq 0$. *Suppose* $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap EP(f) \neq \emptyset$. *Assume that* $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset (0, \infty)$ *and* $\{r_n\} \subset (0, \infty)$ *are real sequences such that*

$$\alpha_n \to 0, \quad 0 < a \leq \lambda_n \leq b < \frac{1}{\|A\|^2} \quad \textit{and} \quad \liminf_{n\to\infty} r_n > 0$$

for some $a, b \in \mathbb{R}$. *Then, the following hold:*

(i) *For any* $n \in \mathbb{N}$, *define* $T_n : H_1 \to H_1$ *by*

$$T_n x = \alpha_n \gamma g(x) + (I - \alpha_n V)S_\lambda J_{\lambda_n}(I - \lambda_n A^*(I - T)A)T_{r_n} x, \quad \forall x \in H_1.$$

Then, T_n *has a unique fixed point* x_n *in* H_1 *and* $\{x_n\}$ *is bounded.*

(ii) *For any nonempty, closed and convex subset* C *of* H_1, $P_C(I - V + \gamma g)$ *has a unique fixed point* z_0 *in* C. *This point* $z_0 \in C$ *is also a unique solution of the variational inequality*

$$\langle (V - \gamma g)z_0, q - z_0\rangle \geq 0, \quad \forall q \in C.$$

In particular, $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap EP(f)$ *is a nonempty, closed and convex subset of* H_1 *and* $P_{F(S)\cap B^{-1}0\cap A^{-1}F(T)\cap EP(f)}(I - V + \gamma g)$ *has a unique fixed point* z_0 *in* $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap EP(f)$*. This point* z_0 *is also a unique solution of the variational inequality*

$$\langle (V - \gamma g)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap EP(f).$$

(iii) *The sequence* $\{x_n\}$ *in* H_1 *converges strongly to a unique element* z_0 *in* $F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap EP(f)$*, where*

$$\langle (V - \gamma g)z_0, q - z_0 \rangle \geq 0, \quad \forall q \in F(S) \cap B^{-1}0 \cap A^{-1}F(T) \cap EP(f).$$

Proof. Take $U = I - T$ in Theorem 3.4, where T is nonexpansive. Then, $U = I - T$ is $\frac{1}{2}$-inverse strongly monotone. For the bifunction $f : C \times C \to \mathbb{R}$, define A_f as in Lemma 4.3. Let T_{r_n} be the resolvent of A_f for $r_n > 0$. Thus, we obtain the desired result by Theorem 3.4. □

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