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INVARIANCE OF THE k-CAUCHY-FUETER EQUATIONS AND HARDY SPACE OVER THE QUATERNIONIC SIEGEL UPPER HALF-SPACE

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Abstract. The k-Cauchy-Fueter operators and k-regular functions on the quaternionic Siegel upper half-space are quaternionic counterparts of the Cauchy-Riemann operator and holomophic functions on the complex Siegel upper half-space in the theory of several complex valuables. 1-regular functions are the usual quaternionic regular functions. In this paper, we give the transformation formulae for the k-Cauchy-Fueter operators under automorphisms of the quaternionic Siegel upper half-space. We also discuss the Hardy space, and the Cauchy-Szegö kernel and their basic properties.

1. Introduction

The classical Hardy space $H^2(\mathbb{R}^2_+)$ consists of holomorphic functions on the upper half-plane $\mathbb{R}^2_+$ such that

$$\sup_{y>0}\int_{-\infty}^{\infty}|f(x+iy)|^2\mathrm{d}y<+\infty.$$

The set of all boundary values forms a closed subspace of $L^2(\mathbb{R})$ and the Cauchy-Szegö integral is the projection operator from $L^2(\mathbb{R})$ to this closed subspace. The Cauchy-Szegö integral is written as a convolution with the Cauchy-Szegö kernel, which is the reproducing kernel for the functions in $H^2(\mathbb{R}^2_+)$. This construction was generalized to several complex variables (cf. [4]). In [2] it was generalized to quaternionic regular functions on the quaternionic Siegel upper half-space, and the Cauchy-Szegö kernel was given explicitly for any dimension n. In the present paper, we give an analogue construction for k-regular functions and give some basic properties of Cauchy-Szegö kernel. 1-regular functions are usual quaternionic regular functions.

The *quaternionic Siegel upper half-space* is

$$\mathcal{U}_n:=\left\{q=(q_1,q_2,\ldots,q_n)=(q_1,q')\in\mathbb{H}^n|\mathrm{Re}q_1>|q'|^2\right\}, \tag{1.1}$$

where we denote $q'=(q_2,\ldots,q_n)\in\mathbb{H}^{n-1}$. We consider the *$k$-Cauchy-Fueter operator* over the quaternionic Siegel upper half-space

$$\mathscr{D}:C^\infty(\mathcal{U}_n,\odot^k\mathbb{C}^2)\to C^\infty(\mathcal{U}_n,\odot^{k-1}\mathbb{C}^2\otimes\mathbb{C}^{2n}) \tag{1.2}$$

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$k = 1, 2, \dots,$ where $\odot^p\mathbb{C}^2$ is the p-th symmetric power of $\mathbb{C}^2$. We can identify $\odot^k\mathbb{C}^2 \cong \mathbb{C}^{k+1}$ by (2.8). A $\mathbb{C}^{k+1}$-valued distribution f on $\mathcal{U}_n$ is called k-regular if $\mathscr{D}f = 0$ in the sense of distributions.

The $(4n-1)$-dimensional *quaternionic Heisenberg group* is $\mathscr{H} := \mathbb{H}^{n-1} \oplus \mathrm{Im}\mathbb{H}$ equipped with the multiplication given by

$$(1.3)\qquad (s, q') \cdot (t, p') = \left(s + t + 2\sum_{j=1}^{n} \mathrm{Im}(\overline{q'_j} p'_j), q' + p' \right),$$

where $q', p' \in \mathbb{H}^{4n-4}$ and $s, t \in \mathrm{Im}\mathbb{H}$. We can identify $\mathscr{H}$ with the boundary $\partial\mathcal{U}_n$ of the Siegel upper half-space

$$(1.4)\qquad \partial\mathcal{U}_n := \left\{ q = (q_1, q_2, \dots, q_n) = (q_1, q') \in \mathbb{H}^n | \mathrm{Re} q_1 = |q'|^2 \right\},$$

using the projection

$$(1.5)\qquad \begin{gathered} \pi : \partial\mathcal{U}_n \longrightarrow \mathscr{H}, \\ (|q'|^2 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}, q') \longmapsto (x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}, q'). \end{gathered}$$

Let $\mathrm{d}\beta(\cdot)$ be the Lebesgue measure on $\partial\mathcal{U}_n$ obtained by pulling back by the projection π, the Haar measure on the group $\mathscr{H}$.

For any function $F : \partial\mathcal{U}_n \to \mathbb{C}^{k+1}$, we write F_ε for its vertical translate. We mean that the vertical direction is given by the positive direction of $\mathrm{Re} q_1 : F_\varepsilon(q) = F(q + \varepsilon\mathbf{e})$, where $\mathbf{e} = (1, 0, 0, \dots, 0)$. If $\varepsilon > 0$, then F_ε is defined in a neighborhood of $\partial\mathcal{U}_n$, in particular, on $\mathcal{U}_n$. The Hardy space $H^2(\mathcal{U}_n)$ consists of all k-regular functions F on $\partial\mathcal{U}_n$ for which

$$(1.6)\qquad \sup_{\varepsilon > 0} \int_{\partial\mathcal{U}_n} |F_\varepsilon(q)|^2 \mathrm{d}\beta(q) < +\infty.$$

The norm $\|F\|_{H^2(\mathcal{U}_n)}$ of F is then the square root of the left-hand side of (1.6).

This paper is organized as follows. In Section 2, we recall definitions of the k-Cauchy-Fueter operator, k-regular functions and the quaternionic Siegel upper Half-space. In Section 3, we give the transformation formulae for the k-Cauchy-Fueter operators under automorphisms of the quaternionic Siegel upper half-space. In Section 4, we discuss the boundary values of k-regular functions in the Siegel upper half-space $\mathcal{U}_n$ and invariance properties of the Hardy space $H^2(\mathcal{U}_n)$. In Section 5, we introduce the notion of the Cauchy-Szegö kernel and shows their basic properties. The explicit form of the Cauchy-Szegö kernel is under investigation in progress.

2. The quaternionic Siegel upper Half-space and k-Cauchy-Fueter operators

Proposition 2.1 (cf. Proposition 2.2 in [2]). *The Siegel upper half space $\mathcal{U}_n$ is invariant under the following transformations.*

(1) dilations*:*

$$\delta_r : (q_1, q') \longrightarrow (r^2 q_1, rq'), \ r > 0. \tag{2.1}$$

(2) left translations*:*

$$\tau_p : (q_1, q') \longrightarrow (p_1, p') \cdot (q_1, q'), \tag{2.2}$$

for $p = (p_1, p') \in \partial\mathcal{U}_n$.

(3) rotations*:*

$$R_{\mathbf{a}} : (q_1, q') \longrightarrow (q_1, \mathbf{a}q'), \ \text{for } \mathbf{a} \in \mathrm{Sp}(n-1), \tag{2.3}$$

where $\mathrm{Sp}(n-1) = \{U \in \mathrm{GL}(n-1, \mathbb{H}) | U\bar{U}^t = I_{n-1}\}$, and

$$R_\sigma : (q_1, q') \longrightarrow (\bar{\sigma} q_1 \sigma, q'\sigma), \ \text{for } \sigma \in \mathbb{H}, \ |\sigma| = 1. \tag{2.4}$$

To write down the k-Cauchy-Fueter operator in terms of complex variables, we will use the well known embedding of quaternionic algebra $\mathbb{H}$ into $\mathbb{C}^{2\times 2}$:

$$\tau : x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k} \mapsto \begin{pmatrix} x_1 + \mathbf{i}x_2 & -x_3 - \mathbf{i}x_4 \\ x_3 - \mathbf{i}x_4 & x_1 - \mathbf{i}x_2 \end{pmatrix}. \tag{2.5}$$

We need complex vector fields $Z_{AA'}$ on $\mathcal{U}_n$ as follows:

$$(Z_{AA'}) = \begin{pmatrix} Z_{00} & Z_{01} \\ Z_{10} & Z_{11} \\ \vdots & \vdots \\ Z_{(2n-2)0} & Z_{(2n-2)1} \\ Z_{(2n-1)0} & Z_{(2n-1)1} \end{pmatrix} := \begin{pmatrix} \partial_{x_1} + \mathbf{i}\partial_{x_2} & -\partial_{x_3} - \mathbf{i}\partial_{x_4} \\ \partial_{x_3} - \mathbf{i}\partial_{x_4} & \partial_{x_1} - \mathbf{i}\partial_{x_2} \\ \vdots & \vdots \\ \partial_{x_{4n-3}} + \mathbf{i}\partial_{x_{4n-2}} & -\partial_{x_{4n-1}} - \mathbf{i}\partial_{x_{4n}} \\ \partial_{x_{4n-1}} - \mathbf{i}\partial_{x_{4n}} & \partial_{x_{4n-3}} - \mathbf{i}\partial_{x_{4n-2}} \end{pmatrix}.$$

The matrices

$$(\varepsilon_{A'B'}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (\varepsilon^{A'B'}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

are used to raise or lower primed indices, e.g.

$$Z_A^{A'} = \sum_{B'=0',1'} Z_{AB'}\varepsilon^{B'A'}.$$

Here $(\varepsilon^{A'B'})$ is the inverse of $(\varepsilon_{A'B'})$. Then $Z_A^{0'} = Z_{A1'}, Z_A^{1'} = -Z_{A0'}$, i.e.

$$\left(Z_A^{A'}\right) = \begin{pmatrix} -\partial_{x_3} - \mathbf{i}\partial_{x_4} & -\partial_{x_1} - \mathbf{i}\partial_{x_2} \\ \partial_{x_1} - \mathbf{i}\partial_{x_2} & -\partial_{x_3} + \mathbf{i}\partial_{x_4} \\ \vdots & \vdots \\ -\partial_{x_{4n-1}} - \mathbf{i}\partial_{x_{4n}} & -\partial_{x_{4n-3}} - \mathbf{i}\partial_{x_{4n-2}} \\ \partial_{x_{4n-3}} - \mathbf{i}\partial_{x_{4n-2}} & -\partial_{x_{4n-1}} + \mathbf{i}\partial_{x_{4n}} \end{pmatrix}$$

where $A = 0, 1, \ldots, 2n-1$, $A' = 0', 1'$. An element of $\mathbb{C}^2$ is denoted by $(f_{A'})$ with $A' = 0', 1'$. The symmetric power $\odot^k\mathbb{C}^2$ is a subspace of $\otimes^k\mathbb{C}^2$, whose element is a 2^k-tuple $(f_{A'_1A'_2\ldots A'_k})$ with $A'_1, A'_2, \ldots, A'_k = 0', 1'$, such that $f_{A'_1A'_2\ldots A'_k} \in \mathbb{C}$ are invariant under permutations of subscripts, i.e.

$$f_{A'_1A'_2\ldots A'_k} = f_{A'_{\sigma(1)}A'_{\sigma(2)}\ldots A'_{\sigma(k)}}$$

for any permutation σ of k letters. The k-Cauchy-Fueter operator $\mathscr{D}$ in (1.2) over the quaternionic Siegel upper half-space is given by

$$(\mathscr{D}f)_{A_2'\ldots A_k'A} := \sum_{A_1'=0',1'} Z_A^{A_1'} f_{A_1'A_2'\ldots A_k'}. \tag{2.6}$$

We have isomorphisms

$$\odot^k\mathbb{C}^2 \cong \mathbb{C}^{k+1}, \qquad \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathbb{C}^{2k}, \tag{2.7}$$

by identifying $f \in \odot^k\mathbb{C}^2$ and $F \in \odot^{k-1}\mathbb{C}^2 \otimes \mathbb{C}^{2n}$ with

$$f := \begin{pmatrix} f_{0'0'0'\ldots 0'} \\ f_{1'0'0'\ldots 0'} \\ f_{1'1'0'\ldots 0'} \\ \vdots \\ f_{1'1'1'\ldots 1'} \end{pmatrix}, \quad F := \begin{pmatrix} F_{0'0'\ldots 0'0} \\ \vdots \\ F_{0'0'\ldots 0'(2n-1)} \\ \vdots \\ F_{1'1'\ldots 1'0} \\ \vdots \\ F_{1'1'\ldots 1'(2n-1)} \end{pmatrix} \tag{2.8}$$

respectively. Thus the k-Cauchy-Fueter operator is a $(2nk) \times (k+1)$-matrix valued differential operator:

(2.9)

$$\mathscr{D} = \begin{pmatrix} -\partial_{x_3}-\mathbf{i}\partial_{x_4} & -\partial_{x_1}-\mathbf{i}\partial_{x_2} & & & & \\ \partial_{x_1}-\mathbf{i}\partial_{x_2} & -\partial_{x_3}+\mathbf{i}\partial_{x_4} & & & & \\ \vdots & \vdots & & & & \\ -\partial_{x_{4n-1}}-\mathbf{i}\partial_{x_{4n}} & -\partial_{x_{4n-3}}-\mathbf{i}\partial_{x_{4n-2}} & & & & \\ \partial_{x_{4n-3}}-\mathbf{i}\partial_{x_{4n-2}} & -\partial_{x_{4n-1}}+\mathbf{i}\partial_{x_{4n}} & & & & \\ & -\partial_{x_3}-\mathbf{i}\partial_{x_4} & -\partial_{x_1}-\mathbf{i}\partial_{x_2} & & & \\ & \partial_{x_1}-\mathbf{i}\partial_{x_2} & -\partial_{x_3}+\mathbf{i}\partial_{x_4} & & & \\ & \vdots & \vdots & & & \\ & -\partial_{x_{4n-1}}-\mathbf{i}\partial_{x_{4n}} & -\partial_{x_{4n-3}}-\mathbf{i}\partial_{x_{4n-2}} & & & \\ & \partial_{x_{4n-3}}-\mathbf{i}\partial_{x_{4n-2}} & -\partial_{x_{4n-1}}+\mathbf{i}\partial_{x_{4n}} & & & \\ & & & \ddots & & \\ & & & & -\partial_{x_3}-\mathbf{i}\partial_{x_4} & -\partial_{x_1}-\mathbf{i}\partial_{x_2} \\ & & & & \partial_{x_1}-\mathbf{i}\partial_{x_2} & -\partial_{x_3}+\mathbf{i}\partial_{x_4} \\ & & & & \vdots & \vdots \\ & & & & -\partial_{x_{4n-1}}-\mathbf{i}\partial_{x_{4n}} & -\partial_{x_{4n-3}}-\mathbf{i}\partial_{x_{4n-2}} \\ & & & & \partial_{x_{4n-3}}-\mathbf{i}\partial_{x_{4n-2}} & -\partial_{x_{4n-1}}+\mathbf{i}\partial_{x_{4n}} \end{pmatrix}.$$

(see (3.2) in [7] for similar operators on the quaternionic space $\mathbb{H}^n$ and (2.11) in [3] on the quagernionic Heisenberg group for case $k = 2$). But here our operators are defined over the quaternionic Siegel upper half-space.

3. The transformation formula for the k-Cauchy-Fueter operators

We can extend the definition of embedding τ in (2.5) to a mapping from quaternionic $(l \times m)$-matrices to complex $(2l \times 2m)$-matrices. Let $A = (A_{jk})_{l\times m}$ be a quaternionic $(l \times m)$-matrix and write

$$A_{jk} = a_{jk}^1 + \mathbf{i}a_{jk}^2 + \mathbf{j}a_{jk}^3 + \mathbf{k}a_{jk}^4 \in \mathbb{H}.$$

We define $\tau(A)$ is the complex $(2l \times 2m)$-matrix $(\tau(A_{jk}))_{j=0,\ldots,l-1}^{k=0,\ldots,m-1}$, i.e.

$$\tau(A) = \begin{pmatrix} \tau(A_{00}) & \tau(A_{01}) & \ldots & \tau(A_{0(m-1)}) \\ \tau(A_{10}) & \tau(A_{11}) & \ldots & \tau(A_{1(m-1)}) \\ \vdots & \vdots & \ldots & \vdots \\ \tau(A_{(l-1)0}) & \tau(A_{(l-1)1}) & \ldots & \tau(A_{(l-1)(m-1)}) \end{pmatrix}, \tag{3.1}$$

where $\tau(A_{jk})$ is the complex (2×2)-matrix

$$\tau(A_{jk}) = \begin{pmatrix} a_{jk}^1 + \mathbf{i}a_{jk}^2 & -a_{jk}^3 - \mathbf{i}a_{jk}^4 \\ a_{jk}^3 - \mathbf{i}a_{jk}^4 & a_{jk}^1 - \mathbf{i}a_{jk}^2 \end{pmatrix}. \tag{3.2}$$

Though our τ is the conjugate of τ in [5], we still have the following proposition.

Proposition 3.1 (Proposition 2.1 in [5])**.** (1) *$\tau(AB) = \tau(A)\tau(B)$ for a quaternionic $(p\times m)$-matrix A and a quaternionic $(m\times l)$-matrix B. In particular, for $q' = Aq$, $q, q' \in \mathbb{H}^n$, $A \in \mathrm{GL}(n, \mathbb{H})$, we have*

$$\tau(q') = \tau(A)\tau(q) \tag{3.3}$$

as complex $(2n \times 2)$-matrix.

(2) *$\tau(\bar{A}^t) = \overline{\tau(A)}^t$ for a quaternionic $(n \times n)$-matrix A.*

Proposition 3.2 (Proposition 3.1.1 in [2])**.** *Let $f = f_0 + \boldsymbol{i}f_1 + \boldsymbol{j}f_2 + \boldsymbol{k}f_3 : D \to \mathbb{H}$ be C^1-smooth function, where D is a domain in $\mathbb{H}^n$.*

(1) *Define the pull-back function $\hat{f}$ of f under the mapping $q \to Q = \mathbf{a}q$ for $\mathbf{a} = (a_{jk}) \in GL(n, \mathbb{H})$ by $\hat{f} := f(\mathbf{a}q)$. Then we have*

$$\partial_{q_j}\hat{f}(q) = \sum_{l=1}^{n} \bar{a}_{lj}\bar{\partial}_{Q_l} f(Q), \tag{3.4}$$

where $\partial_{q_j} = \partial_{x_{4j}} + \boldsymbol{i}\partial_{x_{4j+1}} + \boldsymbol{j}\partial_{x_{4j+2}} + \boldsymbol{k}\partial_{x_{4j+3}}$.

(2) *Define the pull-back function $\tilde{f}$ of f under the mapping $q \to Q = q\sigma$ for $\sigma \in \mathbb{H}$ by $\tilde{f}(q) := f(q_1\sigma, \ldots, q_n\sigma)$. Then we have*

$$\partial_{q_l}\tilde{f}(q) = \bar{\partial}_{Q_l}(\bar{\sigma} f(Q)). \tag{3.5}$$

From Proposition 3.2, we can derive the transformation formula of the k-Cauchy-Fueter operators under automorphisms of the quaternionic Siegel upper half-space.

Proposition 3.3. *Let $f = \left(f_{A_1'A_2'\ldots A_k'}\right) \in \odot^k\mathbb{C}^2$ be C^1-smooth function.*

(1) *For $\mathbf{a} = (a_{jk}) \in GL(n, \mathbb{H})$, we have*

$$(R_{\boldsymbol{a}*}\mathscr{D}f)_{A_2'\ldots A_k'A} = \sum_{B=0}^{2n-1} \tau(\overline{\boldsymbol{a}}^t)_A^B \,(\mathscr{D}f)_{A_2'\ldots A_k'B}\,. \tag{3.6}$$

(2) *For fixed* $\sigma = \sigma_0 + \boldsymbol{i}\sigma_1 + \boldsymbol{j}\sigma_2 + \boldsymbol{k}\sigma_3 \in \mathbb{H}$ *with* $|\sigma| = 1$, *we have*

$$(R_{\sigma *}\mathscr{D}f)_{A'_2\ldots A'_k A} = \left(\mathscr{D}\tilde{f}\right)_{A'_2\ldots A'_k A}, \tag{3.7}$$

where

$$\tilde{f}_{A'_1 A'_2\ldots A'_k}(q) = \sum_{D'=0',1'} \left(\sigma^{\mathbb{C}}\right)_{A'_1}^{D'} f_{D' A'_2\ldots A'_k}, \tag{3.8}$$

with the 2×2 *complex matrix*

$$\sigma^{\mathbb{C}} := \begin{pmatrix} \sigma_0 + \boldsymbol{i}\sigma_1 & \sigma_2 - \boldsymbol{i}\sigma_3 \\ -\sigma_2 - \boldsymbol{i}\sigma_3 & \sigma_0 - \boldsymbol{i}\sigma_1 \end{pmatrix}. \tag{3.9}$$

Proof. (1) Recall that for a differentiable mapping $T : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$, the pushing forward of a vector field X is defined as

$$T_* X f(x) = X[f(T(x))]. \tag{3.10}$$

(3.4) implies

$$R_{\mathbf{a}*}\bar{\partial}_{q_j} f\Big|_{\mathbf{a}q} = \sum_{l=1}^{n} \bar{a}_{lj}\bar{\partial}_{q_l} f\Bigg|_{\mathbf{a}q}. \tag{3.11}$$

Because f is arbitrarily chosen, we have

$$R_{\mathbf{a}*}\bar{\partial}_{q_j} = \sum_{l=1}^{n} \bar{a}_{lj}\bar{\partial}_{q_l}.$$

By applying mapping τ in (3.1)-(3.2), we get

$$R_{\mathbf{a}*} Z_{AA'} = \sum_{B=0}^{2n-1} \tau(\bar{\mathbf{a}}^t)_A^B Z_{BA'}. \tag{3.12}$$

Thus

$$\begin{aligned}(R_{\mathbf{a}*}\mathscr{D}f)_{A'_2\ldots A'_k A} &= \sum_{A'_1,B'=0',1'} R_{\mathbf{a}*} Z_{AB'}\varepsilon^{B'A'_1} f_{A'_1\ldots A'_k} \\ &= \sum_{B=0}^{2n-1}\sum_{A'_1,B'=0',1'} \tau(\bar{\mathbf{a}}^t)_A^B Z_{BB'}\varepsilon^{B'A'_1} f_{A'_1\ldots A'_k} \\ &= \sum_{B=0}^{2n-1} \tau(\bar{\mathbf{a}}^t)_A^B \left(\mathscr{D}f\right)_{A'_2\ldots A'_k B}.\end{aligned}$$

(2) (3.5) implies that

$$R_{\sigma *}\bar{\partial}_{q_l} f\Big|_{q\sigma} = \left(\bar{\partial}_{q_l}\bar{\sigma}\right) f\Big|_{q\sigma}. \tag{3.13}$$

Because f is arbitrarily chosen, we have

$$R_{\sigma *}\bar{\partial}_{q_l} = \bar{\partial}_{q_l}\bar{\sigma}.$$

By applying mapping τ again, we get

$$(R_{\sigma *} Z_{AA'}) = (Z_{AA'})\tau(\bar{\sigma}),$$

i.e.

$$R_{\sigma *} Z_{AA'} = \sum_{C'=0',1'} Z_{AC'} \tau(\bar{\sigma})_{C'}^{A'}. \tag{3.14}$$

So we have

$$\begin{aligned} R_{\sigma *} Z_A^{A_1'} &= \sum_{B',C'=0',1'} Z_{AC'} \tau(\bar{\sigma})_{C'}^{B'} \varepsilon^{B'A_1'} \\ &= \sum_{B',C',D'=0',1'} Z_A^{D'} \varepsilon_{D'C'} \tau(\bar{\sigma})_{C'}^{B'} \varepsilon^{B'A_1'} \\ &= \sum_{D'=0',1'} Z_A^{D'} \left(\sigma^{\mathbb{C}}\right)_{D'}^{A_1'}. \end{aligned} \tag{3.15}$$

The last identity holds because

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \sigma_0 - \mathbf{i}\sigma_1 & \sigma_2 + \mathbf{i}\sigma_3 \\ -\sigma_2 + \mathbf{i}\sigma_3 & \sigma_0 + \mathbf{i}\sigma_1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \sigma^{\mathbb{C}},$$

by the definition of $\sigma^{\mathbb{C}}$ in (3.9). Thus

$$\begin{aligned} (R_{\sigma *} \mathscr{D} f(q))_{A_2' \dots A_k' A} = [\mathscr{D}(f(q\sigma))]_{A_2' \dots A_k' A} &= \sum_{A_1'=0',1'} Z_A^{A_1'} \left(f_{A_1' \dots A_k'}(q\sigma) \right) \\ &= R_{\sigma *} Z_A^{A_1'} f_{A_1' \dots A_k'} \Big|_{q\sigma} \\ &= \sum_{A_1',D'=0',1'} Z_A^{A_1'} \left(\sigma^{\mathbb{C}}\right)_{A_1'}^{D'} f_{D'A_2' \dots A_k'} \Bigg|_{q\sigma}. \end{aligned}$$

The proposition is proved. □

Corollary 3.4. *The space of all k-regular functions on $\mathcal{U}_n$ is invariant under the transformations defined in Proposition 2.1. Namely, if f is k-regular on the Siegel upper half-space $\mathcal{U}_n$, then the functions $f(\tau_p(q)), p \in \partial\mathcal{U}_n$; $f(R_{\mathbf{a}}(q))$, $\mathbf{a} \in \mathrm{Sp}(n-1)$; $\tilde{f}$ given by (3.8) and $f(\delta_r(q))$ are all k-regular on $\mathcal{U}_n$.*

Proof. The k-regularity of $f(R_{\mathbf{a}}(q))$ and $\tilde{f}$ follows directly from Proposition 3.3. As the 1-regular case in [2], the translation τ_p in (2.2) can be represented as a composition of the linear transformation given by the quaternionic matrix

$$\begin{pmatrix} 1 & 2\overline{p}' \\ 0 & I_{n-1} \end{pmatrix},$$

and the Euclidean translation $(q_1, q') \to (q_1 + p_1, q' + p')$. The first transformation preserves the k-regularity of a function by Proposition 3.3, while the later one obviously preserves the k-regularity of a function since the k-Cauchy-Fueter operators have constant coefficients.

It is obviously that $f(\delta_r(q))$ is k-regular on $\mathcal{U}_n$. The corollary is proved. □

4. The Hardy space

The identification of the quaternionic Heisenberg group and the boundary of the quaternionic Siegel upper half-space allows us to define the Lebesgue measure $\mathrm{d}\beta(\cdot)$ on $\partial\mathcal{U}_n$ by pulling back by the projection π defined in (1.5), the Haar measure on $\mathscr{H}$.

Set $\mathscr{D}^* = -\overline{\mathscr{D}}^t$ and $\Box = \mathscr{D}^*\mathscr{D}$. We have the following lemma.

Lemma 4.1. (cf. Lemma 3.3 in [6])

$$\Box = \mathscr{D}^*\mathscr{D} = \Delta \cdot \begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & \ddots & & \\ & & & 2 & \\ & & & & 1 \end{pmatrix}_{(k+1)\times(k+1)}, \tag{4.1}$$

where $\Delta := -\sum_{j=1}^{4n} \partial_{x_j}^2$ *is the Laplacian.*

By (4.1), it is straightforward to see that if f is k-regular, then

$$\Delta f = 0, \quad \text{where } f = \begin{pmatrix} f_{0'0'0'\dots0'} \\ f_{1'0'0'\dots0'} \\ \vdots \\ f_{1'1'1'\dots1'} \end{pmatrix}. \tag{4.2}$$

So each component of a k-regular function F is harmonic. A function $F \in H^2(\mathcal{U}_n)$ has boundary value F^b that belongs to $L^2(\partial\mathcal{U}_n)$ in the following sense.

Theorem 4.2. *Suppose that* $F \in H^2(\mathcal{U}_n)$. *Then*

1. *There exists a function* $F^b \in L^2(\partial\mathcal{U}_n)$ *such that* $F(q+\varepsilon\boldsymbol{e})|_{\partial\mathcal{U}_n} \to F^b(q)$ *as* $\varepsilon \to 0$ *in* $L^2(\partial\mathcal{U}_n)$ *norm.*
2. $\|F^b\|_{L^2(\partial\mathcal{U}_n)} = \|F\|_{H^2(\mathcal{U}_n)}$.
3. *The space of all boundary values forms a closed subspace of the space* $L^2(\partial\mathcal{U}_n)$.

Proof. The proof is just like the 1-regular case in [2]. We omit details. □

Proposition 4.3. *The Hardy space* $H^2(\mathcal{U}_n)$ *is a complex Hilbert space under the inner product* $\langle F, G\rangle = \langle F^b, G^b\rangle_{L^2(\partial\mathcal{U}_n)}$.

Proof. All the real part and imaginary part of $f_{A'_1\dots A'_k}$ for $A'_j = 0', 1', j = 1, \dots k$, are harmonic on $\mathcal{U}_n$. Thus for $q \in \mathcal{U}_n$,

$$f_{A'_1\dots A'_k}(q) = \frac{1}{|B|}\int_B f_{A'_1\dots A'_k}(p)\mathrm{d}V(p),$$

where B is a small ball centered at q and contained in $\mathcal{U}_n$, from which we see that

$$|f(q)|^2 \le \frac{1}{|B|}\int_B |f(p)|^2\mathrm{d}V(p). \tag{4.3}$$

There exist $a, b > 0$ such that $B \in \mathcal{U}_{n;a,b} := \{q \in \mathcal{U}_n | a < \mathrm{Re} q_1 - |q'|^2 < b\}$, and so

$$
\begin{aligned}
|f(q)|^2 &\le \frac{1}{|B|}\int_{\mathcal{U}_{n;a,b}} |f(x_1, \ldots, x_{4n})|^2 \mathrm{d}x_1 \ldots \mathrm{d}x_{4n} \\
&\le \frac{1}{|B|}\int_{(a,b)\times\mathbb{R}^{4n-1}} \left| f\left(x_1 + \sum_{j=5}^{4n} |x_j|^2, x_2, \ldots\right)\right|^2 \mathrm{d}x_1 \ldots \mathrm{d}x_{4n} \\
&\le \frac{1}{|B|}\int_a^b \mathrm{d}x_1 \int_{\partial\mathcal{U}_n} |f(p + x_1\mathbf{e})|^2 \mathrm{d}\beta(p) \le c\|f\|^2_{H^2(\mathcal{U}_n)},
\end{aligned}
\tag{4.4}
$$

where $c = (b-a)/|B|$ is a positive constant depending on q, and independent of the functions $f \in H^2(\mathcal{U}_n)$. We can prove the completeness just like the Cauchy-Fueter case in [2]. We omit the details. □

Corollary 4.4. *The Hardy space $H^2(\mathcal{U}_n)$ is invariant under the transformations of Proposition 2.1.*

Proof. Since the k-regularity property and the hypersurface $\partial\mathcal{U}_n + \varepsilon\mathbf{e}$ $(\varepsilon > 0)$ are invariant under these transformations by Corollary 3.4 and the measure $\mathrm{d}\beta$ either invariant or has a finite distortion, the proof follows. □

5. The Cauchy-Szegö Kernel

Theorem 5.1. *The Cauchy-Szegö kernel $S(q,p)$ is a unique $\odot^k\mathbb{C}^2 \otimes \left(\odot^k\mathbb{C}^2\right)^*$-valued function, defined on $\mathcal{U}_n \times \mathcal{U}_n$ satisfying the following conditions. By the identification $\odot^k\mathbb{C}^2 \cong \mathbb{C}^{k+1}$ in (2.8), $S(q,p)$ is a $(k+1)\times(k+1)$-matrix valued function.*

1. *For each $p \in \mathcal{U}_n$, the function $q \mapsto S(q,p)$ is regular for $p \in \mathcal{U}_n$, and belongs to $H^2(\mathcal{U}_n)$. This allows to define the boundary value $S^b(q,p)$ for each $p \in \mathcal{U}_n$, and for almost all $q \in \partial\mathcal{U}_n$.*
2. *The kernel S is symmetric: $S(q,p) = \overline{S(p,q)}^t$ for each $(q,p) \in \mathcal{U}_n \times \mathcal{U}_n$. The symmetry permits to extend the definition of $S(q,p)$ so that for each $q \in \mathcal{U}_n$, the function $S_b(q,p)$ is defined for almost every $p \in \mathcal{U}_n$ (here we use the subscript b to indicate the boundary value with respect to the second argument).*
3. *The kernel S satisfies the reproducing property in the following sense*

$$
F(q) = \int_{\partial\mathcal{U}_n} S_b(q,Q)F^b(Q)\, d\beta(Q), \quad q \in \mathcal{U}_n, \tag{5.1}
$$

where $F \in H^2(\mathcal{U}_n)$.

Proof. For fixed $q \in \mathcal{U}_n$ and fixed $j = 1, \ldots, k+1$, define a complex functional

$$
\begin{aligned}
l_q : H^2(\mathcal{U}_n) &\longrightarrow \mathbb{C}, \\
F &\longmapsto F_j(q),
\end{aligned}
\tag{5.2}
$$

where F_j is the j-th component of F. It is bounded by estimate (4.4). Apply Riesz's representation theorem to see that there exists an element, denoted by $K_j(\cdot,q) \in H^2(\mathcal{U}_n)$, such that $F_j(q) = \langle F, K_j(\cdot,q)\rangle = \langle F^b, K_j^b(\cdot,q)\rangle_{L^2(\partial\mathcal{U}_n)}$. Here

$K_j(\cdot,\cdot)$ is nontrivial and the boundary value $K^b(p,q)$ exists for almost all $p \in \partial\mathcal{U}_n$. We have

$$F_j(q) = \int_{\partial\mathcal{U}_n} \langle F^b(Q), K^b_j(Q,q)\rangle \mathrm{d}\beta(Q). \tag{5.3}$$

Let $K(q,p)$ be the $(k+1)\times(k+1)$-matrix, whose j-th column is $K_j(q,p)$. Then its (j,k)-th entry is

$$\begin{aligned} K_{jk}(q,p) =& \int_{\partial\mathcal{U}_n} \langle K^b_k(Q,p), K^b_j(Q,q)\rangle \mathrm{d}\beta(Q) \\ =& \overline{\int_{\partial\mathcal{U}_n} \langle K^b_j(Q,q), K^b_k(Q,p)\rangle \mathrm{d}\beta(Q)} = \overline{K_{kj}(p,q)}. \end{aligned}$$

So we have

$$K(q,p) = \overline{K(p,q)}^t. \tag{5.4}$$

Denote $S(q,p) := \overline{K(p,q)}^t$ for $(q,p) \in \mathcal{U}_n \times \mathcal{U}_n$. Then $S(q,p) = K(q,p)$ is regular in q, and $S(q,p) = \overline{K(p,q)}^t = \overline{S(p,q)}^t$. The function S has the boundary values as in Theorem 4.2. Moreover, we have

$$S_b(q,p) = \overline{S^b(p,q)}^t \tag{5.5}$$

for $q \in \mathcal{U}_n$, $p \in \partial\mathcal{U}_n$, which follows from the symmetry $S(q, p+\varepsilon\mathbf{e}) = \overline{S(p+\varepsilon\mathbf{e}, q)}^t$ by taking $\varepsilon \to 0+$.

To show the uniqueness, suppose that $\tilde{S}(\cdot,\cdot)$ is another function satisfying Theorem 5.1. By definition its j-th column $\tilde{S}_j(\cdot,q) \in H^2(\mathcal{U}_n)$ for any fixed $q \in \mathcal{U}_n$. Choose an arbitrary $p \in \mathcal{U}_n$ and apply the reproducing formula (5.1) of $S(\cdot,\cdot)$ and $\tilde{S}(\cdot,\cdot)$ to get

$$\begin{aligned} \tilde{S}(p,q) =& \int_{\partial\mathcal{U}_n} S_b(p,Q)\tilde{S}^b(Q,q)\mathrm{d}\beta(Q) = \int_{\partial\mathcal{U}_n} \overline{S^b(Q,p)}^t \overline{\tilde{S}_b(q,Q)}^t \mathrm{d}\beta(Q) \\ =& \overline{\int_{\partial\mathcal{U}_n} \tilde{S}_b(q,Q) S^b(Q,p)\mathrm{d}\beta(Q)}^t = \overline{S(q,p)}^t = S(p,q). \end{aligned} \tag{5.6}$$

by using (5.5) for $S(\cdot,\cdot)$ and $\tilde{S}(\cdot,\cdot)$. The theorem is proved. □

Since the Siegel upper half-space possesses some invariance properties, it is expected that the Cauchy-Szegö kernel also inherits them. We have the following proposition (see Proposition 5.1 in [2] for the Cauchy-Fueter case). In terms of multi-indices, we write the kernel S as $\left(S^{B'_1\ldots B'_k}_{A'_1\ldots A'_k}\right)$, i.e. (5.1) is written as

$$F_{A'_1\ldots A'_k}(q) = \int_{\partial\mathcal{U}_n} \sum_{B'_1,\ldots,B'_k=0',1'} S^{B'_1\ldots B'_k}_{A'_1\ldots A'_k}(q,Q) F^b_{B'_1\ldots B'_k}(Q)\mathrm{d}\beta(Q).$$

Proposition 5.2. *The Cauchy-Szegö kernel has following invariance properties.*

$$
\begin{aligned}
S(q,Q) =& S(\tau_p(q),\tau_p(Q)),\\
S(q,Q) =& S(R_{\boldsymbol{a}}(q),R_{\boldsymbol{a}}(Q)),\\
S(q,Q) =& S(\delta_r(q),\delta_r(Q))r^{4n+2}, \qquad (5.7)\\
S(q,Q) =& \left(\sum_{C',D'=0',1'}\left(\sigma^{\mathbb{C}}\right)^{C'}_{A_1'}(S_b)^{D'B_2'\ldots B_k'}_{C'A_2'\ldots A_k'}(R_\sigma(q),R_\sigma(Q))\left(\bar{\sigma}^{\mathbb{C}}\right)^{B_1'}_{D'}\right),
\end{aligned}
$$

for $q,Q\in\mathcal{U}_n$, *where* $p\in\partial\mathcal{U}_n$, $\boldsymbol{a}\in\mathrm{Sp}(n-1)$, $r>0$ *and* $|\sigma|=1$.

Proof. The proof of first three identities are just like the 1-regular case in [2]. We omit details. We only prove the last identity.

Since $\sum_{D'=0',1'}\left(\sigma^{\mathbb{C}}\right)^{D'}_{A_1'}f_{D'A_2'\ldots A_k'}(R_\sigma(q))$ is k-regular in q by Corollary 3.4, the function

$$\sum_{D'=0',1'}\left(\bar{\sigma}^{\mathbb{C}}\right)^{D'}_{A_1'}F_{D'A_2'\ldots A_k'}(R_{\sigma^{-1}}(q))$$

is also k-regular in q. As $R_\sigma:(q_1,q')\longrightarrow(\bar{\sigma}q_1\sigma,q'\sigma)$ is orthogonal map (cf. P. 1639 in [2]), it follows that for fixed $A_1'\ldots A_k'$,

$$
\begin{aligned}
&\sum_{D'=0',1'}\left(\bar{\sigma}^{\mathbb{C}}\right)^{D'}_{A_1'}F_{D'A_2'\ldots A_k'}(R_{\sigma^{-1}}(q))\\
=&\int_{\partial\mathcal{U}_n}\sum_{B_1',\ldots,B_k',D'=0',1'}(S_b)^{B_1'\ldots B_k'}_{A_1'\ldots A_k'}(q,Q)\left(\bar{\sigma}^{\mathbb{C}}\right)^{D'}_{B_1'}F^b{}_{D'B_2'\ldots B_k'}(R_{\sigma^{-1}}(Q))\mathrm{d}\beta(Q)\\
=&\int_{\partial\mathcal{U}_n}\sum_{B_1',\ldots,B_k',D'=0',1'}(S_b)^{B_1'\ldots B_k'}_{A_1'\ldots A_k'}(q,R_\sigma(Q))\left(\bar{\sigma}^{\mathbb{C}}\right)^{D'}_{B_1'}F^b{}_{D'B_2'\ldots B_k'}(Q)\mathrm{d}\beta(Q).
\end{aligned}
$$

Since $\mathrm{d}\beta$ is invariant under the orthogonal transformation R_σ. Substituting $R_{\sigma^{-1}}(q)\mapsto q$, we get

$$
\begin{aligned}
F_{A_1'\ldots A_k'}=&\int_{\partial\mathcal{U}_n}\sum_{B_1',\ldots,B_k',C',D'=0',1'}\left(\sigma^{\mathbb{C}}\right)^{C'}_{A_1'}(S_b)^{B_1'\ldots B_k'}_{C'A_2'\ldots A_k'}(R_\sigma(q),R_\sigma(Q))\left(\bar{\sigma}^{\mathbb{C}}\right)^{D'}_{B_1'}\\
&\cdot F^b{}_{D'B_2'\ldots B_k'}(Q)\mathrm{d}\beta(Q),
\end{aligned}
$$

by

$$\sigma^{\mathbb{C}}\bar{\sigma}^{\mathbb{C}}=\begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

The function

$$\left(\sum_{C'=0',1'}\left(\sigma^{\mathbb{C}}\right)^{C'}_{A_1'}(S_b)^{D'B_2'\ldots B_k'}_{C'A_2'\ldots A_k'}(R_\sigma(q),R_\sigma(Q))\right)$$

is also k-regular in q for fixed D', $B_2'\ldots B_k'$ by Corollary 3.4, belongs to $H^2(\mathcal{U}_n)$ and symmetric. The last identity in (5.7) follows by the uniqueness of the reproducing kernel $S(q,p)$ in Theorem 5.1. □

References

[1] D.-C. Chang and I. Markina, *Quaternion H-type group and differential operator* Δ_λ, Science in China (Ser. A) **51** (2008), 523–540.

[2] D.-C. Chang, I. Markina and W. Wang, *On the Cauchy-Szegö kernel for quaternion Siegel upper half-space,* Comp. Anal. Oper. Theory **7** (2013), 1623–1654.

[3] Y. Shi and W. Wang, *The Szegö kernel for k-CF functions on the quaternionic Heisenberg group, Appl. Anal.* **96** (2017), 2474–2492.

[4] E. M. Stein, *Boundary behavior of holomorphic functions of several complex variables,* Princeton mathematical notes, vol. 11. Princeton University Press, Princeton, 1993

[5] D.-R. Wan and W. Wang, *On the quaternionic Monge-Ampère operator, closed positive currents and Lelong-Jensen type formula on the quaternionic space,* Bull. Sci. Math. **141** (2017), 267–311.

[6] H.-Y. Wang and G.-B. Ren, *Bochner-Martinelli formula for k-Cauchy-Fueter operator,* J. Geom. Phys. **84** (2014), 43–54.

[7] W. Wang, *On the weighted* L^2 *estimate for the k-Cauchy-Fueter operator and the weighted k-Bergman kernel,* J. Math. Anal. Appl. **452(1)** (2017), 685–707.

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