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GENERALIZED APPROXIMATE SOLUTIONS FOR MULTIOBJECTIVE OPTIMIZATION WITH INFINITELY MANY CRITERIA

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Abstract. This paper is dedicated to generalized approximate solutions for multiobjective optimization problems (MP) with infinitely many criteria, i.e. (α,ϵ)-quasi-efficient (-properly efficient) solutions. For better understanding of the mentioned solution concept, an illustrative example is given. Relationships between the mentioned solution and generalized approximate solutions to weighted-sum minimization problem (P_λ) associated to (MP) are established. ϵ-Optimality conditions of Karush-Kuhn-Tucker (KKT) type for (P_λ) and then for (MP) are derived.

1. Introduction

Multiobjective optimization along with decision-making has been applied in such fields of science as engineering, economics, logistics and etc., where optimal decisions need to be taken in the presence of trade-offs. These disciplines aim to identify a single or all the best solutions within a set of feasible points. However, by a computation point of view, sometimes it is more meaningful to find not exact solutions but approximate ones. First, such solutions were introduced by Kutateladze [9] and independently defined for multiobjective programming by Loridan [11]. In 1986 White [17] analyzed six different concepts of ϵ-solution. Approximate solutions have got a keen interest by many researches; see, for example, [10, 13, 16, 18] and references therein. In 2008 Beldiman *et al.* [1] suggested a unitary concept of approximate quasi efficient solutions which is the main issue of this research.

One of important solution concepts is marked to be a properly efficient solution which was suggested by Kuhn and Tucker [8] in 1951. Later, a notion of proper efficiency was refined by Geoffrion [5] and very recently was extended to infinitely many criteria case by Engau [4]. Motivated by Engau's idea, we discuss about generalized approximate solutions for (MP) with infinitely many criteria with the help of a special linear space used in semi-infinite programming; see for example [6, 13].

However, it is meaningful not only to find approximate solutions but establish necessary and sufficient optimality conditions. Following Strodiot *et al.* [15] and Liu [10], we extend ϵ-optimality conditions of KKT type for generalized approximate solutions to multiobjective optimization problems with infinitely many criteria by using well-known weighted-sum scalarization method described, for example, in

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Chankong and Haimes [2]. Since one of the main tools for establishing ϵ-optimality conditions is ϵ-subdifferential concept, we would like to refer the reader to Hiriart-Urruty [7] and Dhara and Dutta [3] for better understanding.

This paper is organized as follows. In Section 2, problem statement and main notions are described. Section 3 describes relationships between quasi-(α, ϵ)-efficient (-properly efficient) solution of (MP) and (α, ϵ)-quasi-optimal solution of corresponding scalar problem. Section 4 deduces ϵ-optimality conditions to the multiobjective optimization problem, which is our main result. Finally, we provide conclusions in brief.

2. Preliminaries

Let us consider the following multiobjective optimization problem:

$$\begin{array}{lll}(\mathrm{MP}) & \text{Minimize} & f(x) := \big(f_i(x)\big)_{i\in I} \\ & \text{subject to} & g_t(x) \leqq 0, t \in T := \{1, ..., m\}, \\ & & x \in C,\end{array}$$

where $f_i(x) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i \in I$ (possibly countable infinite) and $g_t(x) : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $t \in T := \{1, ..., m\}$ are proper lower semicontinuous (l.s.c.) convex functions, and C is a closed convex subset of $\mathbb{R}^n$. The feasible set of (MP) is denoted by $F_M := \{x \in C \mid g_t(x) \leqq 0, \ t \in T\}$.

Engau [4] gave definition for efficient and properly efficient solutions for multiobjective optimization with infinitely many criteria. We extend it to approximate solutions for multiobjective constrained optimization problem.

Definition 2.1. Let $\epsilon_i \geqq 0$, $\alpha_i \geqq 0$, $i \in I$. A point $\bar{x} \in F_M$ is said to be

(1) an **ϵ-efficient solution** for (MP), if there is no other $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \epsilon_i, \ \textit{for all } i \in I,$$

with at least one strict inequality.

(2) an **α-quasi-efficient solution** for (MP), if there is no other $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \alpha_i \|x - \bar{x}\|, \ \textit{for all } i \in I,$$

with at least one strict inequality.

(3) an **α-quasi-properly efficient solution** for (MP), if

(a) $\bar{x}$ is α-quasi-efficient

(b) there exists a scalar $M_i > 0$ for each $i \in I$ such that

$$\frac{f_i(\bar{x}) - f_i(x) - \alpha_i \|x - \bar{x}\|}{f_j(x) - f_j(\bar{x}) + \alpha_j \|x - \bar{x}\|} \leqq M_i$$

for some j such that $f_j(\bar{x}) < f_j(x) + \alpha_j \|x - \bar{x}\|$, whenever $x \in F_M$ and $f_i(\bar{x}) > f_i(x) + \alpha_i \|x - \bar{x}\|$.

(4) an **α-quasi-improperly efficient solution** for (MP), if for every scalar $M_i > 0$ there is a point $x \in F_M$ such that $f_i(\bar{x}) > f_i(x) + \alpha_i \|x - \bar{x}\|$ and

$$\frac{f_i(\bar{x}) - f_i(x) - \alpha_i \|x - \bar{x}\|}{f_j(x) - f_j(\bar{x}) + \alpha_j \|x - \bar{x}\|} > M_i,$$

for all j such that $f_j(\bar{x}) < f_j(x) + \alpha_j \|x - \bar{x}\|$

According to Beldiman *et al.* [1] the notions of generalized solutions are as follows:

Definition 2.2. Let $\epsilon_i \geqq 0$, $\alpha_i \geqq 0$, $i \in I$. A point $\bar{x} \in F_M$ is said to be

(1) an **(α, ϵ)-quasi-efficient solution** for (MP), if there is no other $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \alpha_i \|x - \bar{x}\| - \epsilon_i, \text{ for all } i \in I,$$

with at least one strict inequality.

(2) an **(α, ϵ)-quasi-properly efficient solution** for (MP), if
(a) $\bar{x}$ is (α, ϵ)-quasi-efficient
(b) there exists a scalar $M_i > 0$, for each $i \in I$, such that

$$\frac{f_i(\bar{x}) - f_i(x) - \alpha_i \|x - \bar{x}\| - \epsilon_i}{f_j(x) - f_j(\bar{x}) + \alpha_j \|x - \bar{x}\| + \epsilon_j} \leqq M_i$$

for some j such that $f_j(\bar{x}) < f_j(x) + \alpha_j \|x - \bar{x}\| + \epsilon_j$, whenever $x \in F_M$ and $f_i(\bar{x}) > f_i(x) + \alpha_i \|x - \bar{x}\| + \epsilon_i$.

Remark 2.3. If $\epsilon_i = 0$, Def. 2.2 decribes an α-quasi-efficient (-properly efficient) solution. If $\alpha_i = 0$, the above definition reduces to an ϵ-efficient (ϵ-properly efficient) solution. If $\epsilon_i = \alpha_i = 0$ for all $i \in I$, we get the concept of efficient (properly efficient) solution for (MP).

Further on, we will consider $\epsilon > 0$ and $\alpha > 0$ case to deal with the concept of generalized solutions. However, all theorems can be reduced to corresponding approximate solutions by putting ϵ or α equal to zero and still hold true.

Later, we consider an infinite summation of the following form: $\sum_{i \in I} \lambda_i f_i(x)$. If this sum is a converging infinite series, then one can follow Engau [4] approach. In case if the above sum does not converge, we suggest another method.

The following linear space is used for semi-infinite programming [6].

$$\mathbb{R}^{(I)} := \{\lambda = (\lambda_i)_{i \in I} \mid \lambda_i = 0 \text{ for all } i \in I \text{ but only finitely many } \lambda_i \neq 0\}.$$

With $\lambda \in \mathbb{R}^{(I)}$, its supporting set, $I(\lambda) = \{i \in I \mid \lambda_i \neq 0\}$, is a finite subset of I. We define

$$\|\lambda\|_1 = \sum_{i \in I(\lambda)} |\lambda_i|.$$

The norm $\| \cdot \|_1$ was proposed in [14].

The nonnegative cone of $\mathbb{R}^{(I)}$ is denoted by:

$$\mathbb{R}^{(I)}_+ = \{\lambda = (\lambda_i)_{i \in I} \in \mathbb{R}^{(I)} \mid \lambda_i \geqq 0, i \in I\}.$$

With $\lambda \in \mathbb{R}^{(I)}$ and f_i, $i \in I$, we understand that

$$\sum_{i \in I} \lambda_i f_i = \begin{cases} \sum_{i \in I(\lambda)} \lambda_i f_i, & if \quad I(\lambda) \neq \emptyset, \\ 0, & if \quad I(\lambda) = \emptyset. \end{cases}$$

We associate to (MP) the following scalar minimization problem (P_λ)

$$(\mathrm{P}_\lambda) \quad \begin{array}{ll} \text{Minimize} & \lambda^T f(x), \\ \text{subject to} & g_t(x) \leqq 0, t \in T := \{1, ..., m\}, \\ & x \in C, \end{array}$$

where $\lambda \in V_> = \{\lambda \in \mathbb{R}^{(I)} \mid \lambda_i > 0 \quad for\ all\ i \in I(\lambda),\ \sum_{i\in I(\lambda)} \lambda_i = 1\}$ is a non-negative parameter vector and $\lambda^T f(x) = \sum_{i\in I(\lambda)} \lambda_i f_i$. The feasible set of (P_λ) is also denoted by $F_M := \{x \in C \mid g_t(x) \leqq 0,\ t \in T\}$.

Take into consideration that $\lambda \in \mathbb{R}^{(I)}$ and $\epsilon_i,\ i \in I$ (or $\alpha_i, i \in I$)

$$\sum_{i\in I} \lambda_i \epsilon_i = \begin{cases} \sum_{i\in I(\lambda)} \lambda_i \epsilon_i, & if \quad I(\lambda) \neq \emptyset, \\ 0, & if \quad I(\lambda) = \emptyset. \end{cases}$$

Similarly, the concept of unitary solutions for scalar minimization problem are as follows:

Let $\epsilon_i \geqq 0$ and $\alpha_i \geqq 0$, $i \in I$. A point $\bar{x} \in F_M$ is said to be a **$(\boldsymbol{\lambda^T\alpha}, \boldsymbol{\lambda^T\epsilon})$-quasi-optimal solution** for (P_λ), if

$$\sum_{i\in I(\lambda)} \lambda_i f_i(\bar{x}) \leqq \sum_{i\in I(\lambda)} \lambda_i f_i(x) - \sum_{i\in I(\lambda)} \lambda_i \alpha_i \|x - \bar{x}\| - \sum_{i\in I(\lambda)} \lambda_i \epsilon_i,\ for\ all\ x \in F_M.$$

Remark 2.4. If $\epsilon = 0$, the above notion reduces to a $\lambda^T\alpha$-quasi-optimal solution. If $\alpha = 0$, it describes a $\lambda^T\epsilon$-optimal solution. If $\epsilon = \alpha = 0$ we get the concept of optimal solution for (P_λ).

Now we give an example to illustrate the concept of generalized approximate solution.

Example 2.5. Let us consider (MP) as follows:

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & f(x) = (f_i)_{i\in I} = (|x|)_{i\in I}, \\ & \text{subject to} & g(x) = x^2 - 1 \leqq 0, \\ & & x \in \mathbb{R}. \end{array}$$

By using weighted-sum scalarization method we have

$$\begin{array}{lll} (\mathrm{P}_\lambda) & \text{Minimize} & \sum_{i\in I} \lambda_i f_i(x) \\ & \text{subject to} & g(x) = x^2 - 1 \leqq 0, \\ & & x \in \mathbb{R}. \end{array}$$

Let $\lambda_i = \frac{1}{2^i}$, then $\sum_{i\in I} \lambda_i |x| = \sum_1^\infty \frac{1}{2^i}|x| = |x|$. The feasible set is $[-1, 1]$. So, we have the approximate solution sets are as follows:

(i) ϵ-solution set is $[-\epsilon, \epsilon] \cap [-1, 1] = [\max\{-\epsilon, -1\}, \min\{\epsilon, 1\}]$,

(ii) α-quasi-solution set is $\begin{cases} \{0\}, & if \quad 0 < \alpha < 1, \\ [-1, 1], & if \quad \alpha \geq 1. \end{cases}$

(iii) (α, ϵ)-quasi-solution set is $\begin{cases} [\max\{-\epsilon, -1\}, \min\{\epsilon, 1\}], & if \quad 0 < \alpha < 1, \\ [-1, 1], & if \quad \alpha \geq 1. \end{cases}$

It is not difficult to notice that **$(\boldsymbol{\alpha}, \boldsymbol{\epsilon})$-quasi-solution set** is different from **regular $\boldsymbol{\epsilon}$-solution set**, which is an intersection of sets in (i) and (ii) in case $\alpha = \epsilon$.

To establish ϵ-optimality conditions of KKT-type we need some notions related to ϵ-subdifferential concept.

Let $h : \mathbb{R}^n \to \mathbb{R} \bigcup \{+\infty\}$ be a proper l.s.c convex function. **The $\boldsymbol{\epsilon}$-subdifferential** of h at $\bar{x} \in dom\ h$ is the set $\partial_\epsilon h(\bar{x})$ defined by

$$\partial_\epsilon h(\bar{x}) = \{x^* \in \mathbb{R}^n \mid h(y) \geqq h(\bar{x}) - \epsilon + \langle x^*, y - \bar{x}\rangle,\ \forall y \in dom\ h\}.$$

Consider a function $h : \mathbb{R}^n \to \mathbb{R}\bigcup\{+\infty\}$. **The conjugate of h**, $h^* : \mathbb{R}^n \to \mathbb{R}\bigcup\{+\infty\}$ is defined as

$$h^*(x^*) = \sup_{x^* \in \mathbb{R}^n} \{\langle x^*, x\rangle - h(x)\}.$$

The ϵ-subdifferential definition in terms of *conjugate function* h^* of h is as follows:

$$\partial_\epsilon h(\bar{x}) = \{x^* \in \mathbb{R}^n \mid h^*(x^*) + h(\bar{x}) \leqq \langle x^*, \bar{x}\rangle + \epsilon\}.$$

The indicator function δ_K of a subset $K \subset \mathbb{R}^n$ is the function defined as follows:

$$\delta_K = \begin{cases} 0, & if\ x \in K, \\ +\infty, & if\ x \in \mathbb{R}^n \backslash K. \end{cases}$$

Note that if K is convex, then δ_K is also convex.

Let C be a nonempty closed convex subset of $\mathbb{R}^n$. **The ϵ-normal set** of C at $\bar{x}$ is the set

$$N_\epsilon(C;\bar{x}) = \{x^* \in \mathbb{R}^n \mid \langle x^*, y - \bar{x}\rangle \leqq \epsilon,\ \forall y \in C\},$$

where $\epsilon > 0$ and $\bar{x} \in C$.

If $\epsilon = 0$, the ϵ-normal set reduces to the normal cone $N(C;\bar{x})$ to C at $\bar{x}$ that is

$$N(C;\bar{x}) = \{x^* \in \mathbb{R}^n \mid \langle x^*, y - \bar{x}\rangle \leqq 0,\ \forall y \in C\}.$$

It is easy to check that

$$\partial_\epsilon \delta_C(\bar{x}) = N_\epsilon(C;\bar{x}) = \{x^* \in \mathbb{R}^n \mid \delta_C^*(x^*) \leqq \langle x^*, \bar{x}\rangle + \epsilon\}.$$

For ϵ-subdifferential calculus the following propositions (see Theorem 2.115 and Theorem 2.117 in [3]) are very useful.

Proposition 2.6 (Sum Rule). *Consider two proper convex functions $\phi_i : \mathbb{R}^n \to \bar{\mathbb{R}}, i = 1, 2$ such that ri dom $\phi_1 \cap$ ri dom $\phi_2 \neq \emptyset$, where ri denotes the relative interior [see Def. 2.1.13 in [3]]). Then for $\epsilon > 0$,*

$$\partial_\epsilon(\phi_1 + \phi_2)(\bar{x}) = \bigcup_{\epsilon_1 \geqq 0, \epsilon_2 \geqq 0, \epsilon_1 + \epsilon_2 = \epsilon} \big(\partial_{\epsilon_1}\phi_1(\bar{x}) + \partial_{\epsilon_2}\phi_2(\bar{x})\big)$$

for every $\bar{x} \in$ dom $\phi_1 \cap$ dom ϕ_2.

Proposition 2.7 (Scalar Product Rule). *For a proper convex function $\phi : \mathbb{R}^n \to \bar{\mathbb{R}}$ and any $\epsilon \geq 0$,*

$$\partial_\epsilon(\lambda\phi)(\bar{x}) = \lambda\partial_{\epsilon/\lambda}\phi(\bar{x}),\ \forall \lambda > 0.$$

3. Relationship between solutions of (MP) and (P_λ)

In this section, we study the relationships of the solutions between (MP) and (P_λ).

Theorem 3.1. *Let $\epsilon_i > 0,\ \alpha_i > 0,\ i \in I$ and $\lambda \in V_>$ be a given weight parameter. If $\bar{x}$ is a $(\lambda^T\alpha, \lambda^T\epsilon)$-quasi-optimal solution for (P_λ), then $\bar{x}$ is an (α, ϵ)-quasi-efficient solution for (MP).*

Proof. Assume that $\bar{x}$ is not (α,ϵ)-quasi-efficient solution for (MP). Then there exists a $x \in F_M$ such that

$$f_i(x) \leqq f_i(\bar{x}) - \alpha_i\|x-\bar{x}\| - \epsilon_i, \quad i \in I,$$

with at least one strict inequality.
Multiplying by $\lambda \in V_>$, we have

$$\sum_{i\in I(\lambda)} \lambda_i f_i(x) < \sum_{i\in I(\lambda)} \lambda_i f_i(\bar{x}) - \sum_{i\in I(\lambda)} \lambda_i\alpha_i\|x-\bar{x}\| - \sum_{i\in I(\lambda)} \lambda_i\epsilon_i,$$

which contradicts that $\bar{x}$ is a $(\lambda^T\alpha, \lambda^T\epsilon)$-quasi-optimal solution for (P_λ). □

Remark 3.2. Liu [10] provided a counterexample to show that the converse of the above Theorem for finite case is not true. One can extend it to $(\lambda^T\alpha, \lambda^T\epsilon)$-quasi solution case by modifying slightly.

Theorem 3.3. *Let $\epsilon_i > 0$, $\alpha_i > 0$, $i \in I$ and $\lambda \in V_>$ be a given weight parameter. If $\bar{x}$ is a $(\lambda^T\alpha, \lambda^T\epsilon)$-quasi-optimal solution for (P_λ), then $\bar{x}$ is an (α,ϵ)-quasi-properly efficient solution for (MP).*

Proof. First, from Theorem 3.1 we have that $\bar{x}$ is an (α,ϵ)-quasi-efficient solution for (MP). To prove (α,ϵ)-quasi-properly efficiency, assume that $\bar{x}$ is an (α,ϵ)-quasi-improperly efficient point for problem (MP) and choose M_i as follows [Theorem 3.1. in [4]]:

$$M_i = \frac{\sum_{j\neq i}\lambda_j}{\lambda_i} = \frac{1-\lambda_i}{\lambda_i}, \tag{3.1}$$

which is strictly positive for each $i \in I$. Then there exists an index $i_0 \in I$ and $x \in F_M$ such that $f_{i_0}(\bar{x}) > f_{i_0}(x) + \alpha_{i_0}\|x-\bar{x}\| + \epsilon_{i_0}$ and

$$\frac{f_{i_0}(\bar{x}) - f_{i_0}(x) - \alpha_{i_0}\|x-\bar{x}\| - \epsilon_{i_0}}{f_j(x) - f_j(\bar{x}) + \alpha_j\|x-\bar{x}\| + \epsilon_j} > M_{i_0},$$

for all $j \in I$ with $f_j(\bar{x}) < f_j(x) + \alpha_j\|x-\bar{x}\| + \epsilon_j$. It follows that

$$f_{i_0}(\bar{x}) - f_{i_0}(x) - \alpha_{i_0}\|x-\bar{x}\| - \epsilon_{i_0} > M_{i_0}\big(f_j(x) - f_j(\bar{x}) + \alpha_j\|x-\bar{x}\| + \epsilon_j\big)$$

Because the left-hand side and M_{i_0} are positive, we can multiply each of these inequalities by its corresponding λ_j (with $j \in I(\lambda), j \neq i_0$) and add them together:

$$\sum_{\substack{j\in I(\lambda)\\ j\neq i_0}} \lambda_j\Big(f_{i_0}(\bar{x}) - f_{i_0}(x) - \alpha_{i_0}\|x-\bar{x}\| - \epsilon_{i_0}\Big) >$$

$$M_{i_0}\sum_{\substack{j\in I(\lambda)\\ j\neq i_0}} \Big(\lambda_j\big(f_j(x) - f_j(\bar{x}) + \alpha_j\|x-\bar{x}\| + \epsilon_j\big)\Big). \tag{3.2}$$

By definition of M_i with $i = i_0$ in (3.1), we have

$$\sum_{\substack{j\in I(\lambda)\\ j\neq i_0}} \frac{\lambda_j}{M_{i_0}} = \lambda_{i_0}$$

and rearranging of (3.2) yields

$$\sum_{i\in I(\lambda)} \lambda_i f_i(\bar{x}) - \sum_{i\in I(\lambda)} \lambda_i \alpha_i \|x-\bar{x}\| - \sum_{i\in I(\lambda)} \lambda_i \epsilon_i > \sum_{i\in I(\lambda)} \lambda_i f_i(x),$$

which contradicts that $\bar{x}$ is a $(\lambda^T\alpha, \lambda^T\epsilon)$-quasi-optimal solution for (P_λ). □

It is worth mentioning that unlike with the following theorem, Theorem 3.1 and Theorem 3.2 also hold without convexity of objective functions and feasible set. Using a variant of Gordan's Theorem of Alternative which was extented to infinite number of convex functions by Engau [Lemma 1 in [4]], we establish "if and only if" condition for an (α,ϵ)-quasi-properly efficient solution of (MP) to be a $(\lambda^T\alpha, \lambda^T\epsilon)$-quasi-optimal solution of (P_λ).

Theorem 3.4. *Let $\epsilon_i > 0$, $\alpha_i > 0$, $i \in I$. Then $\bar{x} \in F_M$ is an (α,ϵ)-quasi-properly efficient solution for (MP) if and only if there exists $\lambda \in V_>$ such that $\bar{x}$ is a $(\lambda^T\alpha, \lambda^T\epsilon)$-quasi-optimal solution for (P_λ).*

Proof. The statement "if $\bar{x}$ is a $(\lambda^T\alpha, \lambda^T\epsilon)$-quasi-optimal solution for (P_λ), then it is an (α,ϵ)-quasi-properly efficient solution for (MP)" follows from Theorem 3.3. To prove the necessary part one can slightly modify Engau's proof [4] by adding α-quasi part and make summation according to the supporting set $I(\lambda)$. □

4. Optimality conditions

Let us define the following set:

$$S_t = \{x \in \mathbb{R}^n \mid g_t(x) \leqq 0\},\ t \in T.$$

To establish ϵ-complementary slackness condition we need the following Lemma (see Proposition 2.2. in Strodiot et al. [15]):

Lemma 4.1. *Let $\epsilon \geqq 0$. Let $\bar{x} \in S = \{x \in \mathbb{R}^n \mid g(x) \leqq 0\}$ and the following constraint qualification of the Slater type holds true:*

$$(CQ) \qquad \exists x_0 \in C:\ g(x_0) < 0.$$

Then $x^* \in N_\epsilon(S;\bar{x})$, iff there exist $v \geqq 0$ and $\bar{\epsilon} \geqq 0$ such that

$$x^* \in \partial_{\bar{\epsilon}}(vg)(\bar{x}) \qquad \text{and} \qquad \bar{\epsilon} - \epsilon \leqq (vg)(\bar{x}) \leqq 0.$$

Up to now we are ready to establish the ϵ-optimality conditions for (P_λ).

Theorem 4.2. *Let $\bar{x} \in C$, $\epsilon_i > 0$ and $\alpha_i > 0$ for $i \in I$ and the constraint qualification (CQ) hold. $\bar{x}$ is a $(\lambda^T\alpha, \lambda^T\epsilon)$-quasi-optimal solution for (P_λ) if and only if there exist $\bar{\lambda} \in V_>, \bar{\epsilon}_0 \geqq 0, \bar{\epsilon}_t \geqq 0, \bar{\epsilon}_b \geqq 0, \bar{\epsilon}_q \geqq 0$ and $\bar{v}_t \geqq 0$, $t \in T$ such that*

$$0 \in \partial_{\bar{\beta}_0} \sum_{i\in I(\lambda)} \bar{\lambda}_i f_i(\bar{x}) + \partial_{\bar{\beta}_t} \sum_{t\in T} (\bar{v}_t g_t)(\bar{x}) + \sum_{i\in I(\lambda)} \bar{\lambda}_i \alpha_i B + N_{\bar{\beta}_q}(C;\bar{x}), \tag{4.1}$$

$$\bar{\beta}_0 + \sum_{t\in T} \bar{\beta}_t + \bar{\beta}_b + \bar{\beta}_q - \beta \leqq \sum_{t\in T} \bar{v}_t g_t(\bar{x}) \leqq 0, \tag{4.2}$$

where $\bar{\beta}_0 = \sum_{i\in I(\lambda)} \bar{\lambda}_i \bar{\epsilon}_{0_i}$, $\bar{\beta}_t = \sum_{i\in I(\lambda)} \bar{\lambda}_i \bar{\epsilon}_{t_i}$, $t \in T$, $\beta_b = \frac{\sum_{i\in I(\lambda)} \bar{\lambda}_i \bar{\epsilon}_{b_i}}{\sum_{i\in I(\lambda)} \bar{\lambda}_i \alpha_i}$, $\bar{\beta}_q = \bar{\epsilon}_q$, $\beta = \sum_{i\in I(\lambda)} \bar{\lambda}_i \epsilon_i$.

Proof. If $\bar{x}$ is a $(\lambda^T\alpha, \lambda^T\epsilon)$-quasi-optimal solution, then

$$\sum_{i\in I(\lambda)} \lambda_i f_i(\bar{x}) \leqq \sum_{i\in I(\lambda)} \lambda_i f_i(x) - \sum_{i\in I(\lambda)} \lambda_i\alpha_i\|x-\bar{x}\| - \sum_{i\in I(\lambda)} \lambda_i\epsilon_i,\ \forall x\in F_M.$$

We can rewrite it as follows:

$$\sum_{i\in I(\lambda)} \lambda_i f_i(\bar{x}) + \sum_{i\in I(\lambda)} \lambda_i\alpha_i\|\bar{x}-\bar{x}\| \leqq \sum_{i\in I(\lambda)} \lambda_i f_i(x) - \sum_{i\in I(\lambda)} \lambda_i\alpha_i\|x-\bar{x}\| - \sum_{i\in I(\lambda)} \lambda_i\epsilon_i,\ \forall x\in F_M.$$

Hence, $\bar{x}$ is a $\lambda^T\epsilon$-optimal solution of the following problem

$$\begin{array}{ll}\text{Minimize} & \sum_{i\in I(\lambda)} \lambda_i f_i(\cdot) + \sum_{i\in I(\lambda)} \lambda_i\alpha_i\|\cdot-\bar{x}\|.\\ x\in F_M & \end{array}$$

By using indicator functions we can obtain the following unconstrained problem which obviously has the same $\lambda^T\epsilon$-optimal solution:

$$\begin{array}{ll}\text{Minimize} & \sum_{i\in I(\lambda)} \lambda_i f(\cdot) + \sum_{t\in T}\delta_{S_t}(\cdot) + \sum_{i\in I(\lambda)} \lambda_i\alpha_i\|\cdot-\bar{x}\| + \delta_C(\cdot).\\ x\in\mathbb{R}^n & \end{array}$$

So, $\bar{x}$ is a $\lambda^T\epsilon$-optimal solution of the above problem if and only if

$$0\in\partial_{\sum_{i\in I(\lambda)}\lambda_i\epsilon_i}\left(\sum_{i\in I(\lambda)}\lambda_i f_i + \sum_{t\in T}\delta_{S_t} + \sum_{i\in I(\lambda)}\lambda_i\alpha_i\|\cdot-\bar{x}\| + \delta_C\right)(\bar{x}).$$

For convenience, set $\sum_{i\in I(\lambda)}\lambda_i\epsilon_i = \beta$. Since there is at least one point $x_0 \in$ *int* $S_t\bigcap$ *int* C and constraint qualification (CQ) holds, by using the Proposition 2.6 we have

$$\begin{aligned}&\partial_\beta\left(\sum_{i\in I(\lambda)}\lambda_i f_i + \sum_{t\in T}\delta_{S_t} + \sum_{i\in I(\lambda)}\lambda_i\alpha_i\|\cdot-\bar{x}\| + \delta_C\right)(\bar{x}) =\\ &\bigcup_{\substack{\beta_0\geqq0,\beta_t\geqq0,\beta_b\geqq0,\beta_q\geqq0,\\ \beta_0+\sum_{t\in T}\beta_t+\beta_b+\beta_q=\beta}}\Big\{\partial_{\beta_0}\sum_{i\in I(\lambda)}\lambda_i f_i(\bar{x}) + \partial_{\beta_t}\sum_{t\in T}\delta_{S_t}(\bar{x}) +\\ &\qquad\partial_{\beta_b}\sum_{i\in I(\lambda)}\lambda_i\alpha_i\|\cdot-\bar{x}\|(\bar{x}) + \partial_{\beta_q}\delta_C(\bar{x})\Big\}.\end{aligned}$$

By using Proposition 2.7 we can move α outside the ∂_{ϵ_b} and then it is not difficult to check that $\alpha\partial_{\epsilon_b/\alpha}\|\cdot-\bar{x}\|(\bar{x}) = \alpha B$, where B denotes a unit ball. There exist $\bar{\lambda}\in V_>, \bar{v}_t\geqq 0, \bar{\beta}_0\geqq 0, \bar{\beta}_t\geqq 0,\ t\in T, \bar{\beta}_b\geqq 0, \bar{\beta}_q\geqq 0$ such that

$$0\in\partial_{\bar{\beta}_0}\sum_{i\in I(\lambda)}\bar{\lambda}_i f_i(\bar{x}) + \partial_{\bar{\beta}_t}\sum_{t\in T}(\bar{v}_t g_t)(\bar{x}) + \sum_{i\in I(\lambda)}\bar{\lambda}_i\alpha_i B + N_{\bar{\beta}_q}(C;\bar{x}).$$

We can get condition

$$\bar{\beta}_0 + \sum_{t\in T}\bar{\beta}_t + \bar{\beta}_b + \bar{\beta}_q - \beta \leqq \sum_{t\in T}\bar{v}_t g_t(\bar{x}) \leqq 0$$

by applying Lemma 4.1 and summing over $t\in T$. □

By using Theorem 3.3 we can derive ϵ-optimality condition for (MP), which is our main result.

Theorem 4.3. *Let $\bar{x} \in C$, $\epsilon_i > 0$ and $\alpha_i > 0$ for $i \in I$ and the constraint qualification (CQ) hold. Then $\bar{x}$ is an (α, ϵ)-quasi-properly efficient solution for (MP) if and only if there exist $\bar{\lambda} \in V_{>}, \bar{\beta}_{0i} \geqq 0, \bar{\beta}_t \geqq 0, \bar{\beta}_b \geqq 0, \bar{\beta}_q \geqq 0$ and $\bar{v}_t \geqq 0$, $t \in T$ such that*

$$0 \in \sum_{i \in I(\lambda)} \partial_{\bar{\beta}_{0i}} (\bar{\lambda}_i f_i)(\bar{x}) + \partial_{\bar{\beta}_t} \sum_{t \in T} (\bar{v}_t g_t)(\bar{x}) + \sum_{i \in I(\lambda)} \bar{\lambda}_i \alpha_i B + N_{\bar{\beta}_q}(C; \bar{x}), \tag{4.3}$$

$$\sum_{i \in I(\lambda)} \bar{\beta}_{0i} + \sum_{t \in T} \bar{\beta}_t + \bar{\beta}_b + \bar{\beta}_q - \beta \leqq \sum_{t \in T} \bar{v}_t g_t(\bar{x}) \leqq 0. \tag{4.4}$$

Proof. If $\bar{x}$ is an (α, ϵ)-quasi-properly efficient solution for (MP) then, by Theorem 3.3, $\bar{x}$ is a $(\lambda^T \alpha, \lambda^T \epsilon)$-quasi-optimal solution for (P_λ) for some $\lambda \in V_{>}$. Thus, by the necessary condition of Theorem 4.2, there exist $\bar{\lambda} \in V_{>}, \bar{\epsilon}_0 \geqq 0, \bar{\epsilon}_t \geqq 0, t \in T, \bar{\epsilon}_b \geqq 0, \bar{\epsilon}_q \geqq 0$ and $\bar{v}_t \geqq 0$, $t \in T$ such that (4.1) and (4.2) hold. According to [[7], Theorem 2.1]

$$\partial_{\bar{\beta}_0} \left(\sum_{i \in I(\lambda)} \bar{\lambda}_i f_i \right)(\bar{x}) = \bigcup_{\substack{\sum_{i \in I(\lambda)} \bar{\beta}_{0i} = \bar{\beta}_0, \\ \bar{\beta}_{0i} \geqq 0, i \in I(\lambda)}} \left\{ \sum_{i \in I(\lambda)} \partial_{\bar{\beta}_{0i}} (\bar{\lambda}_i f_i)(\bar{x}) \right\}.$$

Thus, we have (4.3) and (4.4). Conversely, if there exist scalars $\bar{\lambda} \in V_{>}, \bar{\beta}_{0i} \geqq 0, \bar{\beta}_t \geqq 0, \bar{\beta}_b \geqq 0, \bar{\beta}_q \geqq 0$ and $\bar{v}_t \geqq 0, t \in T$ such that (4.3) and (4.4) hold, then from [[7], Theorem 2.1]

$$\begin{aligned} 0 &\in \sum_{i \in I(\lambda)} \partial_{\bar{\beta}_{0i}} (\bar{\lambda}_i f_i)(\bar{x}) + \partial_{\bar{\beta}_t} \sum_{t \in T} (\bar{v}_t g_t)(\bar{x}) + \sum_{i \in I(\lambda)} \bar{\lambda}_i \alpha_i B + N_{\bar{\beta}_q}(C; \bar{x}) \\ &\subset \bigcup_{\substack{\sum_{i \in I(\lambda)} \bar{\beta}_{0i} = \bar{\beta}_0, \\ \bar{\beta}_{0i} \geqq 0, i \in I(\lambda)}} \left\{ \textstyle\sum_{i \in I(\lambda)} \partial_{\bar{\beta}_{0i}} (\bar{\lambda}_i f_i)(\bar{x}) \right\} + \partial_{\bar{\beta}_t} \textstyle\sum_{t \in T} (\bar{v}_t g_t)(\bar{x}) \\ &\qquad + \sum_{i \in I(\lambda)} \bar{\lambda}_i \alpha_i B + N_{\bar{\beta}_q}(C; \bar{x}) \\ &= \sum_{i \in I(\lambda)} \partial_{\bar{\beta}_0} \bar{\lambda}_i f_i(\bar{x}) + \partial_{\bar{\beta}_t} \sum_{t \in T} (\bar{v}_t g_t)(\bar{x}) + \sum_{i \in I(\lambda)} \bar{\lambda}_i \alpha_i B + N_{\bar{\beta}_q}(C; \bar{x}). \end{aligned}$$

From the sufficient condition of Theorem 4.2, $\bar{x}$ is a $(\lambda^T \alpha, \lambda^T \epsilon)$-quasi-optimal solution for (P_λ). Thus, by Theorem 3.3, $\bar{x}$ is an (α, ϵ)-quasi-properly efficient solution for (MP). □

Conclusions

In this paper we discussed about generalized approximate solutions for multiobjective optimization problem with infinite number of objective functions. Relationships between (α, ϵ)-quasi-properly efficient solution for (MP) and $(\lambda^T \alpha, \lambda^T \epsilon)$-quasi-optimal solution for corresponding weighted-sum scalar problem (P_λ) were established. Using this equivalence, ϵ-optimality conditions for (MP) were derived under Slater's type constraint qualification.

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