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LOCAL STABILITY OF SOLUTIONS TO PARAMETRIC SEMILINEAR ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. This paper deals with the local stability of solutions for a class of parametric semilinear elliptic optimal control problems with mixed pointwise constraints. We show that if the strictly second-order sufficient conditions for unperturbed problem are valid and the objective function is locally Lipschitz continuous, then the solution map is locally upper Hölder continuous at the reference parameter.

1. Introduction

Let Ω be an open and bounded set in $\mathbb{R}^n$ with $n = 2, 3$ and the boundary $\partial\Omega$ of class C^2. We consider the following parametric semilinear elliptic optimal control problem. For each $w \in L^\infty(\Omega)$, determine a control function $u \in L^2(\Omega)$ and the corresponding state function $y \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ which

$$\text{minimize } J(y,u) = \int_\Omega (\phi(x,y,w) + \varphi(w)u + \psi(w)u^2)dx, \tag{1.1}$$

s.t.

$$-\Delta y + g(x,y,w) = u + w \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

$$a(x) \le h(x, y(x), w(x)) + \lambda u(x) \le b(x), \text{a.e. } x \in \Omega, \tag{1.3}$$

where the mappings $\phi, g, h : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions with $\phi(x,\cdot,\cdot), g(x,\cdot,\cdot)$ and $h(x,\cdot,\cdot)$ are of class C^2 for a.e. $x \in \Omega$; $\varphi, \psi : \mathbb{R} \to \mathbb{R}$ are of class C^2; $a, b \in L^2(\Omega)$ and $\lambda \neq 0$ is a constant.

Throughout the paper we denote by $P(w)$ the problem (1)-(3) corresponding to $w \in L^\infty(\Omega)$, by $\Sigma(w)$ the feasible set of $P(w)$ and by $\mathcal{S}(w)$ the solution set of $P(w)$. For a fixed parameter $\bar{w} \in W$, we call $P(\bar{w})$ the unperturbed problem and assume that $\bar{z} = (\bar{y}, \bar{u}) \in \Sigma(\bar{w})$. Our main concern is to study the solution stability of problems $P(w)$. Namely, we concentrate on investigating the behavior of $\mathcal{S}(w)$ when w varies around $\bar{w}$.

The solution stability of optimization problems as well as optimal control problems has some important applications in *parameter estimation problems* (see for instance [14]) and in numerical methods of finding optimal solutions. The stability

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of solutions guarantees that approximate solutions converge to the original solutions of perturbed problems (see for instance [21]).

The study of solution stability for optimal control problems governed by partial differential equations has been interested several authors recently. For papers which have a close connection to the present work, we refer the readers to [2,3,10,12,18,22] and the references given therein. Let us give briefly some comments on the considered problems and the obtained results of those papers. In [2] and [10], the authors considered problems where the objective functions are quadratic and convex, the state equations are linear and with mixed and pure state poinwise constraints. Under certain conditions, they showed that the solution maps are Lipschitz continuous w.r.t. parameters. In [22], Malanowski considered a family of parameter dependent elliptic optimal control problems with nonlinear boundary control and with pointwise control constraints. He showed that under standard coercivity conditions, the solutions to the problems are Bouligand differentiable functions of the parameter. Particularly, in [12], Hinze and Mayer studied a family of parametric optimal control problems where the objective function likes (1.1), the state equation is a semilinear elliptic equation with Neumann boundary condition and with pure state and control poinwise constraints. This class of perturbed problems comes from the finite element discretization. Hinze and Mayer proved that if the objective function and constraint mapping are Lipschitz continuous in parameters and the optimal solution of the unperturbed problem is a locally strict solution, then the solution map is continuous at the reference parameter and converges to the original solution of the unperturbed problem.

In this paper we study the local stability of solutions for problem (1.1)–(1.3), where the state equations are semilinear elliptic equations with Dirichlet boundary condition, and with mixed pointwise constraints. We show that if the unperturbed problem is good enough, then the solution map is continuous with respect to parameters at the reference point. Namely, we prove that if $\bar{z}$ is a locally strong solution of $P(\bar{w})$ and the objective function is Lipschitz continuous, then the solution map is locally upper Hölder continuous at $\bar{w}$. In order to obtain such a result, we follow the method of [1] and [15], and use techniques as well as recent results on optimality conditions in [19] and [23]. However, in our approach, the critical cone for problem $P(\bar{w})$ is smaller than the critical cone used in [1] and [15]. Here the critical cone is a common cone under which the second-order necessary optimality conditions and second-order sufficiently optimality conditions for unperturbed problem $P(\bar{w})$ are valid. Besides, the assumptions imposing on the unperturbed problem are easy to verify. It is worth pointing out that our method can apply not only for optimal control problems governed by semilinear elliptic equations but also for optimal control problems governed by parabolic equations.

The paper is organized as follows. In Section 2 we set up notation and assumptions. We then state our main result. Section 3 is devoted to some auxiliary results. Section 4 contains the proof of the main result.

2. Assumptions and statement of the main result

Hereafter given a Banach space X, $v \in X$ and $r > 0$, we denote by $B_X(v, r)$ and $\bar{B}_X(v, r)$ the open ball and the closed ball with center v and radius r, respectively.

In some cases, if no confusion, we can write $B(v,r)$ and $\bar{B}(v,r)$. Also, we denote by $B_X, \bar{B}_X$ the open unit ball and the closed unit ball, respectively.

Let $\bar{z} = (\bar{y}, \bar{u}) \in \big(W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\big) \times L^2(\Omega)$. For a number $R > 0$, we define

$$\begin{aligned}
&\Sigma_R(w) = \Sigma(w) \cap B(\bar{z}, R),\\
&\mathcal{S}_R(w) = \{(y_w, u_w) \in \Sigma_R(w) \mid J(y_w, u_w, w) = \inf_{(y,u)\in\Sigma_R(w)} J(y,u,w)\}.
\end{aligned}$$

In this section, $L : \Omega\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a function which stands for ϕ, g and h. Given an admissible couple $(y,u) \in \Sigma(w)$, symbols $L[x], L_y[x], L[\cdot]$, etc., stand respectively for $L(x, y(x), w(x))$, $L_y(x, y(x), w(x))$, $L(\cdot, y(\cdot), w(\cdot))$, etc. Also, given a couple $(\bar{y}, \bar{u}) \in \Sigma(\bar{w})$ $\bar{L}[x], \bar{L}_y[x], \bar{L}[\cdot]$, etc., stand respectively for $L(x, \bar{y}(x), \bar{w}(x))$, $L_y(x, \bar{y}(x), \bar{w}(x))$, $L(\cdot, \bar{y}(\cdot), \bar{w}(\cdot))$, etc.

We now impose the following assumptions on L, φ and ψ.
$(H1)$ $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function of class C^2 with respect to variable (y,w). There exists $\epsilon > 0$ such that $\phi(\cdot, y(\cdot), w(\cdot)) \in L^1(\Omega)$, $g(\cdot, y(\cdot), w(\cdot)) \in L^2(\Omega)$ and $h(\cdot, y(\cdot), w(\cdot)) \in L^2(\Omega)$ for all $y \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $w \in L^\infty(\Omega)$ with $\|w - \bar{w}\|_\infty < \epsilon$. Furthermore, for each $M > 0$, there exists a positive number k_{LM} such that

$$\begin{aligned}
&|L_y(x, y_1, w_1) - L_y(x, y_2, w_2)| \le k_{LM}(|y_1 - y_2| + |w_1 - w_2|),\\
&|L_w(x, y_1, w_1) - L_w(x, y_2, w_2)| \le k_{LM}(|y_1 - y_2| + |w_1 - w_2|)
\end{aligned}$$

for all $y, y_i, w, w_i \in \mathbb{R}$ satisfying $|y_i|, |w_i| \le M$ with $i = 1, 2$. Also for each $M > 0$, there is a number $k_{LM} > 0$ such that

$$\begin{aligned}
&\big|L_{yy}(x, y_1, w_1) - L_{yy}(x, y_2, w_2)\big| \le k_{LM}(|y_1 - y_2| + |w_1 - w_2|),\\
&\big|L_{yw}(x, y_1, w_1) - L_{yw}(x, y_2, w_2)\big| \le k_{LM}(|y_1 - y_2| + |w_1 - w_2|)\\
&|L_{ww}(x, y_1, w_1) - L_{ww}(x, y_2, w_2)| \le k_{LM}(|y_1 - y_2| + |w_1 - w_2|)
\end{aligned}$$

for all $y_i, w_i \in \mathbb{R}$ satisfying $|y_i|, |w_i| \le M$ with $i = 1, 2$.
$(H2)$ The functions φ and ψ are of class C^2 and have a property that for each $M > 0$, there exist numbers $k_\varphi, k_\psi > 0$ such that

$$\begin{aligned}
&|\varphi'(w)| + |\varphi''(w)| \le k_\varphi |w|\\
&|\varphi'(w_1) - \varphi'(w_2)| + |\varphi''(w_1) - \varphi''(w_2)| \le k_\varphi(|w_1 - w_2|)\\
&|\psi'(w)| + |\psi''(w)| \le k_\psi |w|,\\
&|\psi'(w_1) - \psi'(w_2)| + |\psi''(w_1) - \psi''(w_2)| \le k_\psi(|w_1 - w_2|)
\end{aligned}$$

for all $w, w_i \in \mathbb{R}$ satisfying $|w|, |w_i| \le M$ with $i = 1, 2$.
$(H3)$ $g(x, 0, w) = 0$ for all $x \in \Omega, w \in \mathbb{R}$ and $g_y(x, y, w(x)) \ge 0$ for a.e. $x \in \Omega$ and for all $y \in \mathbb{R}$ and $\|w - \bar{w}\|_\infty < \epsilon$.
$(H4)$ There exists $\gamma > 0$ such that $\psi(\bar{w}(x)) \ge \gamma$ and $\dfrac{1}{\lambda}\bar{h}_y[x] \ge 0$ for a.e. $x \in \Omega$.

Let us take a look on assumptions $(H1) - (H4)$. Assumptions $(H1)$ and $(H2)$ guarantee that J, g and h are of class C^2 around $(\bar{z}, \bar{w})$. Under assumption $(H3)$, we have from [7] that, for each $u \in L^2(\Omega)$ and $w \in L^\infty(\Omega)$, equation (1.2) has a unique solution $y \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and there exists an absolute constant $c_0 > 0$ such that

$$\|y\|_{W^{2,2}(\Omega)} \le c_0 \|u + w\|_{L^2(\Omega)}. \tag{2.1}$$

Meanwhile, assumption $(H4)$ makes sure that the Robinson constraint qualification condition is valid. Note that from $(H1), (H2)$, we see that for each fixed parameter $w \in B(\bar{w}, \epsilon)$ and for any $(\hat{y}, \hat{u}) \in \Sigma(w)$, we have

$$\langle \nabla_z J(\hat{y}, \hat{u}, w), (y, u)\rangle =$$
$$\int_\Omega \big(\phi_y(x, \hat{y}(x), w(x))y(x) + \varphi(w(x))u(x) + 2\psi(w(x))\hat{u}(x)u(x)\big)dx.$$

In particular, we have

$$\langle \nabla_z J(\bar{y}, \bar{u}, \bar{w}), (y, u)\rangle =$$
$$\int_\Omega \big(\phi_y(x, \bar{y}(x), \bar{w}(x))y(x) + \varphi(\bar{w}(x))u(x) + 2\psi(\bar{w}(x))\bar{u}(x)u(x)\big)dx.$$

Let us set

$$\Omega_a = \{x \in \Omega \mid \bar{h}[x] + \lambda \bar{u}(x) = a(x)\},\ \Omega_b = \{x \in \Omega \mid \bar{h}[x] + \lambda \bar{u}(x) = b(x)\}. \tag{2.2}$$

Definition 2.1. A couple $d = (y, u) \in (W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)) \times L^2(\Omega)$ is said to be a critical direction at $\bar{z} = (\bar{y}, \bar{u})$ for problem $P(\bar{w})$ if it satisfies the following conditions:

(i) $\langle \nabla_z J(\bar{y}, \bar{u}, \bar{w}), (y, u)\rangle \le 0$;
(ii) $-\Delta y + \bar{g}_y[\cdot]y = u$ in Ω, $y = 0$ on $\partial\Omega$;
(iii)

$$\bar{h}_y[x]y(x) + \lambda u(x) \begin{cases} \ge 0 \text{ if } x \in \Omega_a \\ \le 0 \text{ if } x \in \Omega_b. \end{cases}$$

We denote by $\mathcal{C}(\bar{z})$ the set of critical directions (y, u) at $(\bar{y}, \bar{u})$. It is easy to see that $\mathcal{C}(\bar{z})$ is a closed and convex cone.

Problem $P(\bar{w})$ is associated with the Lagrangian

$$\bar{\mathcal{L}}(z, \vartheta^*, e^*) = J(z, \bar{w}) + \langle \vartheta^*, -\Delta y + g(\cdot, y, \bar{w}) - u - \bar{w}\rangle + \langle e^*, h(\cdot, y, \bar{w}) + \lambda u\rangle$$

with $\vartheta^*, e^* \in L^2(\Omega)$.

Definition 2.2. A couple $(\vartheta^*, e^*) \in L^2(\Omega) \times L^2(\Omega)$ is said to be multipliers of problem $P(\bar{w})$ at $(\bar{y}, \bar{u})$ if

$$\nabla_z \bar{\mathcal{L}}(\bar{z}, \vartheta^*, e^*) = 0,\ e^* \in N\big([a, b], \bar{h}[\cdot] + \lambda \bar{u}\big),$$

where $[a, b] := \{v \in L^2(\Omega) \mid a(x) \le v(x) \le b(x) \text{ a.e. } x \in \Omega\}$ and $N(K, v_0)$ denotes the normal cone to a closed convex set K in $L^2(\Omega)$ at $v_0 \in K$.

We denote by $\Lambda(\bar{z})$ the set of multipliers of $P(\bar{w})$. In Section 4, we will show that $\Lambda(\bar{z}) \ne \emptyset$ and $\Lambda(\bar{z})$ consists of elements $\vartheta^* \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $e^* \in L^2(\Omega)$

such that

$$-\Delta\vartheta^* + \bar{g}_y[\cdot]\vartheta^* = -\bar{\phi}_y[\cdot] - \bar{h}_y[\cdot]^* e^* \text{ in } \Omega,\ \vartheta^* = 0 \text{ on } \partial\Omega, \tag{2.3}$$

$$\varphi(\bar{w}) + 2\psi(\bar{w})\bar{u} - \vartheta^* + \lambda e^* = 0, \tag{2.4}$$

$$e^*(x) \in N\big([a(x), b(x)], \bar{h}[x] + \lambda\bar{u}(x)\big) \text{ a.e. } x \in \Omega. \tag{2.5}$$

We are ready to state the main result of the paper.

Theorem 2.3. *Suppose that assumptions $(H1)-(H4)$ are fulfilled and $(\vartheta^*, e^*) \in \Lambda(\bar{z})$ such that*

$$\bar{\mathcal{L}}_{zz}(\bar{z}, \vartheta^*, e^*)(y,u)^2 =$$
$$\int_\Omega \big(\bar{\phi}_{yy}[x]y^2(x) + 2\bar{\psi}[x]u^2(x) + \vartheta^*(x)\bar{g}_{yy}[x]y^2(x) + e^*(x)\bar{h}_{yy}[x]y^2(x)\big)dx > 0$$

for all $(y,u) \in \mathcal{C}(\bar{z})\setminus\{0\}$. Then $(\bar{y}, \bar{u})$ is a locally strong solution of $P(\bar{w})$ and there exist positive numbers R_0, s_0 and l_0 such that for all $w \in B_{L^\infty(\Omega)}(\bar{w}, s_0)$ and any $(y_w, u_w) \in \mathcal{S}_{R_0}(w)$, (y_w, u_w) is a locally optimal solution of $P(w)$ and

$$\mathcal{S}_{R_0}(w) \subset (\bar{y}, \bar{u}) + l_0\|w - \bar{w}\|_\infty^{1/2}\bar{B}_{(W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega))\times L^2(\Omega)}.$$

As it is mentioned, the proof of Theorem 2.3 is provided in Section 4. For this we need to establish some auxiliary results which are given in the third section below.

3. Some auxiliary results

3.1. **Parametric programming problems.** Let W and Z be Banach spaces. Let $f : Z \times W \to \mathbb{R}$ be a function and $\mathcal{M} : W \rightrightarrows Z$ be a multifunction with closed values. We consider the parametric programming problem

$$P_1(w) \qquad \begin{cases} f(z,w) \to \min \\ z \in \mathcal{M}(w). \end{cases}$$

Hereafter, we assume that $z_0 \in \mathcal{M}(w_0)$. For $r > 0$, the extremal value function $V_r : W \to \bar{\mathbb{R}}$ is defined by

$$V_r(w) = \inf\{f(z,w) \mid z \in \mathcal{M}(w) \cap B_Z(z_0, r)\}$$

and the local solution mapping $S_r : W \rightrightarrows Z$ defined by

$$S_r(w) = \{z \in \mathcal{M}(w) \cap B_Z(z_0, r) \mid f(z,w) = V_r(w)\}.$$

Definition 3.1. A point $z_0 \in \mathcal{M}(w_0)$ is said to be locally strong solution of $P_1(w_0)$ if there exist numbers $\bar{\alpha} > 0$ and $\bar{\gamma} > 0$ such that

$$f(z, w_0) - f(z_0, w_0) \geq \bar{\gamma}\|z - z_0\|^2 \tag{3.1}$$

for all $z \in \mathcal{M}(w_0) \cap B_Z(z_0, \bar{\alpha})$.

We now have the following important result.

Theorem 3.2. *Suppose that z_0 is a locally strong solution of $P_1(w_0)$ and the following assumptions are satisfied:*

(a) *f is Lipschitz continuous around (z_0, w_0), that is, there exist numbers $\alpha > 0$, $\beta > 0$ and $k_f > 0$ such that*

$$|f(z,w) - f(z',w')| \le k_f(\|z - z'\| + \|w - w'\|) \ \forall z, z' \in B_Z(z_0, \alpha)$$

for all , $w, w' \in B_W(w_0, \beta)$;

(b) *$\mathcal{M}(\cdot)$ has the Aubin property around (z_0, w_0), that is, there exist number $\alpha' > 0$, $\beta' > 0$ and $k_M > 0$ such that*

$$\mathcal{M}(w) \cap B_Z(z_0, \alpha') \subset \mathcal{M}(w') + k_M\|w - w'\|\bar{B}_Z \ \forall w, w' \in B_W(w_0, \beta').$$

Then there exist positive numbers k_*, β_* and r such that for all $w \in B_W(w_0, \beta_*)$ and any $z_w \in S_r(w)$, z_w is a locally optimal solution of $P_1(w)$ and

$$S_r(w) \subset z_0 + k_*\|w - w_0\|^{1/2}\bar{B}_Z.$$

Proof. The proof follows the one in [15, Theorem 4.4]. Since z_0 is a local strong solution for $P_1(w_0)$, there exist $\bar{\gamma} > 0$ and $\bar{\alpha} > 0$ such that (3.1) is fulfilled. We choose $\alpha_0 = \min(\alpha, \alpha')$ and $\beta_0 = \min(\beta, \beta')$. Then from (a) and (b), we have for any $w, w' \in B_W(w_0, \beta_0)$ and $z, z' \in B_Z(z_0, \alpha_0)$ that

$$\begin{aligned} &|f(z,w) - f(z',w')| \le k_f(\|z - z'\| + \|w - w'\|), \\ &\mathcal{M}(w) \cap B_Z(z_0, \alpha_0) \subset \mathcal{M}(w') + k_M\|w - w'\|\bar{B}_Z. \end{aligned}$$

Choose $r > 0, \beta_1 > 0$ and $s_1 > 0$ such that $r + 3k_M\beta_1 < \min(\alpha_0, \beta_0, \bar{\alpha})$, $\beta_1 < \beta_0$ and $s_1 = 2k_M\beta_1 + r$. Then we also have

$$\mathcal{M}(w) \cap B_Z(z_0, r) \subset \mathcal{M}(w') \cap B_Z(z_0, s_1) + k_M\|w - w'\| \tag{3.2}$$

for all $w, w' \in B_W(w_0, \beta_1)$. Besides, from the proof of [15, Theorem 4.4]), there exists a number $0 < \beta_1' < \beta_1$ such that

$$|V_r(w) - V_r(w_0)| \le k_1\|w - w_0\| \ \forall w \in B_W(w_0, \beta_1'), \tag{3.3}$$

where $k_1 = k_f(1 + k_M)$.

Put $k_* = k_M\sqrt{\beta_1'} + (2k_1/\bar{\gamma})^{1/2}$ and choose $0 < \beta_* < \min(\beta_1', \frac{r^2}{4k_*^2})$. Fix any $w \in B_W(w_0, \beta_*)$ and take $z_w \in S_r(w)$. Inserting $w' = w_0$ into (3.2), we see that there exists $z \in \mathcal{M}(w_0) \cap B_Z(z_0, s_1)$ such that $\|z_w - z\| \le k_M\|w - w_0\|$. Hence,

$$\|z_w - z_0\| \le \|z_w - z\| + \|z - z_0\| \le k_M\|w - w_0\| + \|z - z_0\|. \tag{3.4}$$

On the other hand from (3.3), we have

$$\begin{aligned} \bar{\gamma}\|z - z_0\|^2 &\le f(z, w_0) - f(z_0, w_0) \\ &= f(z, w_0) - f(z_w, w) + f(z_w, w) - f(z_0, w_0) \\ &= f(z, w_0) - f(z_w, w) + V_r(w) - V_r(w_0) \\ &\le k_f(\|z - z_w\| + \|w - w_0\|) + k_1\|w - w_0\| \\ &\le k_f(k_M\|w - w_0\| + \|w - w_0\|) + k_1\|w - w_0\|, \\ &\le 2k_1\|w - w_0\| \end{aligned}$$

which implies that

$$\|z - z_0\| \le k_2\|w - w_0\|^{1/2},$$

where $k_2 = (2k_1/\bar{\gamma})^{1/2}$. Combining this with (3.4) yields

$$\begin{aligned}\|z_w - z_0\| &\le k_M\|w - w_0\| + k_2\|w - w_0\|^{1/2}\\ &\le k_M\sqrt{\beta_1'}\sqrt{\|w - w_0\|} + k_2\sqrt{\|w - w_0\|}\\ &\le k_*\sqrt{\|w - w_0\|}.\end{aligned}$$

By the choice of β_*, we have $\|z_w - z_0\| < r/2$ for all $w \in B_W(w_0, \beta_*)$. Hence $z_w \in \mathrm{int}B_Z(z_0, r)$. The proof of the theorem is complete. □

3.2. **An abstract optimal control problem.** Let E_0, E, Y and U be reflexive Banach spaces and W is a Banach space as in Subsection 3.1. Here, we suppose that the imbedding $Y \hookrightarrow C(\bar{\Omega})$ is compact, where Ω is an open bounded set in $\mathbb{R}^n$ with $n \ge 1$.

Define $Z = Y \times U$ and assume that $f : C(\bar{\Omega}) \times U \times W \to \mathbb{R}$, $G : Y \times U \times W \to E_0$ and $H : C(\bar{\Omega}) \times U \times W \to E$ are given mappings, D a nonempty closed convex set in E. For each $w \in W$, we consider the following parametric optimal control problem of finding a control $u \in U$ and the corresponding state $y \in Y$ which

$$\text{minimize} \quad f(y, u, w), \tag{3.5}$$

$$\text{s.t.} \quad G(y, u, w) = 0, \tag{3.6}$$

$$H(y, u, w) \in D. \tag{3.7}$$

We denote by $P_2(w)$ the problem (3.5)–(3.7) and by $\mathrm{Sol}(w)$ the solution set of $P_2(w)$. For each $w \in W$ we put

$$\mathcal{A}(w) = \{z \in Z \mid G(z, w) = 0\} \tag{3.8}$$

and denote by $\Phi(w)$ the feasible set of problem (3.5)–(3.7), that is,

$$\Phi(w) = \mathcal{A}(w) \cap H_w^{-1}(D).$$

Then problem $P_2(w)$ can be formulated in the form of $P_1(w)$:

$$P_2(w) \qquad \begin{cases} f(z, w) \to \min \\ z \in \Phi(w). \end{cases}$$

Fixing a parameter $w_0 \in W$, we call $P_2(w_0)$ the unperturbed problem and assume $z_0 = (y_0, u_0) \in \Phi(w_0)$. For each $r > 0$, we define

$$\begin{aligned}&\Phi_r(w) = \Phi(w) \cap B_Z(z_0, r),\\ &\mathrm{Sol}_r(w) := \{z_w \in \Phi_r(w) \mid f(z_w, w) = \min_{z \in \Phi_r(w)} f(z, w)\}.\end{aligned}$$

Given a closed set C in Z and a point $z \in C$, the sets

$$\begin{aligned}T^\flat(C, z) &= \{h \in Z \mid \liminf_{t \to 0^+} \mathrm{dist}(h, \frac{C - z}{t}) = 0\}\\ &= \{h \in Z | \forall t_n \to 0^+, \exists h_n \to h, z + t_n h_n \in C\},\\ T(C, z) &= \{h \in Z \mid \limsup_{t \to 0^+} \mathrm{dist}(h, \frac{C - z}{t}) = 0\}\\ &= \{h \in Z | \exists t_n \to 0^+, \exists h_n \to h, z + t_n h_n \in C\}\end{aligned}$$

are called *the adjacent tangent cone* and *the contingent cone* to C at z, respectively. These cones are closed and $T^\flat(C,z) \subseteq T(C,z)$. It is well known that when C is convex, then

$$T^\flat(C,z) = T(C,z) = \overline{\text{cone}}(C-z)$$

and the normal cone to C at z is given by

$$N(C,z) = \{z^* \in Z^* \mid \langle z^*, c-z\rangle \le 0 \ \forall c \in C\}.$$

Definition 3.3. A point $z_0 \in \Phi(w_0)$ is said to be a regular point of $P_2(w_0)$ if there exist numbers $\eta > 0$ and $\varrho > 0$ such that

(3.9)

$$\eta B_E \subset \Big\{ \bigcap_{z\in B(z_0,\varrho)\cap\mathcal{A}(w_0)} \nabla_z H(z_0,w_0)(T(\mathcal{A}(w_0),z)\cap B_Z) - (D - H(z_0,w_0))\cap B_E \Big\}.$$

It is known that this constraint qualification condition is an extension of the so-called Robinson constraint qualification condition. By [17, Theorem 2.5], condition (3.9) is equivalent to the following:

$$E = \bigcap_{z\in B(z_0,\varrho)\cap\mathcal{A}(w_0)} \nabla_z H(z_0,w_0)(T(\mathcal{A}(w_0),z)) - \text{cone}(D - H(z_0,w_0)). \tag{3.10}$$

The following assumptions will be needed throughout the paper.

$(A1)$ There exist positive numbers r_1, r_1', r_1'' such that for any $w \in B_W(w_0, r_1'')$, the mapping $f(\cdot,\cdot,w)$ and $H(\cdot,\cdot,w)$ are Fréchet differentiable on $B_Y(y_0,r_1)\times B_U(u_0,r_1')$. The mapping $G(\cdot,\cdot,\cdot)$ is continuously Fréchet differentiable on $B_Y(y_0,r_1)\times B_U(u_0,r_1')\times B_W(w_0,r_1'')$.
$(A2)$ The mappings f and H are Lipschitz continuous on $B_Y(y_0,r_1)\times B_U(u_0,r_1')\times B_W(w_0,r_1'')$ i.e., there exist constants $L_f, L_H > 0$ such that

$$|f(z_1,w_1) - f(z_1,w_2)| \le L_f(\|z_1 - z_2\| + \|w_1 - w_2\|),$$
$$\|H(z_1,w_1) - H(z_2,w_2)\| \le L_H(\|z_1 - z_2\| + \|w_1 - w_2\|)$$

for all $z_1, z_2 \in B_Y(y_0,r_1)\times B_U(u_0,r_1')$ and $w_1, w_2 \in B_W(w_0,r_1'')$.
$(A3)$ The mapping $G_y(z_0,w_0)$ is bijective.
$(A4)$ The mappings $f(\cdot,\cdot,w_0)$, $G(\cdot,\cdot,w_0)$ and $H(\cdot,\cdot,w_0)$ are twice continuously Fréchet differentiable on $B_Y(y_0,r_1)\times B_U(u_0,r_1')$.
$(A5)$ $\nabla_z H(z_0,w_0)(T^\flat(\mathcal{A}(w_0),z_0)) = E$.

From $(A1)$ and $(A3)$, we have that $G(\cdot,\cdot,\cdot)$ is continuously differentiable on $B_Y(y_0,r_1)\times B_U(u_0,r_1')\times B_W(w_0,r_1'')$ and $G_y(z_0,w_0)$ is bijective. By the implicit function theorem, there exist balls $B_Y(y_0,r_2)$, $B_U(u_0,r_2')$ and $B_W(w_0,r_2'')$ with $r_2 < r_1, r_2' < r_1'$ and $r_2'' < r_1''$ such that for each $(u,w) \in B_U(u_0,r_2')\times B_W(w_0,r_2'')$, the equation

$$G(y,u,w) = 0$$

has a unique solution $y = \zeta(u,w) \in B_Y(y_0,r_2)$. Moreover, the mapping

$$\zeta : B_U(u_0,r_2')\times B_W(w_0,r_2'') \to B_Y(y_0,r_2)$$

is continuously Fréchet differentiable and $\zeta(u_0,w_0) = y_0$. Thus

$$G(\zeta(u,w),u,w) = 0 \ \forall (u,w) \in B_U(u_0,r_2')\times B_W(w_0,r_2'').$$

Since ζ_u and ζ_w are continuous at (u_0, w_0), we can find positive numbers $r_3' < r_2'$ and $r_3'' < r_2''$ such that

$$\|\zeta_u(u,w)\|, \|\zeta_w(u,w)\| \le \gamma, \ \forall (u,w) \in B_U(u_0, r_3') \times B_W(w_0, r_3''),$$

where $\gamma > 0$ independent of (u, w). By the Taylor expansion, we have

$$\zeta(u_1, w_1) - \zeta(u_2, w_2) = \nabla_{u,w}\zeta(t(u_1, w_1) + (1-t)(u_1, w_2))(u_1 - u_2, w_1 - w_2)$$

for some $0 < t < 1$ and for all $u_1, u_2 \in B_U(u_0, r_3')$ and $w_1, w_2 \in B_W(w_0, r_3'')$. By the boundedness of ζ_u and ζ_w, we have

$$\|\zeta(u_1, w_1) - \zeta(u_2, w_2)\| \le \gamma(\|u_1 - u_2\| + \|w_1 - w_2\|). \tag{3.11}$$

Thus ζ is Lipschitz continuous on $B_U(u_0, r_3') \times B_W(w_0, r_3'')$.

We now have the following result.

Lemma 3.4. *Suppose $z_0 \in \Phi(w_0)$ is a regular point and $(A1) - (A3)$ are fulfilled. Then Φ has the Aubin property around $(z_0, w_0) \in \mathrm{Graph}(\Phi)$.*

Proof. Let $\zeta : B_U(u_0, r_3') \times B_W(w_0, r_3'') \to B_Y(y_0, r_2)$ be such that

$$\zeta(u_0, w_0) = y_0,$$
$$G(\zeta(u,w), u, w) = 0 \ \forall (u,w) \in B_U(u_0, r_3') \times B_W(w_0, r_3'').$$

Then we have

$$G_y(y_0, u_0, w_0)\zeta_u(u_0, w_0) + G_u(y_0, u_0, w_0) = 0$$

and so

$$G_y(y_0, u_0, w_0)\zeta_u(u_0, w_0)v + G_u(y_0, u_0, w_0)v = 0 \tag{3.12}$$

for all $v \in U$. Since $G_y(z_0, w_0)$ is bijective, $\zeta_u(u_0, w_0)v$ is the unique solution of (3.12).

Also, since $G_y(z_0, w_0)$ is bijective, the operator $\nabla_z G(z_0, w_0)$ is surjective. By [17, Lemma 2.2], we get

$$\begin{aligned} T(\mathcal{A}(w_0), z_0) &= \{h = (y, v) \mid \nabla_z G(z_0, w_0)h = 0\} \\ &= \{(y, v) \mid G_y(z_0, w_0)y + G_u(z_0, w_0)v = 0\} \\ &= \{(\zeta_u(u_0, w_0)v, v) \mid v \in U\}. \end{aligned}$$

From this and (3.9), we have

$$\begin{aligned} \eta B_E &\subset \nabla_z H(z_0, w_0)(T(\mathcal{A}(w_0), z_0)) - (D - H(z_0, w_0))\} \\ &= \Big\{ H_y(z_0, w_0)\zeta_u(u_0, w_0)v + H_u(z_0, w_0)v - (D - H(z_0, w_0)) \mid v \in U \Big\} \\ &\subset \Big\{ F_u(u_0, w_0)U + (F(u_0, w_0) - D) \Big\}, \end{aligned}$$

where $F(u, w) := H(\zeta(u, w), u, w)$. It is clear that $F(\cdot, \cdot)$ is of class C^1 on a neighborhood of (u_0, w_0). By [4, Theorem 4.2], we see that the mapping $\Theta(u, w) := -F(u, w) + D$ is metrically regular at $(u_0, w_0, 0) \in \mathrm{Graph}(\Theta)$, that is, there are numbers $a > 0$ and $r > 0$ such that

$$\mathrm{dist}(u, \Theta_w^{-1}(e)) \le a\,\mathrm{dist}(e, \Theta(u, w))$$

for all (u, w, e) satisfying $\|u - u_0\| \le r, \|e - 0\|_E \le r$ and $\|w, w_0\| \le r$. In other words, $\Theta(\cdot, w)$ is metrically regular at $(u_0, 0)$ uniform in w.

• Claim 1. The multifunction $\Theta(u, \cdot)$ is Lipchitz continuous w.r.t. w uniformly in u around (u_0, w_0).

In fact, fix any $u \in B(u_0, r_3')$ and take $w, w' \in B(w_0, r_3'')$. Then for any $e \in \Theta(u, w)$, we have $e = -F(u, w) + d$ for some $d \in D$. Choosing $e' = -F(u, w') + d$, we have $e' \in \Theta(u, w')$. Since H and ζ are Lipschitz continuous, we have

$$\begin{aligned}\|e - e'\| = \|F(u, w) - F(u, w')\| &= \|H(\zeta(u, w), u, w) - H(\zeta(u, w'), u, w')\|\\ &\le L_H(\|\zeta(u, w) - \zeta(u, w')\| + \|w - w'\|)\\ &\le L_H(\gamma\|w - w'\| + \|w - w'\|)\\ &= L_H(1 + \gamma)\|w - w'\|.\end{aligned}$$

This implies that

$$\Theta(u, w) \subset \Theta(u, w') + L_H(1 + \gamma)\|w - w'\|\bar{B}_E$$

for all $u \in B_U(u_0, r_3')$ and $w, w' \in B_W(w_0, r_3'')$. Hence Claim 1 is justified.

According to [9, Proposition 4.1], the multifunction $\Gamma : W \rightrightarrows U$ which is given by

$$\Gamma(w) = \{u \in U \mid 0 \in \Theta(u, w)\} = \{u \in U \mid F(u, w) \in D\}$$

has the Aubin property around (u_0, w_0). Hence, there are positive numbers $r_4' < r_3'$, $r_4'' < r_3''$ and a constant $l > 0$ such that

$$\Gamma(w) \cap B_U(u_0, r_4') \subset \Gamma(w') + l\|w - w'\|\bar{B}_U \tag{3.13}$$

for all $w, w' \in B_W(w_0, r_4'')$.

• Claim 2. The multifunction Φ has the Aubin property. Namely,

$$\Phi(w) \cap (B(y_0, r_2) \times B(u_0, r_4')) \subset \Phi(w') + (\gamma l + \gamma + l)\|w - w'\|\bar{B}_Z$$

for all $w, w' \in B_W(w_0, r_4'')$.

Indeed, take any $(y, u) \in \Phi(w) \cap ((B(y_0, r_2) \times B(u_0, r_4')))$. Then we have $y = \zeta(u, w)$ and $H(\zeta(u, w), u, w) \in D$. Hence $u \in \Gamma(w) \cap B(u_0, r_4')$. By (3.13), there exists $u' \in \Gamma(w')$ such that $\|u - u'\| \le l\|w - w'\|$. Since $u' \in \Gamma(w')$, we have $H(\zeta(u', w'), u', w') \in D$. By putting $y' = \zeta(u', w')$, we have $(y', u') \in \Phi(w')$. It follows that

$$\begin{aligned}\|(y, u) - (y', u')\| &= \|\zeta(u, w) - \zeta(u', w')\| + \|u - u'\|\\ &\le \gamma(\|u - u'\| + \|w - w'\|) + \|u - u'\|\\ &\le \gamma(l\|w - w'\| + \|w - w'\|) + l\|w - w'\|\\ &= (\gamma l + \gamma + l)\|w - w'\|.\end{aligned}$$

Therefore, Claim 2 is justified and the proof is complete. □

Problem $P_2(w_0)$ is associated with the following Lagrangian:

$$\mathcal{L}(z, v^*, e^*) = f(z, w_0) + \langle v^*, G(z, w_0)\rangle + \langle e^*, H(z, w_0)\rangle \text{ with } v^* \in E_0^*, e^* \in E^*.$$

We denoted by $\Lambda_2(z_0)$ the set of multipliers $(v^*, e^*) \in E_0^* \times E^*$ such that

$$\nabla_z\mathcal{L}(z, v^*, e^*) = 0,\ e^* \in N(D, H(z, w_0))$$

and by

$$\begin{aligned}\mathcal{C}_2(z_0) = \big\{ & d \in Z | \langle \nabla_z f(z_0, w_0), d\rangle \le 0, \nabla_z G(z_0, w_0)d = 0, \\ & \nabla_z H(z_0, w_0)d \in T^\flat(D, H(z_0, w_0))\big\}\end{aligned}$$

the set of *critical directions* at z_0 of $P_2(w_0)$. It is clear that $\mathcal{C}_2(z_0)$ is a closed convex cone.

In the sequel we need the so-called polyhedric property of D. This property plays an important role in deriving second-order optimality conditions. According to Bonnans and Shapiro [3, chapter 3], the set D is said to be *polyhedric* at $u \in D$ if for all $q^* \in N(D, u)$ then

$$T^\flat(D, u) \cap (q^*)^\perp = cl[\text{cone}(D - u) \cap (q^*)^\perp],$$

where $(q^*)^\perp := \{x \in E \mid \langle q^*, x\rangle = 0\}$.

The following proposition provides second-order necessary optimality conditions for $P_2(w_0)$.

Proposition 3.5 ([23, Theorem 6])**.** *Suppose that z_0 is a regular point of $P_2(w_0)$, assumptions $(A1)$, $(A3)$, $(A4)$ and $(A5)$ are satisfied, and D is polyhedric at $H(z_0, w_0)$. Then if z_0 is a locally optimal solution of $P_2(w_0)$, then for each $d \in \mathcal{C}_2(z_0)$, there exists $(v^*, e^*) \in \Lambda_2(z_0)$ such that*

$$\begin{aligned}&\mathcal{L}_{zz}(z_0, e^*, v^*)(d, d) = \\ &f_{zz}(z_0, w_0)d^2 + \langle v^*, G_{zz}(z_0, w_0)d^2\rangle + \langle e^*, H_{zz}(z_0, w_0)d^2\rangle \ge 0.\end{aligned}$$

Remark 3.6. In the case where $\Lambda_2(z_0)$ is singleton, the conclusion of Proposition 3.5 becomes:

$$f_{zz}(z_0, w_0)d^2 + \langle v^*, G_{zz}(z_0, w_0)d^2\rangle + \langle e^*, H_{zz}(z_0, w_0)d^2\rangle \ge 0.$$

for all $d \in \mathcal{C}_2(z_0)$.

The following lemma gives sufficient conditions under which z_0 is a locally strong solution of $P_2(w_0)$.

Lemma 3.7. *Suppose that Y and U are reflexive Banach space, z_0 is a regular point of $P_2(w_0)$, assumptions $(A1), (A3), A(4)$ and $(A5)$ are satisfied and D is polyhedric at $H(z_0, w_0)$. Assume that $(v^*, e^*) \in \Lambda_2(z_0)$ such that*

(3.14)
$$\mathcal{L}_{zz}(z_0, v^*, e^*)(d, d) > 0 \ \forall d \in \mathcal{C}_2(z_0) \setminus \{0\},$$

(3.15)
$$\langle v^*, G_{zz}(z_0, w_0)d_k^2\rangle + \langle e^*, H_{zz}(z_0, w_0)d_k^2\rangle \to \langle v^*, G_{zz}(z_0, w_0)d^2\rangle + \langle v^*, H_{zz}(z_0, w_0)d^2\rangle$$

whenever $d_k \rightharpoonup d$, and there exists a number $\gamma_0 > 0$ satisfying

$$f_{uu}(y_0, u_0, w_0)(u, u) \ge \gamma_0\|u\|^2 \ \forall u \in U. \tag{3.16}$$

Then $z_0 = (y_0, u_0)$ is a locally strong solution of $P_2(w_0)$, that is, there exist numbers $\gamma_0 > 0$ and $r > 0$ such that

$$f(y, u, w_0) \ge f(y_0, u_0, w_0) + \gamma_0\|(y, u) - (y_0, u_0)\|^2 \ \forall (y, u) \in \Phi(w_0) \cap B_Z(z_0, r).$$

Proof. By contradiction, we assume that the conclusion of theorem is false. Then, there exists a sequence $\{z_k = (y_k, u_k)\} \subset \Phi(w_0)$, $z_k \to z_0 = (y_0, u_0)$ such that

$$f(z_k, w_0) < f(z_0, w_0) + \frac{1}{k}\|z_k - z_0\|_Z^2 = f(z_0, w_0) + o(t_k^2), \tag{3.17}$$

where $t_k := \|z_k - z_0\|_Z \to 0$ as $k \to \infty$. Let us put $\hat{z}_k = (\hat{y}_k, \hat{u}_k), \hat{y}_k = \frac{y_k - y_0}{t_k}, \hat{u}_k = \frac{u_k - u_0}{t_k}$, then $\|\hat{z}_k\|_Z = \|\hat{y}_k\|_Y + \|\hat{u}_k\|_U = 1$. Since Z is reflexive, we may assume that $\hat{z}_k \rightharpoonup \hat{z} = (\hat{y}, \hat{u})$ in Z, that is, $\hat{y}_k \rightharpoonup \hat{y}$ in Y and $\hat{u}_k \rightharpoonup \hat{u}$ in U.

• Claim 1. $\hat{z} \in \mathcal{C}_2(z_0)$.

In what follows, we write $f(\cdot)$, $G(\cdot)$ and $H(\cdot)$ instead of $f(\cdot, w_0)$, $G(\cdot, w_0)$ and $H(\cdot, w_0)$, respectively. Writing $z_k = z_0 + t_k\hat{z}_k$ and using the first-order Taylor expansion, we get from (3.17) that

$$\nabla f(z_0)\hat{z}_k + \frac{o(t_k)}{t_k} \le \frac{o(t_k^2)}{t_k}.$$

By letting $k \to \infty$, we get $\nabla f(z_0)\hat{z} \le 0$. Since $z_k \in \Phi(w_0)$, $G(z_k) = G(z_0 + t_k\hat{z}_k) = 0$ for all $k \ge 1$. Using a first-order Taylor expansion, we have

$$\nabla G(z_0)\hat{z}_k + \alpha_1(t_k) = 0, \quad \forall k \ge 1$$

where $\alpha_1(t_k) \to 0$ as $k \to \infty$. By Theorem 3.10 in [6], $\nabla G(z_0)$ is weakly continuous. Hence, $\nabla G(z_0)(\hat{z}_k) \rightharpoonup \nabla G(z_0)(\hat{z})$ as $k \to \infty$ and so $\nabla G(z_0)(\hat{z}) = 0$.

Since $H(z_k) - H(z_0) \in D - H(z_0)$ for all $k \ge 1$ and using the first order Taylor expansion, we get $\nabla H(z_0)\hat{z}_k + \frac{o(t_k)}{t_k} \in T^\flat(D, H(z_0))$. By similar arguments as the above and noting that $T^\flat(D, H(z_0))$ is a weakly closed and convex subset, we have $\nabla H(z_0)\hat{z} \in T^\flat(D, H(z_0))$. Therefore, $\hat{z} \in \mathcal{C}_2(z_0)$ and Claim 1 is justified.

• Claim 2. $\hat{z} = 0$.

Since $(v^*, e^*) \in \Lambda_2(z_0)$, we have

$$\nabla_z \mathcal{L}(z_0, v^*, e^*) = \nabla f(z_0) + \nabla G(z_0)^* v^* + \nabla H(z_0)^* e^* = 0, \quad e^* \in N(D, H(z_0)).$$

By using the second-order Taylor expansion for $\mathcal{L}$, we get

$$\begin{aligned}
&\mathcal{L}(z_k, v^*, e^*) - \mathcal{L}(z_0, v^*, e^*) \\
&= t_k \nabla_z \mathcal{L}(z_0, v^*, e^*)\hat{z}_k + \frac{t_k^2}{2}\nabla_{zz}^2 \mathcal{L}(z_0, v^*, e^*)(\hat{z}_k, \hat{z}_k) + o(t_k^2) \\
&= 0 + \frac{t_k^2}{2}\nabla_{zz}^2 \mathcal{L}(z_0, v^*, e^*)(\hat{z}_k, \hat{z}_k) + o(t_k^2).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathcal{L}(z_k, v^*, e^*) - \mathcal{L}(z_0, v^*, e^*) \\
&= f(z_k) - f(z_0) + \langle v^*, G(z_k) - G(z_0)\rangle + \langle e^*, H(z_k) - H(z_0)\rangle \\
&\le f(z_k) - f(z_0) \le o(t_k^2).
\end{aligned}$$

Here we used the fact that $e^* \in N(D, H(z_0))$, $G(z_k) = G(z_0) = 0$ and (3.17). Therefore, we have

$$\frac{t_k^2}{2}\nabla_{zz}^2 \mathcal{L}(z_0, v^*, e^*)(\hat{z}_k, \hat{z}_k) + o(t_k^2) \le o(t_k^2).$$

This is equivalent to

$$\nabla^2_{zz}\mathcal{L}(z_0, v^*, e^*)(\hat{z}_k, \hat{z}_k) \le \frac{o(t_k^2)}{t_k^2}. \tag{3.18}$$

By letting $k \to \infty$ and using the fact that $\hat{y}_k \to \hat{y}$ strongly in $C(\bar{\Omega})$ together with (3.15), we obtain

$$\nabla^2_{zz}\mathcal{L}(z_0, v^*, e^*)(\hat{z}, \hat{z}) \le 0.$$

Combining this with (3.14), we get $\hat{z} = 0$. Claim 2 is proved.

Since $G(z_k) = G(y_k, u_k) = 0$ and $z_k \to z_0$, we have $y_k = \zeta(u_k, w_0)$ for k large enough. From (3.11) and definition of $(\hat{y}_k, \hat{u}_k)$, we have

$$\|\hat{y}_k\|_Y = \frac{\|\zeta(u_k, w_0) - \zeta(u_0, w_0)\|_Y}{t_k} \le \frac{\gamma\|u_k - u_0\|_U}{t_k} = \gamma\|\hat{u}_k\|_U. \tag{3.19}$$

This implies that

$$1 = \|\hat{y}_k\|_Y + \|\hat{u}_k\|_U \le (1+\gamma)\|\hat{u}_k\|_U$$

or, equivalently, $\|\hat{u}_k\| \ge \dfrac{1}{1+\gamma}$. Combining this with (3.16) and (3.18), we have

$$\begin{aligned}
&\frac{o(t_k^2)}{t_k^2} \ge \nabla^2_{zz}\mathcal{L}(z_0, e^*, v^*)(\hat{z}_k, \hat{z}_k) \\
&= \nabla^2 f(z_0)\hat{z}_k^2 + \langle v^*, \nabla^2 G(z_0)\hat{z}_k^2\rangle + \langle e^*, \nabla^2 H(z_0)\hat{z}_k^2\rangle \\
&\ge f_{uu}(z_0)\hat{u}_k^2 + 2f_{yu}(z_0)\hat{u}_k\hat{y}_k + f_{yy}(z_0)\hat{y}_k^2 + \langle v^*, \nabla^2 G(z_0)\hat{z}_k^2\rangle + \langle e^*, \nabla^2 H(z_0)\hat{z}_k^2\rangle \\
&\ge \gamma_0\frac{1}{(1+\gamma)^2} + 2f_{yu}(z_0)\hat{u}_k\hat{y}_k + f_{yy}(z_0)\hat{y}_k^2 + \langle v^*, \nabla^2 G(z_0)\hat{z}_k^2\rangle + \langle e^*, \nabla^2 H(z_0)\hat{z}_k^2\rangle.
\end{aligned}$$

Since the embedding $Y \hookrightarrow C(\bar{\Omega})$ is compact, we have $\hat{y}_k \to \hat{y}$ strongly in $C(\bar{\Omega})$. By letting $k \to \infty$ and using Claim 2, we obtain $0 \ge \frac{\gamma_0}{(1+\gamma)^2}$ which is impossible. The proof of the lemma is complete. □

The following theorem is a main result of this section

Theorem 3.8. *Suppose that z_0 is a regular point of $P_2(w_0)$, assumptions $(A1)$-$(A5)$ are satisfied and D is polyhedric at $H(z_0, w_0)$. Assume that $(v^*, e^*) \in \Lambda_2(z_0)$ such that conditions (3.14)–(3.16) are fulfilled. Then z_0 is a locally strong solution of $P_2(w_0)$ and there exist numbers $r_0 > 0$, $s_0 > 0$ and $l_0 > 0$ such that for all $w \in B_W(w_0, s_0)$ and any $z_w \in \mathrm{Sol}_{r_0}(w)$, z_w is a locally optimal solution of $P_2(w)$ and*

$$\mathrm{Sol}_{r_0}(w) \subset z_0 + l_0\|w - w_0\|^{1/2}\bar{B}_Z.$$

Proof. By Lemma 3.7, z_0 is a locally strong solution of $P_2(w_0)$. By $(A2)$, $f(\cdot,\cdot)$ is Lipschitz continuous around (z_0, w_0). From Lemma 3.4, we see that the feasible set $\Phi(w)$ has the Aubin property around (z_0, w_0). Therefore, the conclusion of the theorem follows directly from Theorem 3.2. The proof is complete. □

4. Proof of the main result

Let us set

$$Y = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega,),\ U = L^2(\Omega),\ Z = Y \times U,\ W = L^\infty(\Omega).$$

Define mappings

$$\begin{aligned} &G : Z \times W \to U,\ G(z,w) = -\Delta y + g(\cdot, y, w) - u - w,\\ &H : Z \times W \to U,\ H(z,w) = h(\cdot, y, w) + \lambda u \end{aligned}$$

with $z = (y,u)$, and define the set

$$D = [a,b] = \{v \in L^2(\Omega) \mid a(x) \le v(x) \le b(x) \text{ a.e. } x \in \Omega\}.$$

Then for each $w \in W$, problem $P(w)$ can be formulated in the form of $P_2(w)$:

$$P(w) \qquad \begin{cases} J(z,w) \to \min,\\ G(z,w) = 0,\\ H(z,w) \in D. \end{cases}$$

In what follows, we show that assumptions $(A1) - (A5)$ are fulfilled and then apply Theorem 3.8 for problem $P(w)$. Here z_0 and w_0 are replaced by $\bar z$ and $\bar w$, respectively and $E = E_0 = U = L^2(\Omega)$.
• Verification of $(A1)$. By $(H1)$ and $(H2)$, we have $J(\cdot,\cdot)$, $G(\cdot,\cdot)$ and $H(\cdot,\cdot)$ are continuously Fréchet differentiable in $B_Z(\bar z, \epsilon) \times B_W(\bar w, \epsilon)$.
• Verification of $(A2)$. Since $\nabla J(\cdot,\cdot)$ and $\nabla H(\cdot,\cdot)$ are continuous at $(\bar z, \bar w)$, $J(\cdot,\cdot)$ and $H(\cdot,\cdot)$ are Lipschitz continuous in $B_Z(\bar z, \epsilon') \times B_W(\bar w, \epsilon')$ for some $\epsilon' < \epsilon$.
• Verification of $(A3)$. For this we have

$$G_y(\bar z, \bar w) = -\Delta + g_y(\cdot, \bar y, \bar w),\ G_u(\bar z, \bar w) = -I,$$

where I is the identify mapping in $L^2(\Omega)$. By $(H3)$ we have $g_y(x, \bar y(x), \bar w(x)) \ge 0$. By [11, Theorem 2.4.2.5, p. 124], for each $v \in L^2(\Omega)$, the linear elliptic equation

$$-\Delta y + \bar g_y[\cdot] y = v \text{ in } \Omega,\ y = 0 \text{ on } \partial\Omega$$

has a unique solution $y \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Hence $(A3)$ is valid.
• Verification of $(A4)$. This follows from $(H1)$ and $(H2)$.
• Verification of $(A5)$. It is based on the following lemma.

Lemma 4.1. *$\bar z \in \Sigma(\bar w)$ is a regular point and $(A5)$ is valid.*

Proof. According to (3.10), we only need to show that

$$U = \nabla_z H(\bar z, \bar w)(T(\Sigma(\bar w), \hat z)) - \text{cone}(D - H(\bar z, \bar w)). \tag{4.1}$$

for all $\hat z \in \Sigma(\bar w)$ with $\|\hat z - \bar z\| < \epsilon'$. In fact, we have $G(\hat z, \bar w) = 0$ and $G_y(\hat z, \bar w) = -\Delta + g_y(\cdot, \hat y, \bar w)$. Since $g_y(x, \hat y(x), \bar w(x)) \ge 0$ for a.e. $x \in \Omega$, for each $v \in U = L^2(\Omega)$, the equation

$$-\Delta y + g_y(\cdot, \hat y, \bar w) y = v$$

has a unique solution $y \in Y$. Hence $G_y(\hat{z}, \bar{w})$ is bijective. Consequently, $\nabla G(\hat{z}, \bar{w})$ is surjective. It follows that

$$\begin{aligned} T(\Sigma(\bar{w}), \hat{z}) &= \{(y,u) \mid G_y(\hat{z}, \bar{w})y + G_u(\hat{z}, \bar{w})u = 0\} \\ &= \{(y,u) \mid G_y(\hat{z}, \bar{w})y = u\}. \end{aligned}$$

Since $\nabla_z H(\bar{z}, \bar{w}) = (\bar{h}_y[\cdot], \lambda I)$, we have

$$\nabla_z H(\bar{z}, \bar{w})(T(\Sigma(\bar{w}), \hat{z})) = \{\bar{h}_y[\cdot]y + \lambda u \mid G_y(\hat{z}, \bar{w})y = u, u \in U\}. \tag{4.2}$$

Take any $v \in U$ and consider equation

$$-\Delta y + \frac{(\lambda g_y(\cdot, \hat{y}, \bar{w}) + \bar{h}_y[\cdot])y}{\lambda} = \frac{v}{\lambda}.$$

By $(H3)$ and $(H4)$, $\dfrac{\lambda g_y(x, \hat{y}(x), \bar{w}(x)) + \bar{h}_y[x]}{\lambda} \geq 0$ a.e. Hence, the above equation has a unique solution $y \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Putting $u = -\Delta y + g_y(\cdot, \hat{y}, \bar{w})y$, we see that $u \in U$ and $(y, u) \in T(\Sigma(\bar{w}), \hat{z})$. This implies that $v = \bar{h}_y[\cdot]y + \lambda u$ and so $v \in \nabla_z H(\bar{z}, \bar{w})(T(\Sigma(\bar{w}), \hat{z}))$ because of (4.2). Hence $U \subseteq \nabla_z H(\bar{z}, \bar{w})(T(\Sigma(\bar{w}), \hat{z}))$. In particular, for $\hat{z} = \bar{z}$, we have $U \subseteq \nabla_z H(\bar{z}, \bar{w})(T(\Sigma(\bar{w}), \bar{z}))$ and $(A5)$ is valid. Since $0 \in \text{cone}(D - H(\bar{z}, \bar{w}))$, we have

$$U \subseteq \nabla_z H(\bar{z}, \bar{w})(T(\Sigma(\bar{w}), \hat{z})) - \text{cone}(D - H(\bar{z}, \bar{w})).$$

The proof of the lemma is complete. □

Recall that problem $P(\bar{w})$ is associated with the Lagrangian

$$\bar{\mathcal{L}}(z, \vartheta^*, e^*) = J(z, \bar{w}) + \langle \vartheta^*, G(z, \bar{w})\rangle + \langle e^*, H(z, \bar{w})\rangle, \ \vartheta^*, e^* \in L^2(\Omega)$$

and the critical directional set

$$\begin{aligned} \mathcal{C}(\bar{z}) = \big\{ & d \in Z \mid \langle \nabla_z J(\bar{z}, \bar{w}), d\rangle \leq 0, \ \nabla_z G(\bar{z}, \bar{w})d = 0, \\ & \nabla_z H(\bar{z}, \bar{w})d \in T^\flat(D, H(\bar{z}, \bar{w}))\big\}. \end{aligned}$$

By a simple computation, we have

$$\begin{aligned} &\nabla_z J(\bar{z}, \bar{w}) = (\bar{\phi}_y[\cdot], \varphi(\bar{w}) + 2\psi(\bar{w})\bar{u}), \\ &\nabla_z G(\bar{z}, \bar{w}) = (-\Delta + \bar{g}_y[\cdot], I), \\ &\nabla_z H(\bar{z}, \bar{w}) = (\bar{h}_y[\cdot], \lambda I), \end{aligned}$$

where I is the identify mapping on $L^2(\Omega)$. By [19, Lemma 2.4], D is polyhedric at $H(\bar{z}, \bar{w}) \in D$ and

$$T^\flat(D, H(\bar{z}, \bar{w})) = \left\{ v \in L^2(\Omega) \mid v(x) \begin{cases} \geq 0 \text{ if } x \in \Omega_a \\ \leq 0 \text{ if } x \in \Omega_b \end{cases} \right\},$$

where Ω_a and Ω_b are defined by (2.2). Also we have

$$N(D, H(\bar{z}, \bar{w})) = \Big\{ e^* \in L^2(\Omega) \mid e^*(x) \in N\big([a(x), b(x)], \bar{h}[x] + \lambda\bar{u}(x)\big) \text{ a.e.}\Big\}$$

It follows that $\mathcal{C}(\bar{z})$ consists of critical directions as in Definition 2.1. According to Proposition 3.5, if $\bar{z}$ is a locally optimal solution, then for each $d = (y, u) \in \mathcal{C}(\bar{z})$, there exists $(\vartheta^*, e^*) \in L^2(\Omega) \times L^2(\Omega)$ such that

$$D_z\bar{\mathcal{L}}(\bar{z}, v^*, e^*) = 0, \ e^* \in N(D, H(\bar{z}, \bar{w})) \tag{4.3}$$

and

$$\bar{\mathcal{L}}_{zz}(\bar{z}, \vartheta^*, e^*)(y, u)^2 = \int_\Omega \big(\bar{\phi}_{yy}[x]y^2(x) + 2\psi(\bar{w}(x))u^2(x) + \vartheta^*(x)\bar{g}_{yy}[x]y^2(x) + e^*(x)\bar{h}_{yy}[x]y^2(x)\big)dx \geq 0.$$

Note that the first relation of (4.3) is equivalent to

$$\begin{cases} D_y\bar{\mathcal{L}}(\bar{z}, v^*, e^*) = 0 \\ D_u\bar{\mathcal{L}}(\bar{z}, v^*, e^*) = 0 \end{cases} \Leftrightarrow \begin{cases} \langle \vartheta^*, -\Delta y + \bar{g}_y[\cdot]y\rangle = \langle -\bar{\phi}_y[\cdot] - \bar{h}_y[\cdot]e^*, y\rangle \ \forall y \in Y, \\ \varphi(\bar{w}) + 2\psi(\bar{w})\bar{u} - \vartheta^* + \lambda e^* = 0. \end{cases} \tag{4.4}$$

Here $\langle\cdot,\cdot\rangle$ denotes the scalar product in $L^2(\Omega)$. Let $A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ be a linear operator defined by

$$Ay = -\Delta y + \bar{g}_y[\cdot]y \text{ for } y \in \mathcal{D}(A) \text{ with } \mathcal{D}(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega).$$

We denote by A^* is the adjoint operator of A and by $\mathcal{D}(A^*)$ the domain of A^*.

Lemma 4.2. *$\vartheta^* \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and $e^* \in L^2(\Omega)$ which satisfy*

$$-\Delta\vartheta^* + \bar{g}_y[\cdot]\vartheta^* = -\bar{\phi}_y[\cdot] - \bar{h}_y[\cdot]e^* \text{ in } \Omega, \ \vartheta^* = 0 \text{ on } \partial\Omega, \tag{4.5}$$

$$\varphi(\bar{w}) + 2\psi(\bar{w})\bar{u} - \vartheta^* + \lambda e^* = 0. \tag{4.6}$$

Moreover $\Lambda(\bar{z})$ is singleton and $\Lambda(\bar{z}) = \{(\vartheta^*, e^*)\}$ satisfies (2.3)–(2.5).

Proof. Relation (4.6) is the second condition of (4.4). It remains to prove (4.5). Since $-\bar{\phi}_y[\cdot] - \bar{h}_y[\cdot]e^* \in L^2(\Omega)$, we have from the first relation in (4.4) that $\vartheta^* \in \mathcal{D}(A^*)$. Let us claim that $\mathcal{D}(A) = \mathcal{D}(A^*) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. It is sufficient to prove that $\mathcal{D}(A^*) \subseteq \mathcal{D}(A)$.

By Green's formula, we can show that A is symmetric, that is

$$\langle Ay, \tilde{y}\rangle = \langle y, A\tilde{y}\rangle \ \forall y, \tilde{y} \in \mathcal{D}(A). \tag{4.7}$$

We now take any $\vartheta \in \mathcal{D}(A^*)$. By definition of $\mathcal{D}(A^*)$, there exists $\xi_1 \in L^2(\Omega)$ such that $\langle Ay, \vartheta\rangle = \langle y, \xi_1\rangle$ for all $y \in \mathcal{D}(A)$. Since $\xi_1 \in L^2(\Omega)$ and $\bar{g}_y[x] \geq 0$, [11, Theorem 2.4.2.5, p. 124] implies that there exists $y_1 \in \mathcal{D}(A)$ such that $Ay_1 = \xi_1$. Also, take any $\xi \in L^2(\Omega)$. Then there is $y \in \mathcal{D}(A)$ such that $Ay = \xi$. From this and (4.7) we have

$$\begin{aligned}\langle \xi, \vartheta - y_1\rangle &= \langle Ay, \vartheta\rangle - \langle Ay, y_1\rangle = \langle y, \xi_1\rangle - \langle Ay, y_1\rangle \\ &= \langle y, \xi_1\rangle - \langle y, Ay_1\rangle = \langle y, \xi_1\rangle - \langle y, \xi_1\rangle = 0.\end{aligned}$$

Since ξ is arbitrary in $L^2(\Omega)$, we have $\vartheta - y_1 = 0$ and so $\vartheta = y_1 \in \mathcal{D}(A)$. The claim is justified. Since $\vartheta^* \in \mathcal{D}(A^*)$ and by definition of A^*, we have for any $\vartheta \in C_0^\infty(\Omega) \subset \mathcal{D}(A)$ that

$$\begin{aligned}\langle A^*\vartheta^*, \vartheta\rangle &= \langle \vartheta^*, A\vartheta\rangle = \langle \vartheta^*, -\Delta\vartheta + \bar{g}_y[\cdot]\vartheta\rangle \\ &= \langle -\Delta\vartheta^* + \bar{g}_y[\cdot]\vartheta^*, \vartheta\rangle.\end{aligned}$$

It follows that $A^*\vartheta^* = -\Delta\vartheta^* + \bar{g}_y[\cdot]\vartheta^*$ and the first relation of (4.4) is equivalent to

$$-\Delta\vartheta^* + \bar{g}_y[\cdot]\vartheta^* = -\bar{\phi}_y[\cdot] - \bar{h}_y[\cdot]e^*.$$

It remains to show that $\Lambda(\bar{z}) = \{\vartheta^*, e^*)\}$. Assume that there is another couple $(\vartheta_1^*, e_1^*) \in \Lambda(\bar{z})$ satisfying (4.5) and (4.6). Then we have

$$\begin{aligned} &-\Delta(\vartheta^* - \vartheta_1^*) + \bar{g}_y[\cdot](\vartheta^* - \vartheta_1^*) = -\bar{h}_y[\cdot](e^* - e_1^*), \\ &-(\vartheta^* - \vartheta_1^*) + \lambda(e^* - e_1^*) = 0. \end{aligned}$$

This implies that

$$-\Delta(\vartheta^* - \vartheta_1^*) + \frac{\lambda\bar{g}_y[\cdot] + \bar{h}_y[\cdot]}{\lambda}(\vartheta^* - \vartheta_1^*) = 0.$$

By $(H3)$ and $(H4)$, $\dfrac{\lambda\bar{g}_y[\cdot] + \bar{h}_y[\cdot]}{\lambda} \geq 0$. By taking the scalar product both sides of the above equation with $\vartheta^* - \vartheta_1^*$ in $L^2(\Omega)$, we obtain $\vartheta^* = \vartheta_1^*$. Hence $e^* = e_1^*$ and the proof of the lemma is complete. □

• Complete proof of Theorem 2.3.

So far, we have shown that assumptions $(A1) - (A5)$ are fulfilled, $\bar{z}$ is a regular point of $P(\bar{w})$ and $\Lambda(\bar{z})$ consists of vectors (ϑ^*, e^*) satisfying (2.3)–(2.5). Besides, we have

$$\begin{aligned} J_{uu}(\bar{z}, \bar{w})(u, u) &= \int_\Omega 2\psi(\bar{w}(x))u^2(x)dx \geq 2\gamma\|u\|^2_{L^2(\Omega)}, \\ \langle\vartheta^*, \nabla_z^2 G(\bar{z}, \bar{w})d^2\rangle &= \int_\Omega \vartheta^*(x)\bar{g}_{yy}[x]y^2(x)dx \\ \langle e*, \nabla_z^2 H(\bar{z}, \bar{w})d^2\rangle &= \int_\Omega e^*(x)\bar{h}_{yy}[x]y^2(x)dx \end{aligned}$$

with $d = (y, u)$. Since $Y \hookrightarrow C(\bar{\Omega})$ is compact, we see that

$$\begin{aligned} \langle\vartheta^*, \nabla_z^2 G(\bar{z}, \bar{w})d_k^2\rangle &\to \langle\vartheta^*, \nabla_z^2 G(\bar{z}, \bar{w})d^2\rangle, \\ \langle e^*, \nabla_z^2 H(\bar{z}, \bar{w})d_k^2\rangle &\to \langle e^*, \nabla_z^2 H(\bar{z}, \bar{w})d^2\rangle \end{aligned}$$

whenever $d_k = (y_k, u_k) \rightharpoonup d = (y, u)$. Therefore, the conclusion of Theorem 2.3 is obtained from Theorem 3.8. The proof of Theorem 2.3 is complete. □

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