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DOUBLE SUBMERSIONS AND HAMILTON FLOWS

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Abstract. We discuss solution curves of Hamilton systems in a framework of a double submersion. Typical examples of double submersions are those defined by right and left coset spaces of closed subgroups of a Lie group, where Hamiltonians are homogeneous functions defined as a "co-norm" function of a subbundle in the tangent bundle, a principal symbol of a (sub)-Laplacian or a Grushin type operator. As examples we show that some geodesics on one base space are constructed from the known geodesics on another base space of a double submersion.

1. Introduction

In this short note we explain a relation between Hamilton flows defined on the cotangent bundles of the total space and the base space of a submersion and a double submersion together with an interpretation to geodesics (cf. [2, 4, 7, 13]).

We are interested in Hamilton flows whose Hamiltonians are given by principal symbols of a Laplacian, a sub-Laplacian or a Grushin type operator. The last one is defined through a submersion (cf. [3, 15]). These Hamiltonians are also thought as *norm* functions of covectors with respect to dual inner products (we call *one half of it a co-norm function* of a subbundle) and such Hamilton flows are sometimes called bi-characteristic flows. There are many study on the bi-characteristic flows or geodesics defined by Grushin or Grushin-type operators and geodesics on the sub-Riemannian manifolds, like [8–11] and so on. In the theory, one of the main topics is to solve the Hamilton system in relation to the explicit construction of the heat kernel, where the Hamiltonians are co-norm functions including the cases that they might be *singular* metrics (here it means that the metric is defined only on an open dense subset or is defined on a subbundle in the tangent bundle). In particular, the paper [14] proves basic properties on the relation (or equivalence) of geodesics (= space components of solution curves of a Hamilton system by a co-norm function) and length minimizing curves in a sub-Riemannian setting under a somewhat strong condition on the bracket generating property.

In general, it is not possible to solve the Hamilton system explicitly and then one of the next problems will be to prove the complete integrability.

Our main concern in this note is to find their relations through a submersion and a double submersion, when two Hamilton systems are given on the total space and the base space of a submersion. Especially, the second case is our main concern and

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includes double fiberations given by left and right coset spaces of Lie groups, which are the most interesting cases.

In §2 we explain basic properties of Hamilton systems defined by a submersion-related Hamiltonians and treat the case of "co-norm functions" as Hamiltonians.

Then in §3 we show a correspondence of Hamilton curves with respect to two pairs of double submersion-related Hamiltonians arising from a double submersion. At the end of this section we summarize as a theorem of our method to construct geodesics with respect to "Grushin metric" from known geodesics of the Riemannian sense of one base manifold.

In §4 we show concrete examples, left and right coset spaces of the Heisenberg group and $SL(2,\mathbb{R})$ where in the first case one coset space is called the Grushin plane, and for the second case Grushin upper half plane with a metric different from the Poincaré metric.

In §5 as a final remark we shortly explain the case of an embedding-related Hamiltonians, an opposite situation to the submersion cases (see [5] for the method employed there).

2. Submersion and Hamilton flow

In this section we discuss some basic relations of Hamilton systems and their solution curves with respect to the Hamiltonians defined on the total space and the base space of a submersion under some relation. Here Hamiltonians are given as the co-norm functions on the horizontal subbundle and the submersion metric on the base manifold.

2.1. **Submersion-related pair of Hamiltonians.** Let $\varphi : M \to N$ be a surjective submersion between orientable manifolds M and N ($\dim M = m$, $\dim N = n$ and $d = m - n \geq 0$).

The dual of the differential $d\varphi : T(M) \to T(N)$ induces an injective bundle map $(d\varphi)^* : \varphi^*(T^*(N)) \to T^*(M)$ and it can be seen as a natural embedding of the manifold $\varphi^*(T^*(N))$ (= the induced bundle of $T^*(N)$ to M by the submersion $\varphi : M \to N$). So, we regard the induced bundle $\Sigma_N := \varphi^*(T^*(N))$ as a submanifold in $T^*(M)$ through the map $(d\varphi)^*$. Then we have the following intrinsic relations among their cotangent bundles, which can be expressed in the form of a commutative diagram:

$$
\begin{tikzcd}
T^*(N) \arrow[d, "\pi^N"'] & \varphi^*(T^*(N)) \arrow[l, "\chi"'] \arrow[r, hook, "(d\varphi)^*"] \arrow[d, "\pi^M"'] & T^*(M) \arrow[dl, "\pi^M"] \\
N & M \arrow[l, "\varphi"'] &
\end{tikzcd}
\tag{2.1}
$$

In the diagram the map π^M and π^N denote the natural projection maps to the base manifolds M and N respectively. We denote the natural projection map $\varphi^*(T^*(N)) \xrightarrow{\chi} T^*(N)$ by χ. The map χ is a submersion too.

Hereafter, we denote the Hamilton vector field corresponding to a function $f \in C^\infty(T^*(M))$ (also to functions in $C^\infty(T^*(N))$) by H_f.

Now we assume that there exist a smooth function Φ on $T^*(M)$ and a smooth function Ψ on $T^*(N)$ satisfying the condition that *the restriction of Φ to the submanifold $\varphi^*(T^*(N))$ coincides with the function $\chi^*(\Psi)$*, i.e.,

$$\Phi\Big|_{\varphi^*(T^*(N))} = \chi^*(\Psi). \tag{2.2}$$

We call such a pair of functions a *"submersion-related pair of Hamiltonians"* or in the case that the submersion is specified with a map φ, then we call *"φ-related pair of Hamiltonians."*

Then

Theorem 2.1 (see [4, Proposition 2.8]). *The Hamilton flow $\{\exp tH_\Phi\}$ of the Hamiltonian Φ leaves the submanifold $\varphi^*(T^*(N))$ invariant. Moreover the following diagram is commutative:*

$$\begin{tikzcd} \varphi^*(T^*(N)) \arrow[r, "\exp tH_\Phi"] \arrow[d, "\chi"'] & \varphi^*(T^*(N)) \arrow[d, "\chi"] \\ T^*(N) \arrow[r, "\exp tH_\Psi"'] & T^*(N). \end{tikzcd} \tag{2.3}$$

Proof. By the implicit function theorem, it is enough to give the proof on a local coordinate neighborhood of the form $W = U \times V \subset M$ with coordinates $U \times V \ni q \mapsto (x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_d) \in \mathbb{R}^n \times \mathbb{R}^d$ such that the submersion φ is given as the projection map $\varphi : (x, y) \mapsto x$. So U can be seen as a local coordinate neighborhood in N with the local coordinate $U \ni \varphi(q) \mapsto (x_1, \ldots, x_n)$. Then we also have local coordinates on $T^*(W) = \{\pi^M\}^{-1}(W)$ and $T^*(U) = \{\pi^N\}^{-1}(U)$ by the correspondences defined as

$$T^*(W) \ni \sum \xi_i dx_i + \eta_j dy_j \longleftrightarrow (x, y\,; \xi, \eta) \in U \times V \times \mathbb{R}^n \times \mathbb{R}^d,$$
$$T^*(U) \ni \sum \xi_i dx_i \longleftrightarrow (x\,; \xi) \in U \times \mathbb{R}^n.$$

Then we can express the map $(d\varphi)^*$ in this coordinates as

$$(d\varphi)^* : \varphi^*(T^*(U)) \cong U \times V \times \mathbb{R}^n \ni (x, y\,; \xi) \mapsto (x, y\,; \xi, 0) \in U \times V \times \mathbb{R}^n \times \mathbb{R}^d \cong T^*(W)$$

and the assumption (2.2) is expressed on the submanifold $T^*(W)$ in the form that

$$\Phi(x, y\,; \xi, 0) = \Psi(x\,; \xi).$$

Since the Hamilton vector fields H_Φ and H_Ψ are expressed as

$$H_\Phi = \sum \left(\frac{\partial \Phi}{\partial \xi_i} \frac{\partial}{\partial x_i} + \frac{\partial \Phi}{\partial \eta_j} \frac{\partial}{\partial y_j} \right) - \sum \left(\frac{\partial \Phi}{\partial x_i} \frac{\partial}{\partial \xi_i} + \frac{\partial \Phi}{\partial y_j} \frac{\partial}{\partial \eta_j} \right),$$
$$H_\Psi = \sum \left(\frac{\partial \Psi}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial \Psi}{\partial x_i} \frac{\partial}{\partial \xi_i} \right),$$

the Hamilton vector field H_Φ on $\varphi^*(T^*(U))$ is of the form

$$H_\Phi = \sum \left(\frac{\partial \Phi}{\partial \xi_i} \frac{\partial}{\partial x_i} + \frac{\partial \Phi}{\partial \eta_j} \frac{\partial}{\partial y_j} \right) - \sum \left(\frac{\partial \Phi}{\partial x_i} \frac{\partial}{\partial \xi_i} \right) = H_\Psi + \sum \left(\frac{\partial \Phi}{\partial \eta_j} \frac{\partial}{\partial y_j} \right).$$

Hence the solution of the equation

$$\frac{d\eta_j}{dt} = -\frac{\partial\Phi(x,y;\xi,0)}{\partial y_j} = -\frac{\partial\Psi(x;\xi)}{\partial y_j} \equiv 0$$

with the initial condition $\eta = 0$ is identically zero, which says that the solution curves of the Hamilton vector field H_Φ starting from points in $\varphi^*(T^*(U))$ stay in this submanifold. Moreover after we solve the equations

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial\Phi}{\partial\xi_i}(x,y\,;\xi,0) = \dfrac{\partial\Psi}{\partial\xi_i}(x\,;\xi), \\[2ex] \dfrac{d\xi_i}{dt} = -\dfrac{\partial\Phi}{\partial x_i}(x,y\,;\xi,0) = -\dfrac{\partial\Psi}{\partial x_i}(x\,;\xi), \end{cases} \tag{2.4}$$

with the initial point in $\varphi^*(T^*(U))$, we can solve the equation

$$\frac{dy_j}{dt} = \frac{\partial\Phi}{\partial\eta_j}(x(t),y\,;\xi(t),0) \tag{2.5}$$

independently. Here $x(t)$ and $\xi(t)$ are solutions of (2.4). Hence we have the commutative diagram (2.3). □

Remark 2.2. Even if we delete the zero sections of the bundles in the diagram (2.1), still holds the diagram:

$$\begin{tikzcd} T_0^*(N) \arrow[d, "\pi^N"'] & \varphi^*(T_0^*(N)) \arrow[l, "\chi"'] \arrow[r, hook, "(d\varphi)^*"] \arrow[d, "\pi^M"'] & T_0^*(M) \arrow[dl, "\pi^M"] \\ N & M \arrow[l, "\varphi"'] & \end{tikzcd} , \tag{2.6}$$

since the natural map $\chi : \varphi^*(T_0^*(N)) \to T_0^*(N)$ is isomorphic on each fiber and the map $(d\varphi)^*$ is injective. Hence the assertions in Theorem 2.1 hold in such a case too. In some cases the Hamiltonians might not be defined on the zero covectors.

2.2. **Co-norm functions as Hamiltonians and submersion.** Hamiltonians we are concerning are those defined as the co-norm functions of the given subbundles or Riemannian metrics, or in some case it will be understood as a principal symbol of a (pseudo-)differential operator. In this section we define one of such a submersion-related pair of Hamiltonians and remark the metric tensor in relation to a submersion.

First we remark the following property as Proposition 2.3 (cf. [14]).

Let $\imath_{\mathcal{H}} : \mathcal{H} \hookrightarrow T(M)$ be a subbundle on which an inner product $Q_{\mathcal{H}}$ is installed. Then there is a natural map defined by the equality

$$\mathcal{L} : \mathcal{H}^* \to \mathcal{H},\ \xi(Y) = Q_{\mathcal{H}}(\mathcal{L}(\xi), Y),$$

where $\xi \in \mathcal{H}_q^*$ and $Y \in \mathcal{H}_q$ and we can equip the dual bundle $\mathcal{H}^*$ with the inner product $Q_{\mathcal{H}^*}$ through this relation. Then by composing the dual map $\imath_{\mathcal{H}}^* : T^*(M) \to \mathcal{H}^*$ we have

$$g : T^*(M) \xrightarrow{\imath_{\mathcal{H}}^*} \mathcal{H}^* \xrightarrow{\mathcal{L}} \mathcal{H},\ g_q : T_q^*(M) \to \mathcal{H}_q$$

and define a positive bilinear form $Q_{T^*(M)}$ on $T^*(M)$ such that

$$Q_{T^*(M)}(\xi,\eta) := Q_{\mathcal{H}^*}\big(\imath_{\mathcal{H}}^*(\xi)\,,\,\imath_{\mathcal{H}}^*(\eta)\big).$$

Now, we define a co-norm function $\Phi_{\mathcal{H}^*} \in C^\infty(T^*(M))$ by

$$\Phi_{\mathcal{H}^*}(\xi) := \frac{1}{2} Q_{T^*(M)}(\xi, \xi) = \frac{1}{2}\langle g(\xi),\, \xi\rangle,$$

where $\langle Y,\, \xi\rangle$ denotes the natural pairing of $Y \in T_q(M)$ and $\xi \in T_q^*(M)$. Let $x = (x_1, \ldots, x_m) \leftrightarrow q \in M$ be a local coordinates around a point $q \in M$, and we put

$$g_q(dx_i) = \sum g^{i\,j}(x)\frac{\partial}{\partial x_j}.$$

Then

$$\Phi_{\mathcal{H}^*}\left(\sum \xi_i dx_i\right) = \frac{1}{2}\sum_i \xi_i \left\langle g_q\big(\sum_k \xi_k dx_k\big),\, dx_i \right\rangle = \frac{1}{2}\sum \xi_i \xi_k\, g^{k\,i}(x).$$

The Hamilton vector field $H_{\Phi_{\mathcal{H}^*}}$ is

$$\begin{aligned} H_{\Phi_{\mathcal{H}^*}} &= \sum \left(\frac{\partial \Phi_{\mathcal{H}^*}}{\partial \xi_\ell}\frac{\partial}{\partial x_\ell} - \frac{\partial \Phi_{\mathcal{H}^*}}{\partial x_\ell}\frac{\partial}{\partial \xi_\ell}\right) \\ &= \sum_{i,\,\ell} \xi_i\, g^{i\,\ell}(x)\frac{\partial}{\partial x_\ell} - \sum_{i,\,k,\,\ell} \xi_i \xi_k\, \frac{\partial g^{k\,i}(x)}{\partial x_\ell}\frac{\partial}{\partial \xi_\ell}. \end{aligned}$$

So

$$\sum \frac{dx_\ell}{dt}\frac{\partial}{\partial x_\ell} = \sum_{i,\,\ell} \xi_i g^{i\,\ell}(x)\,\frac{\partial}{\partial x_\ell} = g_q\big(\, \sum \xi_i dx_i \,\big) \in \mathcal{H}_q. \tag{2.7}$$

As before we denote by $\pi^M : T^*(M) \to M$ the projection map to the base manifold. Let $\{\gamma(t)\} \in T^*(M)$ be a solution curve of the Hamilton vector field $H_{\Phi_{\mathcal{H}^*}}$, then by the expression (2.7) we have

Proposition 2.3. *The curve $\{\pi^M(\gamma(t))\}$ on M is tangent to $\mathcal{H}$, i.e., the tangent vectors $\left\{\frac{d\pi^M(\gamma)(t)}{dt}\right\}$ of the curve belong to the subbundle $\mathcal{H}$.*

We call the curve $\{\pi^M(\gamma(t))\}$ the space component of the solution curve of the Hamilton vector field $H_{\Phi_{\mathcal{H}^*}}$.

Remark 2.4. Let $\{X_j\}$ be a local orthonormal basis of the subbundle $\mathcal{H}$. We put $D = -\sum_j {X_j}^2$ a locally defined second order differential operator and $\mathcal{L} \circ \imath_{\mathcal{H}}{}^*(\theta) = \sum a_j(\theta) X_j$. Then $a_j(\theta) = \langle X_j\,,\, \theta\rangle$ and

$$||\,\theta\,||^2 = \theta(\mathcal{L} \circ \imath^*_{\mathcal{H}}(\theta)) = \sum_j \langle X_j\,,\,\theta\rangle^2 = \sigma_D(q, \theta),\ \ \theta \in T_q^*(M),$$

where the first equality is by definition of the norm and the last quantity is the principal symbol of the differential operator D.

Now, let $\varphi : M \to N$ be a surjective submersion and we fix a decomposition

$$T(M) = \mathcal{V} \oplus \mathcal{H}, \tag{2.8}$$

where $\mathcal{V} = \mathrm{Ker}\,(d\varphi)$ is the vertical subbundle and $\mathcal{H}$ is a horizontal subbundle. Also we assume that the horizontal subbundle $\mathcal{H}$ is equipped with an inner product $Q_{\mathcal{H}}$

which can be descended to the base manifold N by the submersion φ, that is we assume that

$$\begin{array}{l} \text{If } \varphi(x) = \varphi(x'), \text{ the map } d\varphi_x{}^{-1} \circ d\varphi_{x'} : \mathcal{H}_{x'} \longrightarrow \mathcal{H}_x \text{ is isometric,} \\ \text{where } d\varphi_x{}^{-1} : T_{\varphi_x}(N) \to \mathcal{H}_x \text{ is the inverse map of the restriction} \\ d\varphi_{x|\mathcal{H}_x}. \end{array} \tag{2.9}$$

Then it will be clear that the manifold N can be equipped with a Riemannian metric by the obvious way. Such a Riemannian metric is called a *submersion metric.*

Under these assumptions we consider the commutative diagram:

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\imath_{\mathcal{H}}"] \arrow[dr] & T(M) \arrow[r, "d\varphi"] \arrow[d, "\pi_M"'] & \varphi^*(T(N)) \arrow[r] \arrow[dl] & T(N) \arrow[dl, "\pi_N"] \\
 & M \arrow[r, "\varphi"] & N & \\
\mathcal{H}^* \arrow[ur] & T^*(M) \arrow[l, "\imath_{\mathcal{H}}{}^*"] \arrow[u, "\pi^M"] & \varphi^*(T^*(N)) \arrow[l, "(d\varphi)^*"] \arrow[ul] \arrow[r] & T^*(N) \arrow[ul, "\pi^N"']
\end{tikzcd} \quad . \tag{2.10}$$

In the commutative diagram above, the bundles $\varphi^*(T(N))$ and $\varphi^*(T^*(N))$ denote the induced bundles of the tangent and the cotangent bundle of the manifold N to the total space M by the map φ, respectively. Also the map π_M is the natural projection maps to the base space M of the tangent bundle (also $\pi_N : T(N) \to N$). Other maps without notations are all natural maps.

The composition $d\varphi \circ \imath_{\mathcal{H}} : \mathcal{H} \to \varphi^*(T(N))$ is isomorphic (between vector bundles on M) and we transfer the inner product on $\mathcal{H}$ to $\varphi^*(T(N))$. The condition (2.9) allows us to descend the inner product on $\varphi^*(T(N))$ to $T(N)$ and we consider the manifold N is equipped with this Riemannian metric. We denote it by g_N. Then we consider the duals of these bundles, especially we denote the dual metric on $T^*(N)$ by $Q_{T^*(N)}$.

The transferred metric of the dual metric $Q_{\mathcal{H}^*}$ on $\mathcal{H}^*$ to the dual bundle $\varphi^*(T^*(N))$ through the dual isomorphism $\imath_{\mathcal{H}}{}^* \circ \imath_{\mathcal{H}}{}^* \circ (d\varphi)^*$ of $d\varphi \circ \imath_{\mathcal{H}}$ also can be descended to the cotangent bundle $T^*(N)$, which coincides with the dual metric of the submersion metric on $T(N)$. Hence,

Theorem 2.5. *The pair of functions*

$$\begin{aligned} \Phi_{\mathcal{H}^*}(\theta) &:= \frac{1}{2} Q_{\mathcal{H}^*}(\imath_{\mathcal{H}}{}^*(\theta), \imath_{\mathcal{H}}{}^*(\theta)), \ \theta \in T^*(M), \\ \Psi(\alpha) &:= \frac{1}{2} Q_{T^*(N)}(\alpha, \alpha), \ \alpha \in T^*(N) \end{aligned}$$

is a φ-related pair of Hamiltonians.

Corollary 2.6. *Let $w = w(\theta)$ be a smooth function on $T^*(M)$ such that it vanishes on $\varphi^*(T^*(N))$. Then $\Phi_{\mathcal{H}^*} + w$ and Ψ is also a pair of φ-related Hamiltonians.*

Especially if $g_{\mathcal{V}^*}$ is an inner product on the dual of the vertical subbundle $\mathcal{V}$ and we put $w_{\mathcal{V}^*}(\eta) := g_{\mathcal{V}^*}(\eta, \eta)$, $\eta \in \mathcal{V}^*$. Since the bundle map $\mathcal{V} \overset{\imath_{\mathcal{V}}}{\hookrightarrow} T(M) \overset{d\varphi}{\to} \varphi^*(T(N))$ is the zero map onto $\varphi^*(T(N))$, the function $(d\varphi \circ \imath_{\mathcal{V}})^*(w_{\mathcal{V}^*})$ vanishes identically *on the submanifold $\varphi^*(T^*(N))$. Hence the pair $\Phi_{\mathcal{H}^*} + (\imath_{\mathcal{V}})^*(w_{\mathcal{V}^*})$ and Ψ is φ-related Hamiltonians. Moreover in this case the solution curves on $\varphi^*(T^*(N))$ of the Hamilton vector field $H_{\Phi_{\mathcal{H}^*}+(\imath_{\mathcal{V}})^*(w_{\mathcal{V}^*})}$ coincide with those of the Hamilton vector field $H_{\Phi_{\mathcal{H}^*}}$.*

2.3. **Horizontal lifts.** Let $\varphi : M \to N$ be a surjective submersion between orientable manifolds M to N.

We assume a decomposition $T(M) = \mathcal{V} \oplus \mathcal{H}$ by the vertical bundle $\mathcal{V}$ and a horizontal subbundle $\mathcal{H}$ as in the last section.

Consider the diagram of vector bundles on M in which the low is exact.

$$\begin{tikzcd}
\{0\} \arrow[r] & \mathcal{V} \arrow[r, hook] & T(M) \arrow[r, "d\varphi"] & \varphi^*(T(N)) \arrow[r] & \{0\} \\
 & & \mathcal{H} \arrow[u, hook, "\imath_{\mathcal{H}}:inclusion"'] \arrow[ur] & &
\end{tikzcd} \tag{2.11}$$

The composition $d\varphi \circ \imath_{\mathcal{H}}$ is an isomorphism between the vector bundles $\mathcal{H}$ and $\varphi^*(T(N))$ on M.

The composition of the natural map $\Gamma(T(N)) := \mathcal{X}(N) \to \Gamma(\varphi^*(T(N)))$ from the space of vector fields on N and the inverse map of $d\varphi \circ \imath_{\mathcal{H}} : \mathcal{H} \to \varphi^*(T(N))$ defines a map $\lambda : \mathcal{X}(N) \to \Gamma(\mathcal{H})$, which gives the horizontal lift $\lambda(X)$ of a vector field $X \in \mathcal{X}(N)$ as a vector field on M which takes values in $\mathcal{H}$.

Now we assume that the manifold M is equipped with a Riemannian metric g_M such that the vertical subbundle $\mathcal{V}$ and the horizontal subbundle $\mathcal{H}$ are orthogonal. So, let $Q_{\mathcal{V}}$ and $Q_{\mathcal{H}}$ be the inner product on $\mathcal{V}$ and $\mathcal{H}$, the restrictions of the Riemannian metric, then g_M can be written as $g_M = Q_{\mathcal{H}} + Q_{\mathcal{V}}$.

Moreover we assume as before the condition (2.9)*, that is the inner product $Q_{\mathcal{H}}$ defines a submersion metric g_N on N.*

Let's denote the metric tensor

$$G_M = G_M(x, y) = \left\{ \begin{pmatrix} g_{i\,j} & s_{i\,\alpha} \\ s_{\alpha\,i} & v_{\alpha\,\beta} \end{pmatrix} \right\} := \left\{ \begin{pmatrix} g_N & S \\ {}^tS & V \end{pmatrix} \right\}$$

in terms of the local coordinates (x, y) (see the proof of Theorem 2.1 of the coordinates), where we put

$$\begin{aligned}
g_{ij} &= g_M\left(\left(\frac{\partial}{\partial x_i}\right), \left(\frac{\partial}{\partial x_j}\right)\right) = Q_{\mathcal{H}}\left(\left(\frac{\partial}{\partial x_i}\right), \left(\frac{\partial}{\partial x_j}\right)\right), \\
s_{i\alpha} &= g_M\left(\left(\frac{\partial}{\partial x_i}\right), \left(\frac{\partial}{\partial y_\alpha}\right)\right), \\
v_{\alpha\beta} &= g_M\left(\left(\frac{\partial}{\partial y_\alpha}\right), \left(\frac{\partial}{\partial y_\beta}\right)\right) = Q_{\mathcal{V}}\left(\left(\frac{\partial}{\partial y_\alpha}\right), \left(\frac{\partial}{\partial y_\beta}\right)\right).
\end{aligned}$$

For each $X \in T_{\varphi(q)}(N)$ its horizontal lift $\lambda(X)_q \in \mathcal{H}_q$ at the point $q \in M$ satisfies the condition that

$$\text{for } {}^\forall\gamma,\ g_M\left(\lambda(X), \left(\frac{\partial}{\partial y_\gamma}\right)\right) = 0,$$

especially if we put $L\left(\left(\frac{\partial}{\partial x_i}\right)\right) = \left(\frac{\partial}{\partial x_i}\right) + \sum b_{i\alpha}\left(\frac{\partial}{\partial y_\alpha}\right)$, then

$$g_M\left(\left(\frac{\partial}{\partial x_i}\right) + \sum b_{i\alpha}\left(\frac{\partial}{\partial y_\alpha}\right),\ \left(\frac{\partial}{\partial y_\gamma}\right)\right) = s_{i\gamma} + \sum b_{i\alpha} v_{\alpha\gamma} = 0.$$

Hence

$$B = -S \cdot V^{-1}.$$

The condition (2.9) and the property that the tensors $g_{ij}(x, y)$ for $1 \le i, j \le n$ do not depend on the variables y, are equivalent.

Now let's assume the subbundle $\mathcal{H}$ is bracket generating. Space components of solution curves of the Hamilton vector field $H_{\Phi_{\mathcal{H}^*}}$ are called geodesics. In the sub-Riemannian setting, it is not true that every geodesic is a locally length minimizing curve in the sense of the Carnot-Carathéodory metric (cf. [12, 14]).

However from the above arguments we know that space components (= geodesics) of solution curves of the Hamilton vector field $H_{\Phi_{\mathcal{H}^*}}$ included in $\varphi^*(T^*(N)) \overset{(d\varphi)^*}{\hookrightarrow} T^*(M)$ are locally length minimizing curves. Since if $\tilde{\gamma}$ is such a curve in M, then it is a horizontal lift of a space component γ in N, which is a locally length minimizing curve as a space component in a Riemannian manifold, so that $\tilde{\gamma}$ must be also a (locally)length minimizing curve.

3. Double submersion and bi-characteristic curves

Let

$$\begin{array}{ccc} & M & \\ {}^{\varphi_L}\swarrow & & \searrow^{\varphi_R} \\ N_L & & N_R \end{array} \tag{3.1}$$

be two surjective submersions between a total space M and two base manifolds N_R and N_L. All manifolds are assumed to be orientable and M is equipped with a Riemannian metric g_M.

We call such a pair of submersions a double submersion.

Based on the properties proved on the Hamiltonian flows in the previous section we discuss a relation of Hamiltonian curves of a φ_R-related pair of Hamiltonians (Φ, Ψ_R) of a submersion $\varphi_R : M \to N_R$, and a φ_L-related pair of Hamiltonians (Φ, Ψ_L) of a submersion $\varphi_L : M \to N_L$.

Let

$$\mathcal{V}_R = \mathrm{Ker}\,(d\varphi_R) \text{ and } \mathcal{V}_L = \mathrm{Ker}\,(d\varphi_L)$$

be the vertical subbundles of the submersions φ_R and φ_L, respectively.

By Proposition 2.1, solution curves of the Hamilton vector field H_Φ passing through

$$\begin{aligned} \Sigma :=& {\varphi_R}^*(T^*(N_R)) \cap {\varphi_L}^*(T^*(N_L)) = \Sigma_{N_R} \bigcap \Sigma_{N_L} \\ =& \big\{\theta \in T^*(M) \mid \theta = 0 \text{ on } \mathcal{V}_R + \mathcal{V}_L\big\} \end{aligned} \tag{3.2}$$

are descended to both of solution curves of the Hamilton vector fields H_{Ψ_R} and H_{Ψ_L}. In a certain case, it gives us a length minimizing curve on one base manifold from geodesics on another base manifold.

We restrict ourselves to a special case of those Hamiltonians defined as a co-norm function or a principal symbol of an operator. So, we consider the orthogonal decomposition

$$T(M) = \mathcal{V}_R \oplus \mathcal{H}_R$$

with respect to the metric g_M, where $\mathcal{H}_R$ is the horizontal subbundle. We assume the following 3 conditions:

(1) *The inner product $Q_{\mathcal{H}_R}$ on $\mathcal{H}_R$, the restriction of the Riemannian metric g_M, is descended to the base manifold N_R. We denote the resulting submersion metric on N_R by g_{N_R}.*

(2) *We assume the horizontal subbundle $\mathcal{H} := \mathcal{H}_R$ is bracket generating of 2 step, that is the naturally induced bundle map*

$$\rho := \rho^{\mathcal{H}\otimes\mathcal{H}}_{T(M)/\mathcal{H}} : \mathcal{H}\otimes\mathcal{H} \to T(M)/\mathcal{H}$$

from the the bracket operation

$$\Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}) \ni (X\,,\,Y) \mapsto [X\,,\,Y] \in \Gamma(T(M)) := \mathcal{X}(M)$$

is surjective.

The tensor product $\mathcal{H}\otimes\mathcal{H}$ is equipped with a natural inner product and by the assumption (2) that the map

$$\rho := \rho^{\mathcal{H}\otimes\mathcal{H}}_{T(M)/\mathcal{H}} : \mathcal{H}\otimes\mathcal{H} \to T(M)/\mathcal{H}$$

is surjective, so that we can install an inner product on the quotient bundle $T(M)/\mathcal{H}$ by assuming that it is isometric with the orthogonal complement of the kernel $\mathrm{Ker}(\rho)$ of the map ρ. Hence we transfer this inner product to the subbundle $\mathcal{V}_R$ through the isomorphism $T(M)/\mathcal{H} \cong \mathcal{V}_R$.

(3) *We assume that the inner product on $\mathcal{V}_R$ installed above coincides with the restriction of the Riemannian metric g_M on $\mathcal{V}_R$.*

Then under the assumptions (1), (2) and (3) the Popp's measure coincides with the Riemannian volume form (cf. [1, 2, 12]), so that we denote them by dg_M.

Let $\mathrm{grad}_{\mathcal{H}}(f)$ be the *gradient vector field* of a function $f \in C^\infty(M)$ along the subbundle $\mathcal{H} = \mathcal{H}_R$, which is defined by the relation

$$df(X) = Q_{\mathcal{H}}(\mathrm{grad}_{\mathcal{H}}(f)\,,\,X),\ \ X \in \mathcal{H}.$$

Also we denote by $\mathrm{grad}(f)$ the gradient vector field of the function f in the usual Riemannian sense, that is it is defined by the relation

$$df(X) = g_M(\mathrm{grad}(f),\,X),\ \ X \in T(M).$$

We denote by L_X the *Lie derivative* with respect to a vector field $X \in \mathcal{X}(M)$. Then the divergence $\mathrm{div}_{d\mu}(X) \in C^\infty(M)$ of a vector field $X \in \mathcal{X}(M)$ with respect to a smooth measure $d\mu$ on M is defined by the equation

$$L_X(d\mu) = \mathrm{div}_{d\mu}(X)d\mu = (d\circ i_X + i_X\circ d)(d\mu) = d\circ i_X(d\mu),$$

where i_X *is the interior product by the vector field* X. One way to define the Laplacian Δ is given in terms of grad and div_{dg_M} operations in the following way that

$$\Delta(f) = \mathrm{div}_{dg_M} \circ \mathrm{grad}(f),\ f \in C^\infty(M).$$

We can also define an operator Δ_{sub}, called a *sub-Laplacian* by a similar way as the Laplacian that

$$\Delta_{sub}(f) = \mathrm{div}_{dg_M} \circ \mathrm{grad}_{\mathcal{H}}(f),\ f \in C^\infty(M).$$

This is a second order differential operator defined by an intrinsic way in a sense that it is defined solely based on the assumptions (1), (2) and (3). It is sub-elliptic and not elliptic unless $\mathcal{H} = T(M)$.

The principal symbol $\sigma_{\Delta_{sub}} \in C^\infty(T^*(M))$ coincides with the co-norm function $2\Phi_{\mathcal{H}^*}$. Let Δ_{N_R} be the Laplacian on N_R with respect to the submersion metric. Then the principal symbol $\sigma_{\Delta_{N_R}}$ coincides with the co-norm function $2\Psi_{T^*(N_R)}$.

Proposition 3.1. *The pair of functions $\sigma_{\Delta_{sub}}$ and $\sigma_{\Delta_{N_R}}$ is φ_R-related Hamiltonians. Also the pair of functions σ_Δ and $\sigma_{\Delta_{N_R}}$ is φ_R-related too.*

Now in addition to the above three assumptions (1), (2) and (3), we assume

(4) *there exists a second order* (*differential or pseudo-differential*) *operator* $G : C_0^\infty(N_L) \to C^\infty(N_L)$ *such that*

$$\Delta_{sub} \circ {\varphi_L}^*(f) = {\varphi_L}^* \circ G(f),\ f \in C_0^\infty(N_L).$$

Then

Proposition 3.2. *The functions $\Phi_{\mathcal{H}^*} = \frac{1}{2}\sigma_{\Delta_{sub}}$ and $\Psi_L := \frac{1}{2}\sigma_G$ are a φ_L-related pair of Hamiltonians.*

This is a special case and we show a general property for the case of the differential operators.

Proposition 3.3. *Let $\varphi : M \to N$ be a submersion. If we are given two differential operator D_M on M and D_N on N of the same order, say second order and these satisfy the condition that $\varphi^* \circ D_N = D_M \circ \varphi^*$ on $C_0^\infty(N)$. Then the principal symbols σ_{D_M} and σ_{D_N} are a φ-related pair of Hamiltonians.*

Proof. Let (x, y) be a local coordinates around a point $q \in M$ appearing in the proof of Theorem 2.1, so that the submersion φ is expressed as the projection: $\varphi : (x, y) \mapsto x$. Then the operators D_M and D_N are expressed as

$$\begin{aligned} D_M = &\sum_{i,j} a_{ij}(x,y)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i(x,y)\frac{\partial}{\partial x_i} + c(x,y) \\ &+ \sum_{k,\ell} \tilde{a}_{k,\ell}(x,y)\frac{\partial^2}{\partial y_k \partial y_\ell} + \sum_{i,k} \tilde{b}_{ik}(x,y)\frac{\partial^2}{\partial x_i \partial y_k} + \sum_k \tilde{c}_k(x,y)\frac{\partial}{\partial y_k}, \end{aligned}$$

$$D_N = \sum_{i,j} \alpha_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i \beta_i(x)\frac{\partial}{\partial x_i} + \gamma(x).$$

Let f be a smooth function defined around a point $\varphi(q)$ having the properties that $f(\varphi(q)) = 0$ and $df_{\varphi(q)} = \sum \xi_i dx_i \neq 0$. Then by the assumption

$$D_M(\varphi^*(f^2))(q) = 2\sum_{i,j} a_{ij}(x,y)\xi_i\xi_j = \varphi^* \circ D_N(f^2)(q) = 2\sum_{i,j} \alpha_{ij}(x)\xi_i\xi_j. \tag{3.3}$$

Hence $a_{ij}(x,y) = \alpha_{ij}(x)$ and the equality can be understood as the coincidence

$$\sigma_{D_M}(q, d(\varphi^*(f))_q) = \sigma_{D_N}(\varphi(q), df_{\varphi(q)}),$$

that is the pair of principal symbols satisfy the condition (2.2). □

Under these assumptions (1), (2), (3) and (4), bi-characteristic curves of the sub-Laplacian on M passing through

$$\Sigma = {\varphi_R}^*(T^*(N_R)) \bigcap {\varphi_L}^*(T^*(N_L)) \neq \{0\} \tag{3.4}$$

correspond to the both of the bi-characteristic curves of the operator G and the Laplacian on N_R according to the diagram (2.3) (note that the embedding $\varphi_R^*(T^*(N_R)) \subset T^*(M)$ and $\varphi_L^*(T^*(N_L)) \subset T^*(M)$ are given by the map $(d\varphi_R)^*$ or $(d\varphi_L)^*$).

In the Riemannian case, (locally) length minimizing curves and the space components of bi-characteristic curves of the Laplacian coincide.

However in the sub-Riemannian case, even for our cases of the assumption (2) above, geodesics (= space components of the solution curves of the Hamilton vector field $H_{\Phi_{\mathcal{H}^*}}$) will not be always (local) length minimizing curves (cf. [14]).

One of our purpose is to obtain a special curve of bi-characteristic curves of a Grushin type operator on N_L from known geodesics on N_R.

All the examples in the next section satisfy an additional property to (2) that the principal symbol of the operator G defines a Riemannian metric at least on an open dense subset in N_L(see the expression (3.3) in the proof of Proposition 3.3). Then such a special curve, that is coming from geodesics on N_R gives a singular geodesics on N_L. These are given as examples in the next section.

Finally, under assumptions (1) $\sim$ (3) and an additional assumption on the operator G explained in (4), we sum up a method to obtain a length minimizing curve as a

Theorem 3.4. *We assume that a double submersion* (3.1) *satisfies the assumptions* (1) *to* (4)*. Let $\{c(t)\}$ be a geodesic in N_R, then there is a bi-characteristic curve $\{\tilde{c}(t)\}$ in $T^*(N_R)$ whose space component $\{\pi^N(\tilde{c}(t))\} = \{c(t)\}$ and also there are bi-characteristic curves $\{\tilde{a}(t)\}$ of the operator Δ_{sub} satisfying the relation given in the diagram* (2.3)*. Moreover the space components $\{\pi^M(\tilde{a}(t))\}$ are horizontal lifts of $\{c(t)\}$ according to an arbitrary given initial point in M. If this curve $\{\tilde{a}(t)\}$ is included in $\Sigma \neq \{0\}$ (see* (3.4)*), then the space component $\{\varphi_L(\tilde{a}(t))\}$ gives a local length minimizing curve on the non-singular part of the metric in N_L.*

4. Examples

In this section we deal with two examples for which we determine the bi-characteristic curves of the Grushin (type) operator based on the known geodesics with respect to the submersion metric following the procedure explained in Theorem 3.4.

Total space of our examples are

(1) the Heisenberg group H_{2n+1} and

(2) the group $SL(2,\mathbb{R})$.

The first example is rather elementary, but shows the typical procedure to express certain bi-characteristic curves and geodesics with respect to a singular metric by solving the Hamilton system explicitly. Here by singular Riemannian metric, we mean that it is defined only on an open dense subset. For this example it is sometimes called Grushin plane with such a metric.

We treated in the previous papers [4, 7] on the completely integrability of bi-characteristic flows of the several sub-Laplcians on $SL(2,\mathbb{R})$. In this note we can only give a special geodesic curve on the Grushin upper half plane based on Theorem 3.4.

4.1. **Heisenberg group case.** Let $H_{2n+1} \cong \mathbb{R}^{2n+1}$ be the $2n+1$ dimensional Heisenberg group with the group law given by

$$H_{2n+1} \times H_{2n+1} \ni (x,y,z) \times (x',y',z') \mapsto \left(x+x', y+y', z+z' + \frac{\langle x,y'\rangle - \langle y,x'\rangle}{2}\right) \in H_{2n+1},$$

where $x = (x_1,\ldots,x_n) \in \mathbb{R}^n,\ y = (y_1,\ldots,y_n) \in \mathbb{R}^n,\ z \in \mathbb{R}$ and $\langle x,y\rangle = \sum x_i y_i$ and so on.

Let X_i and Y_i $(i = 1,\ldots,n)$ be left invariant vector fields defined by

$$X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2}\frac{\partial}{\partial z},$$

$$Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2}\frac{\partial}{\partial z}.$$

The group H_{2n+1} is equipped with the left invariant Riemannian metric defined by assuming that the left invariant vector fields $\left\{X_i, Y_i, Z = \frac{\partial}{\partial z}\right\}$ are orthonormal at each point.

We consider a left invariant subbundle $\mathcal{H} \subset T(H_{2n+1})$ spanned by the vector fields $\{X_i,\ Y_i\}_{i=1}^n$. Since $[X_i,\ Y_i] = Z$, it is bracket generating and defines a sub-Riemannian structure. Also it gives a connection of the principal bundle $\varphi_R : H_{2n+1} \to H_{2n+1}/\mathbf{Z} \cong \mathbb{R}^{2n}$, where $\mathbf{Z} = \{tZ \mid t \in \mathbb{R}\}$ is the center.

Let $\mathbf{Y}$ be a subgroup generated by $\{Y_i\}_{i=1}^n$. We consider the double fiberation

$$\begin{array}{ccc} & H_{2n+1} & \\ \overset{\varphi_L}{\swarrow} & & \overset{\varphi_R}{\searrow} \\ \mathbb{R}^{n+1} \cong \mathbf{Y}\backslash H_{2n+1} & & H_{2n+1}/\mathbf{Z} \cong \mathbb{R}^{2n}, \end{array} \tag{4.1}$$

where $\varphi_L(x,y,z) = \left(x,\ z + \frac{\langle x,y\rangle}{2}\right) =: (u,v)$ and $\varphi_R(x,y,z) = (x,y)$.

The vector fields $d\varphi_L(X_i)$ and $d\varphi_L(Y_i)$ are given by

$$d\varphi_L(X_i) = \frac{\partial}{\partial u_i}, \qquad \varphi_L(Y_i) = u_i \frac{\partial}{\partial v}.$$

In this case the Grushin operator $\mathcal{G}$ is defined by

$$-\mathcal{G} =: \sum \frac{\partial^2}{\partial u_i^2} + \sum u_i^2 \cdot \frac{\partial^2}{\partial v^2}.$$

Let's denote the Euclidean Laplacian on $\mathbb{R}^{2n}$ by $-\Delta_{\mathbb{R}^{2n}} = \sum \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2}$ and the sub-Laplacian Δ_{sub} on H_{2n+1} by $-\Delta_{sub} = \sum X_i^2 + Y_i^2$. Then their principal symbols are given by

$$\begin{aligned}\sigma_{\Delta_{sub}}(x,y,z;\xi,\eta,\tau) &= \sum \left(\xi_i - \frac{y_i\tau}{2}\right)^2 + \left(\eta_i + \frac{x_i\tau}{2}\right)^2, \\ &\qquad (x,y,z;\xi,\eta,\tau) \in T^*(H_{2n+1}) \cong \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1}, \\ \sigma_{\mathcal{G}}(u,v;\alpha,\beta) &= \left(\sum \alpha_i^2 + |u|^2\beta^2\right), \\ &\qquad (u,v;\alpha,\beta) \in T^*(\mathbf{Y})\backslash H_{2n+1} \cong \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}.\end{aligned}$$

So we have a φ_L-related pair of Hamiltonians $\{\sigma_{\Delta_{sub}}, \sigma_{\mathcal{G}}\}$ and φ_R-related pair of Hamiltonians $\{\sigma_{\Delta_{sub}}, \sigma_{\Delta_{\mathbb{R}^{2n}}}\}$.

The horizontal lift of the line $(x(s), y(s)) = (\xi_0 s + x^0, \eta_0 s + y^0)$ in $H_{2n+1}/\mathbf{Z} \cong \mathbb{R}^{2n}$ with respect to the connection $\mathcal{H}$ is given as lines $\{\tilde{\gamma}(s) = (x(s), y(s), z(s))\}$ with

$$z(s) = \frac{1}{2}\left(\langle \eta, x^0\rangle - \langle \xi, y^0\rangle\right) s + z^0.$$

Then there is a bi-characteristic curve of the Grushin operator $\{g(s) = (u(s), v(s); \alpha(s), \beta(s))\}$ such that the curve $\{\varphi_L(\tilde{\gamma})\}$ coincides with the projection $\{\pi_{\mathbf{Y}\backslash H_{2+1}}(g(s))\}$, that is

$$\begin{cases} \dfrac{du}{ds} = \alpha, & \dfrac{d\alpha}{ds} = -u_i\beta^2, \\ \dfrac{dv}{ds} = |u|^2\beta, & \dfrac{d\beta}{ds} = 0. \end{cases}.$$

Here the map $\pi_{\mathbf{Y}\backslash H_{2n+1}} : T(\mathbf{Y}\backslash H_{2n+1}) \to \mathbf{Y}\backslash H_{2n+1}$ is the natural projection map to the base space. Hence we have the possible line in $H_{2n+1}/\mathbf{Z}$ which corresponds to a projection to the base manifold of the bi-characteristic curve of the Grushin operator. These are given by

$$u(s) = \alpha_0 s + u^0, v(s) = constant.$$

4.2. *SL*(2, $\mathbb{R}$) **case.** Let $M = SL(2,\mathbb{R})$ and $\mathbf{K} = \mathrm{SO}(2)$ a compact subgroup. We consider the double fiberation (4.2) with the left and right coset spaces by the subgroup $\mathbf{K}$ with the projection maps $\varphi_L : SL(2,\mathbb{R}) \to \mathbf{G}^+ = \mathbf{K}\backslash SL(2,\mathbb{R})$ and $\varphi_R : SL(2,\mathbb{R}) \to \mathbf{H}^+ = SL(2,\mathbb{R})/\mathbf{K}$:

$$\begin{tikzcd} & SL(2,\mathbb{R}) \arrow[dl, "\varphi_L"'] \arrow[dr, "\varphi_R"] & \\ \mathbf{K}\backslash SL(2,\mathbb{R}) = \mathbf{G}^+ & & \mathbf{H}^+ = SL(2,\mathbb{R})/\mathbf{K}. \end{tikzcd} \tag{4.2}$$

For the realizations of the maps φ_R and φ_L, we consider the decompositions of $g = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \in SL(2,\mathbb{R})$ as

$$\begin{aligned} g &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in \mathbf{ANK} \\ &= \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix} \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix} \in \mathbf{KAN}, \end{aligned}$$

where $\mathbf{A} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a > 0 \right\}$ and $\mathbf{N} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{R} \right\}$.

Then the map φ_R defined by

$$\varphi_R : g \longmapsto (a,b) \in \mathbf{H}^+ \cong \mathbb{R}^+ \times \mathbb{R}$$

and the map φ_L defined by

$$\varphi_L : g \longmapsto (p,q) \in \mathbf{G}^+ \cong \mathbb{R}^+ \times \mathbb{R}.$$

give the realizations of the quotient spaces.

The right coset space $SL(2,\mathbb{R})/\mathbf{K}$ is also given by the well-known action of $SL(2,\mathbb{R})$ on the upper half plane $\mathbf{H}^+ \cong \{u + \sqrt{-1}v \mid v > 0\}$:

$$g \cdot \sqrt{-1} = u + \sqrt{-1}v = \frac{x\sqrt{-1} + y}{w\sqrt{-1} + z} = \frac{xw + yz + \sqrt{-1}}{w^2 + z^2}.$$

Then the correspondence

$$g \longmapsto (u,v) = \left(\frac{xw + yz}{w^2 + z^2}, \frac{1}{w^2 + z^2} \right) = (a^2 b, a^2) \longleftrightarrow (a,\, b)$$

also gives a realization of the map φ_R.

We identify the tangent bundle $T(SL(2,\mathbb{R}))$ with the product bundle $SL(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ by left invariant vector fields and consider a left invariant subbundle $\mathcal{H}$ spanned by $X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$\mathcal{H} = SL(2,\mathbb{R}) \times [\{X, Y\}] \hookrightarrow T(SL(2,\mathbb{R})).$$

We denote the left invariant vector fields defined by X and Y by $\tilde{X}$ and $\tilde{Y}$ respectively and define a left invariant inner product on $\mathcal{H}$ by assuming that the vector fields $\tilde{X}$ and $\tilde{Y}$ are orthonormal at each point.

Then for $k(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$,

$$\begin{cases} Ad_{k(\theta)}(X) = (\cos\theta^2 - \sin\theta^2)\, X + 2\sin\theta\cos\theta \cdot Y, \\ Ad_{k(\theta)}(Y) = -2\cos\theta\sin\theta \cdot X + (\cos\theta^2 - \sin\theta^2)\, Y. \end{cases} \tag{4.3}$$

Hence the action by $\mathbf{K}$ leaves $\mathcal{H}$ invariant and the action is orthogonal so that the submersion φ_R defines the submersion metric g^+ on $\mathbf{H}^+$. Also this subbundle $\mathcal{H}$ defines a connection on the principal bundle $SL(2,\mathbb{R}) \overset{\varphi_R}{\to} \mathbf{H}^+$.

In terms of the coordinates (u,v) the metric tensor are given by

$$\begin{pmatrix} g^+(\frac{\partial}{\partial u},\frac{\partial}{\partial u}) & g^+(\frac{\partial}{\partial u},\frac{\partial}{\partial v}) \\ g^+(\frac{\partial}{\partial v},\frac{\partial}{\partial u}) & g^+(\frac{\partial}{\partial v},\frac{\partial}{\partial v}) \end{pmatrix} = \begin{pmatrix} \frac{1}{4v^2} & 0 \\ 0 & \frac{1}{4v^2} \end{pmatrix}$$

and is the well-known as the *Poincaré metric*. In the following we consider the metric as $g^+ := \frac{du^2+dv^2}{v^2}$ (constant is omitted).

Let $\Delta_{sub} = -\big(\tilde{X}^2+\tilde{Y}^2\big)$ be the sub-Laplacian on $SL(2,\mathbb{R})$ with respect to the sub-Riemannian structure $\mathcal{H}$, $\Delta_{\mathbf{H}^+}$ the Laplacian on $\mathbf{H}^+$ with respect to the Riemannian metric g^+.

Proposition 4.1. *The functions $\sigma_{\Delta_{sub}}$ and $\sigma_{\Delta_{\mathbf{H}^+}}$, the principal symbols of the operators Δ_{sub} and $\Delta_{\mathbf{H}^+}$, are a pair of φ_R-related Hamiltonians.*

The left invariant vector fields $\tilde{X}$ and $\tilde{Y}$ are descended to the left coset space $\mathbf{G}^+$ and they are linearly independent on the whole space $\mathbf{G}^+$, since the vectors $X, Y, Ad_g(K) \in \mathfrak{sl}(2,\mathbb{R})$ are linearly independent for any $g \in SL(2,\mathbb{R})$. So we can introduce a metric on $\mathbf{G}^+$ by assuming that the two vector fields $d\varphi_L(\tilde{X})$ and $d\varphi_L(\tilde{Y})$ are orthonormal everywhere. We call $\mathbf{G}^+$ equipped with this metric *Grushin upper half plane*(cf. [3]).

Then the operator

$$\mathcal{G} = -\left(d\varphi_L(\tilde{X})^2 + d\varphi_L(\tilde{Y})^2\right)$$

is a Grushin type operator and satisfies the condition in Proposition 3.3:

$$\varphi_L^* \circ \mathcal{G} = \Delta_{sub} \circ \varphi_L^*.$$

Hence,

Proposition 4.2. *The functions $\sigma_{\Delta_{sub}} \in C^\infty(T^*(SL(2,\mathbb{R})))$ and $\sigma_{\mathcal{G}} \in C^\infty(T^*(\mathbf{G}^+))$ are a φ_L-related pair of Hamiltonians.*

Using the relations

$$a^{-2} = w^2 + z^2 = \frac{1}{2}(p^{-2} - p^2 - p^2q^2)\cos 2\zeta + q\sin 2\zeta + \frac{1}{2}(p^{-2} + p^2 + p^2q^2),$$
$$b = xw + yz = \frac{1}{2}(p^2 + p^2q^2 - p^{-2})\sin 2\zeta + q\cos 2\zeta,$$

the intersection

$$\Sigma = \varphi_L^*(T^*(\mathbf{G}^+)) \bigcap \varphi_R^*(T^*(\mathbf{H}^+)) := \Sigma_L \bigcap \Sigma_R$$

is characterized as follows:

Lemma 4.3.

$$\begin{aligned} &\Sigma_L \cap \Sigma_R \ni \xi_1 da + \xi_2 db \\ &\Leftrightarrow \xi_1 \frac{\partial a}{\partial \zeta} + \xi_2 \frac{\partial b}{\partial \zeta} = 0 \\ &\Leftrightarrow -\frac{1}{2}\xi_1 (w^2+z^2)^{-\frac{3}{2}}\{(p^2 + p^2q^2 - p^{-2})\sin 2\zeta + 2q\cos 2\zeta\} \\ &\qquad + \xi_2\{(p^2 + p^2q^2 - p^{-2})\cos 2\zeta - 2q\sin 2\zeta\} = 0. \end{aligned}$$

The geodesics in $\mathbf{H}^+$ are well-known and they are described in the following way in terms of the coordinates (u, v):

Since all the geodesics are given as space components of solution curves of the Hamilton vector field $H_{\sigma_{\Delta_{\mathbf{H}^+}}}$, we consider a solution curve $\{\tilde{\gamma}\}$ of $H_{\sigma_{\Delta_{\mathbf{H}^+}}}$ with the initial condition $\varpi_0 = (u_0, v_0; \alpha_0, \beta_0) \in T^*(\mathbf{H}^+)$. Then the space component $\{\gamma\}$ of $\{\tilde{\gamma}\}$ is expressed as

$$(4.4)\qquad \begin{cases} \ell_{\varpi_0} = \{(u_0,\ v_0 e^{t\beta_0 v_0})\},\ \text{or} \\ C_{\varpi_0} = \{(u - \tilde{u}_0,\ v) \mid (u - \tilde{u}_0)^2 + v^2 = r_0^2,\ v > 0\}, \\ \qquad \text{where } \alpha_0 \neq 0 \text{ and } \tilde{u}_0 = u_0 + \frac{\beta_0}{\alpha_0} v_0,\ r_0^2 = \left(\frac{\alpha_0^2+\beta_0^2}{\alpha_0^2}\right) v_0^2. \end{cases}$$

First, we deal with the second case. The upper semi-circle is parameterized as

$$u = \tilde{u}_0 + r_0 \tanh(t + t_0),\ v(t) = r_0 \cosh(t + t_0),\ t \in \mathbb{R},$$

and the value t_0 is uniquely determined by the equations

$$\cosh t_0 = \frac{\alpha_0^2 + \beta_0^2}{\alpha_0},\ \text{and}\ \sinh t_0 = \frac{\beta_0}{\alpha_0}.$$

The horizontal lift $\lambda(C_{\varpi_0})$ of the upper semi-circle C_{ϖ_0} to $SL(2, \mathbb{R})$ along the subbundle $\mathcal{H}$ with the initial point

$$g_0 = \begin{pmatrix} \sqrt{v_0} & 0 \\ 0 & 1/\sqrt{v_0} \end{pmatrix} \begin{pmatrix} 1 & u_0/v_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{pmatrix} := \left(\sqrt{v_0},\ \frac{u_0}{v_0},\ \theta_0\right)$$

is given by

$$\left(\sqrt{\frac{r_0}{\cosh(t+t_0)}},\ \sinh(t+t_0) - \frac{\alpha_0 u_0 + \beta_0 v_0}{v_0\sqrt{\alpha_0^2 + \beta_0^2}} \cdot \cosh(t+t_0),\ \theta(t)\right).$$

The horizontality condition requires that $\dot{\theta}(t) = \frac{\dot{u}(t)}{2v(t)}$. Hence with a suitable $t_0 \in \mathbb{R}$

$$(4.5)\qquad \theta(t) = \int_{t_0}^{t} \frac{\dot{u}(s)}{2v(s)} ds = \int_{t_0}^{t} \frac{1}{2\cosh s} ds = \arctan e^t - \arctan e^{t_0}.$$

Moreover there is a solution curve $\widetilde{\lambda(C_{\varpi_0})}$ of the Hamilton vector field $H_{\sigma_{\Delta_{sub}}}$ on $T^*(SL(2,\mathbb{R}))$ such that $\chi(\widetilde{\lambda(C_{\varpi_0})}) = \tilde{\gamma}$. It is given by the curve:

$$\begin{aligned} \widetilde{\lambda(C_{\varpi_0})} =& \big\{(a(t), b(t), \theta(t); \xi_1(t), \xi_2(t), \xi_3(t))\big\} \\ =& \left(\sqrt{\frac{r_0}{\cosh(t+t_0)}},\ \sinh(t+t_0) - \frac{\alpha_0 u_0 + \beta_0 v_0}{v_0\sqrt{\alpha^2+\beta^2}} \cdot \cosh(t+t_0),\ \theta(t+t_0);\right. \\ & \left. -\frac{\sinh(t+t_0) + 2\tilde{u}_0 r_0^{-2}\cosh(t+t_0)}{4\sqrt{r_0\cosh(t+t_0)}},\ -\frac{1}{4\cosh(t+t_0)},\ 0\right). \end{aligned}$$

Among these curves, curves with the condition $\tilde{u}_0^2 + 1 = r_0^2$ are included in $\Sigma_L \cap \Sigma_R$.

It will be hard to describe all the geodesic curves on the Grushin upper half plane explicitly. Here by the above argument the projection $\{(p(t), q(t))\}$ of the curve

$\{(a(t), b(t), \theta(t))\}$ to $\mathbf{G}^+$ is a geodesic curve with respect to the Grushin metric and are given as (we put $t_0 = 0$ for simplicity)

$$\begin{aligned}
p(t)^2 &= a^2\cos^2\theta + 2a^2 b\cos\theta\sin\theta + a^2b^2\sin^2\theta + a^{-2}\sin^2\theta \\
&= (1+\sinh(t)^2)^{-1}\times \\
&\left[\frac{r_0}{\cosh(t)} + \frac{2r_0\sinh(t)}{\cosh(t)}\left(\sinh(t) + \frac{\tilde{u}_0}{r_0}\cosh(t)\right)\right. \\
&\qquad \left. + \left\{\frac{r_0}{\cosh(t)}\left(\sinh(t) + \frac{\tilde{u}_0}{r_0}\cosh(t)\right)^2 + \frac{\cosh(t)}{r_0}\right\}\sinh(t)^2\right] \\
2q(t)p(t)^2 &= (a^2b^2 - a^2 + a^{-2})\sin 2\theta + 2a^2 b\cos 2\theta \\
&= (1+\sinh(t)^2)^{-1}\times \\
&\left[\left\{\frac{r_0}{\cosh(t)}\left(\sinh(t) + \frac{\tilde{u}_0}{r_0}\cosh(t)\right)^2 - \frac{r_0}{\cosh(t)} + \frac{\cosh(t)}{r_0}\right\}(2\sinh(t))\right. \\
&\qquad \left. + \frac{2r_0(1-\sinh(t)^2)}{\cosh(t)}\left(\sinh(t) + \frac{\tilde{u}_0}{r_0}\cosh(t)\right)\right].
\end{aligned}$$

As for the line ℓ_{ϖ_0}, the line with $u_0 = 0$ is only the geodesic whose horizontal lift to $SL(2,\mathbb{R})$ is the space component of a curve of the Hamilton vector field $H_{\Phi_{\mathcal{H}^*}}$ and as a geodesic in $\mathbf{G}^+$ it is given by

$$(p, q) = (\sqrt{v_0}e^{\frac{t\beta_0}{2v_0}}, 0).$$

5. Embedding and final remarks

So far, our main interest was a relation between Hamilton flows in the framework of a submersion. In this last section we discuss a relation between Hamilton flows and embeddings.

Let $\varepsilon : M \to L$ be an embedding or we assume M is a closed submanifold in L, then we may consider that the natural projection $\varepsilon^*(T^*(L)) \to T^*(L)$, where $\varepsilon^*(T^*(L))$ is the induced bundle by the map ε, is also an embedding so that we regard it as a submanifold of $T^*(L)$. The bundle map $(d\varepsilon)^* : \varepsilon^*(T^*(L)) \to T^*(M)$ on M, the dual map of the differential $d\varepsilon$, is a submersion.

We assume that there exist functions $\Xi \in C^\infty(T^*(L))$ *and* $\Phi \in C^\infty(T^*(M))$ *such that*

$$\Xi_{|\varepsilon^*(T^*(L))} = \Phi \circ (d\varepsilon)^*.$$

Then

Proposition 5.1. *The Hamilton flow* $\{e^{tH_\Xi}\}$ *leaves the submanifold* $\varepsilon^*(T^*(L))$ *invariant and the following diagram is commutative:*

$$\begin{tikzcd}
\varepsilon^*(T^*(L)) \arrow[r, "\exp tH_\Xi"] \arrow[d, "(d\varepsilon)^*"'] & \varepsilon^*(T^*(L)) \arrow[d, "(d\varepsilon)^*"] \\
T^*(M) \arrow[r, "\exp tH_\Phi"'] & T^*(M).
\end{tikzcd} \tag{5.1}$$

The proof is similar to the case of Theorem 2.1(cf. [5]).

We also call such a pair of Hamiltonians an embedding-related pair of Hamiltonians, or in case the embedding is specified by a map ε, then ε-related pair of Hamiltonians.

Then we assume an extension of a submersion, that is let $\varepsilon : M \to L$ be an embedding, $\varphi : M \to N$ and $\pi : L \to N$ submersions satisfying the condition that $\pi \circ \varepsilon = \varphi$. The functions Ξ and Φ are as above and assume there is a function $\Psi \in C^\infty(T^*(N))$ such that the pair $\{\Phi, \Psi\}$ is φ-related Hamiltonians, then

Proposition 5.2. *the pair $\{\Xi,\, \Psi\}$ is π-related Hamiltonians.*

Let $\mathcal{V}_\pi$ be the vertical subbundle of the submersion $\pi : L \to N$ and $\mathcal{H}$ a horizontal subbundle in $T(L)$:

$$T(L) = \mathcal{V}_\pi \oplus \mathcal{H},$$

then the induced bundle $\varepsilon^*(\mathcal{H})$ is a horizontal subbundle of $T(M)$. If $\mathcal{H}$ is equipped with an inner product $Q_{\mathcal{H}}$ such that it defines a submersion metric on $T(N)$, then the induced inner product on $\varepsilon^*(\mathcal{H})$ defines the same submersion metric on $T(N)$. Hence the co-norm functions $\left\{\Xi_{\mathcal{H}^*},\, \Phi_{\varepsilon^*(\mathcal{H}^*)}\right\}$ are a ε-related pair of Hamiltonians, and $\left\{\Xi_{\mathcal{H}^*},\, \Psi_{T^*(N)}\right\}$ is a π-related pair of Hamiltonians.

Now we treat the case $SL(2,\mathbb{R})$ again as a simple example. We consider the inclusion $SL(2,\mathbb{R}) \subset GL^+(2,\mathbb{R})$ and the map

$$\pi_R : GL^+(2,\mathbb{R}) \ni \begin{pmatrix} x & y \\ w & z \end{pmatrix} \longmapsto (u,v) \in H^+,$$

where

$$u = \frac{xw+yz}{w^2+z^2}, \quad v = \frac{xz-yw}{w^2+z^2}. \tag{5.2}$$

This map is the natural extension of the map φ_R and gives the realization of the right coset space of $GL^+(2,\mathbb{R})$ by the subgroup $SO(2) \times \mathbb{R}^+$. The group $\mathbb{R}^+$ is identified with the diagonal matrix in $GL^+(2,\mathbb{R})$.

The left invariant vector fields $\tilde{X}$ and $\tilde{Y}$ are tangent to the each hypersurface defined by the equation $\det \begin{pmatrix} x & y \\ w & z \end{pmatrix} = xz - yw = constant$.

As before let $\mathcal{H}$ be the left invariant subbundle in $T(GL^+(2,\mathbb{R}))$ spanned by $\tilde{X}$ and $\tilde{Y}$. In this case the subbundle $\mathcal{H}$ is not bracket generating on $GL^+(2,\mathbb{R})$ and so does not define a sub-Riemannian structure, but we define an inner product on $\mathcal{H}$ as before. Then the right action by the subgroup $\mathbb{R}^+$ leaves these vector fields(in fact all left or right invariant vector fields), the submersion metric on H^+ can be defined with the same metric g^+ given by the submersion $\varphi_R : SL(2,\mathbb{R}) \to SL(2,\mathbb{R})/SO(2)$.

Let G^+ be the left coset space of $GL^+(2,\mathbb{R})$ by the subgroup $SO(2) \times \mathbb{R}^+$. Then the map $\pi_L : GL^+(2,\mathbb{R}) \to \mathbb{R} \times \mathbb{R}^+ \cong G^+$ given by

$$\pi_L \ni g = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \longmapsto (\mu,\, \nu),\ \mu = -\frac{xw+yz}{x^2+w^2},\ \nu = \frac{xz-yw}{x^2+w^2}, \tag{5.3}$$

is the realization of the left coset space G^+ and the natural extension of the map φ_L. We can install the metric on G^+ by assuming that the descended vector fields $d\pi_L(\tilde{X})$ and $d\pi_L(\tilde{Y})$ are orthonormal. The resulting metric is same with the metric defined in the $SL(2,\mathbb{R})$ case (=*Grushin metric*). We denote its co-norm function by Ψ_{G^+}.

$$
\begin{tikzcd}
 & GL^+(2,\mathbb{R}) \arrow[dl, "\pi_L"'] \arrow[dr, "\pi_R"] & \\
\mathbb{R}\times\mathbb{R}^+ \cong G^+ & SL(2,\mathbb{R}) \arrow[l, "\varphi_L"] \arrow[r, "\varphi_R"'] \arrow[u, "\text{inclusion}"] & H^+ \cong \mathbb{R}\times\mathbb{R}^+
\end{tikzcd}
\tag{5.4}
$$

Proposition 5.3. *The pair of the Hamiltonians* $\{\,\Xi_{\mathcal{H}^*},\ \Psi_{H^+}\,\}$ *is* π_R*-related and the pair of Hamiltonians* $\{\,\Xi_{\mathcal{H}^*}, \Psi_{G^+}\,\}$ *is* π_L*-related.*

Finally we remark that the Hamilton system $H_{\Xi_{\mathcal{H}^*}}$ can be expressed in a symmetric way on $T^*(GL^+(2,\mathbb{R})) \cong GL^+(2,\mathbb{R})\times\mathbb{R}^4$ by using the coordinates $(x,y,w,z) = g = \begin{pmatrix} x & y \\ w & z \end{pmatrix} \in GL^+(2,\mathbb{R})$ and it gives invariants of the system rather easily, but we do not enter the details.

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