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A NEW SCALARIZING FUNCTIONAL IN SET OPTIMIZATION WITH RESPECT TO VARIABLE DOMINATION STRUCTURES

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Abstract. We introduce a new nonlinear scalarizing functional in set optimization with respect to variable domination structures. By means of this functional, we characterize solutions of set optimization problems, where the solution concept is given by the set approach. We also investigate the relationship between the well-posedness property of a set-valued problem and the Tykhonov well-posedness property of the scalarized problem by means of the proposed scalarizing functional. Also, two classes of well-posed set optimization problems with respect to variable domination structures are identified. Finally, we apply our results to uncertain vector optimization problems.

1. Introduction

Set optimization has developed as an extension of vector optimization where the objective map is a set-valued map acting between abstract spaces. It is useful in various applications ranging from economics and engineering to medicine and thus has an essential role in optimization. For an overview and more detailed investigations, we refer the reader to [26].

Although there are different solution concepts for the set optimization problem, nowadays it seems more appropriate when one works with a more natural approach (called set approach) which has been introduced by Kuroiwa in [27, 28]. Several authors have already investigated this problem equipped with a constant cone, see [17, 18, 26] and references therein. Recently, Köbis [33], Durea, Strugariu and Tammer [7], Eichfelder and Pilecka [12] have studied set optimization problems with respect to (w.r.t.) variable ordering structures. They also provided scalarization results for obtaining optimality conditions for the solutions of these problems. These results extend an approach given by Jahn [22] to the variable domination structure. A nonlinear scalarization is also introduced in [13] when the images of ordering maps are Bishop-Phelps cones. Köbis, Le and Tammer [32] introduced nonlinear scalarizing methods to characterize several set relations and minimal solutions for set-valued problems w.r.t. general domination structures as well.

The main purpose of this paper is investigating the well-posedness property for set-valued optimization problems w.r.t. variable domination structures. This well-posedness property is studied by many authors in the literature not only for vector optimization but also for set optimization w.r.t. a fixed ordering structure (see for instance [6,18,35–37]). In this paper, we introduce a new nonlinear scalarizing functional modified from the well-known Gerstewitz functional [16] for a set optimization

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problem w.r.t. variable domination structures. We use a set relation generalizing the lower set less order which has been used widely in the literature and applied in many practical problems, see [17, 18, 35, 40]. Our approach can be considered as an extension of [18] in which the authors studied Gerstewitz scalarization for set optimization problems equipped with constant cones. However, in this paper, the assumption of cone-proper sets in [18] is relaxed. In addition, our new functional will avoid a drawback of nonlinear scalarizing functionals given in [32], that is, the values of these functionals at the minimal point are not necessarily zero. This is also beneficial for us to prove the equivalence between well-posedness property of a set-valued problem and the Tykhonov well-posedness property of a scalar problem. Moreover, one can find some classes of pointwise well-posedness sets for set optimization based on this equivalence.

The paper is organized as follows. Section 2 presents properties of the variable generalized lower set less relation, which will be concerned throughout our work. In Section 3, we introduce a new nonlinear scalarizing functional and present various important properties of this functional. By means of this functional, we characterize minimal solutions for a family of sets in Section 4. In Section 5, we prove the equivalence between the well-posedness property of a set-valued optimization problem and the Tykhonov well-posedness property of a scalar problems in which the objective map of the original problem is involved. Also, we identify two classes of pointwise well-posed set optimization problems w.r.t. a cone-valued ordering structure. An application in uncertain optimization is considered in the last section of this paper.

2. Preliminaries

Throughout this paper, let Y be a linear topological space and let

$$\mathcal{P}(Y) := \{A \subseteq Y | A \text{ is nonempty}\}$$

denote the power set of Y without the empty set. A set $Q \subset Y$ is a cone, if for every $q \in Q$ and $\lambda \geq 0$ it holds that $\lambda q \in Q$. A cone Q is convex, if $Q + Q \subseteq Q$. In addition, a set Q is pointed, if $Q \cap (-Q) = \{0\}$, and a set Q is proper, if $Q \neq Y$, $Q \neq \{0\}$ and $Q \neq \emptyset$. For a subset A of Y, we denote by $\operatorname{cl} A$ the closure of A and by $\operatorname{int} A$ the interior of A. A is called Q-bounded if for each neighborhood U of zero in Y there exists a constant $r > 0$ such that $A \subseteq rU + Q$. For every $A, A_1, A_2 \in \mathcal{P}(Y)$ and $\lambda \in \mathbb{R}$ we denote

$$A_1 + A_2 = \{a_1 + a_2 | a_1 \in A_1, a_2 \in A_2\},\ \lambda A = \{\lambda a | a \in A\}.$$

In addition, $A + \emptyset = \emptyset + A = A$, $\lambda\emptyset = \emptyset$, and for convenience we write $y + A$ instead of $\{y\} + A$ for all $y \in Y$. Let $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued mapping. For each $A \in \mathcal{P}(Y)$, we set $\mathcal{K}(A) := \bigcup\limits_{a \in A} (a + \mathcal{K}(a))$. Let X be a linear space, $\emptyset \neq S \subseteq X$ and let the set-valued mapping $F : S \rightrightarrows Y$ be given. We denote

$$\operatorname{Dom} F := \{x \in S \mid F(x) \neq \emptyset\},\ \operatorname{Im} F := \{F(x) \mid x \in \operatorname{Dom} F\}.$$

Definition 2.1. Let $A, B, C \in \mathcal{P}(Y)$ and a binary relation $\preceq$ be given. $\preceq$ is said to be

(i) reflexive, if $A \preceq A$.

(ii) transitive, if $A \preceq B$, $B \preceq C$ implies $A \preceq C$.
(iii) symmetric, if $A \preceq B$ implies $B \preceq A$.
(iv) antisymmetric, if $A \preceq B$, $B \preceq A$ implies $A = B$.

There exist various types of set order relations which were introduced by Kuroiwa ([27, 28]), Jahn and Ha ([23]), Kuroiwa, Tanaka and Ha ([29]). Although each of these set order relations serves its own purpose in different applications, in this paper, we are concerned with the generalized lower set less relation introduced in [34]. The reason of this choice is that the lower set less order is used widely in many references concerning set-valued optimization (see, for instance, [26] and references therein). In addition, this relation is very important in applications, as it can be used by the decision maker for obtaining solutions of an uncertain multi-objective optimization problem ([20,21]). The following definition is concerned with the generalized lower set relations w.r.t. a constant set.

Definition 2.2 (Generalized Lower Set Less Relation, [34])**.** Let $Q \in \mathcal{P}(Y)$. Then the generalized lower set less relation for two sets $A, B \in \mathcal{P}(Y)$ is given by

$$\begin{aligned} A \preceq_l^Q B &\iff \forall\, b \in B\ \exists\, a \in A : b \in a + Q \\ &\iff B \subseteq A + Q. \end{aligned}$$

Notice that $\preceq_l^Q$ is reflexive if $0 \in Q$ and it is transitive if $Q + Q \subseteq Q$. If the set Q is replaced by a convex cone in Y, then this definition reduces to the definition of the lower set less order given by Kuroiwa ([27, 28]). Now, we recall the generalized lower set less order relation, which is introduced by Eichfelder and Pilecka [12], namely the *l-less order relation of type* $\mathcal{D}$, denoted by $\preceq_l^{\mathcal{D}}$, where $\mathcal{D}$ is a set-valued map from Y to Y. This relation is also further discussed in [31, 32].

Definition 2.3 (Variable generalized lower set less relation)**.** Let $A, B \in \mathcal{P}(Y)$ and let $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map. The **variable generalized lower set less relation** $\preceq_l^{\mathcal{K}}$ is defined as

$$A \preceq_l^{\mathcal{K}} B \iff \forall\, b \in B,\ \exists\, a \in A : b \in a + \mathcal{K}(a) \tag{2.1}$$

Note that (2.1) can be written by $B \subseteq \cup_{a \in A}(a + \mathcal{K}(a))$. For $A, B, C \in \mathcal{P}(Y)$, we write $A \not\preceq_l^{\mathcal{K}} B$ if $B \not\subseteq \cup_{a \in A}(a + \mathcal{K}(a))$, $A \sim B$ if $A \preceq_l^{\mathcal{K}} B$ and $B \preceq_l^{\mathcal{K}} A$.

In the following, we prove some properties of the relation $\preceq_l^{\mathcal{K}}$ given in Definition 2.3. Note that the assertions (i) and (ii) are presented in [12, Lemma 4.1] for the case $\mathcal{K} : Y \rightrightarrows Y$ is a cone-valued map, whereas (iii) and (iv) are given in [11] without proof.

Theorem 2.4. *The relation* $\preceq_l^{\mathcal{K}}$ *satisfies the following properties:*

(i) $\preceq_l^{\mathcal{K}}$ *is reflexive, if*

$$0 \in \mathcal{K}(y) \text{ for all } y \in Y. \tag{2.2}$$

(ii) $\preceq_l^{\mathcal{K}}$ *is transitive, if for all* $y \in Y$ *and for all* $d \in \mathcal{K}(y)$*, it holds that*

$$\mathcal{K}(y + d) \subseteq \mathcal{K}(y) \tag{2.3}$$

$$\text{and } \mathcal{K}(y) + \mathcal{K}(y) \subseteq \mathcal{K}(y). \tag{2.4}$$

(iii) *Suppose that $A, B \in \mathcal{P}(Y)$ and $\mathcal{K}$ satisfies the condition (2.3). Then,*

$$A \preceq_l^{\mathcal{K}} B \quad \Longrightarrow \quad \cup_{b\in B}\mathcal{K}(b) \subseteq \cup_{a\in A}\mathcal{K}(a).$$

(iv) *Suppose that $A, B, C \in \mathcal{P}(Y)$. Then,*

$$A \preceq_l^{\mathcal{K}} B \text{ and } A \subset C \quad \Longrightarrow \quad C \preceq_l^{\mathcal{K}} B.$$

Proof. (i) Let $A \in \mathcal{P}(Y)$. We have that

$$A = \cup_{a\in A}(a + 0) \subseteq \cup_{a\in A}(a + \mathcal{K}(a)).$$

Thus $A \preceq_l^{\mathcal{K}} A$, i.e., $\preceq_l^{\mathcal{K}}$ is reflexive.

(ii) Suppose that $A, B, C \in \mathcal{P}(Y)$ satisfying $A \preceq_l^{\mathcal{K}} B$ and $B \preceq_l^{\mathcal{K}} C$. The definition of $\preceq_l^{\mathcal{K}}$ implies that

$$C \subseteq \cup_{b\in B}(b + \mathcal{K}(b)) \text{ and } B \subseteq \cup_{a\in A}(a + \mathcal{K}(a)).$$

Choose $b \in B$ arbitrarily. This yields that there exists $a_b \in A$ such that $b = a_b + d$ with some $d \in \mathcal{K}(a_b)$. We have that

$$\begin{aligned} b + \mathcal{K}(b) &= a_b + d + \mathcal{K}(a_b + d) \\ &\subseteq a_b + \mathcal{K}(a_b) + \mathcal{K}(a_b + d) \\ &\subseteq a_b + \mathcal{K}(a_b) + \mathcal{K}(a_b) \\ &\subseteq a_b + \mathcal{K}(a_b). \end{aligned}$$

Therefore,

$$C \subseteq \cup_{b\in B}(b + \mathcal{K}(b)) \subseteq \cup_{a\in A}(a + \mathcal{K}(a)).$$

This means that $A \preceq_l^{\mathcal{K}} C$ and the transitivity of $\preceq_l^{\mathcal{K}}$ is satisfied.

(iii) Assume that $A, B \in \mathcal{P}(Y)$ such that $A \preceq_l^{\mathcal{K}} B$. We get that $B \subseteq \cup_{a\in A}(a + \mathcal{K}(a))$ i.e., for each $b \in B$, there is $a_b \in A$ satisfying $b = a_b + d$ where $d \in \mathcal{K}(a_b)$. Since $\mathcal{K}(\cdot)$ satisfies (2.3), we get $\mathcal{K}(b) = \mathcal{K}(a_b + d) \subseteq \mathcal{K}(a_b) \subseteq \cup_{a\in A}\mathcal{K}(a)$.
Thus $\cup_{b\in B}\mathcal{K}(b) \subseteq \cup_{a\in A}\mathcal{K}(a)$, which is the desired conclusion.

(iv) Obviously, we have that $B \subseteq \cup_{a\in A}(a + \mathcal{K}(a)) \subseteq \cup_{c\in C}(c + \mathcal{K}(c))$. Therefore, $B \subseteq \cup_{c\in C}(c + \mathcal{K}(c))$, i.e., $C \preceq_l^{\mathcal{K}} B$.

□

3. Set optimization with respect to variable domination structures

3.1. **Optimality definitions.** We begin this section with the definition of minimal elements of a family of sets by using the relation $\preceq_l^{\mathcal{K}}$ given in the previous part. The notion 'minimal' in set optimization is introduced by Kuroiwa [28] for the case $\mathcal{K}(\cdot) = K$ where K is a fixed, solid (i.e., $\operatorname{int} K \neq \emptyset$), pointed, convex cone, while the notion 'strict minimizer' is given by Ha [19] for the case $\mathcal{K}(\cdot) = K$ where K is a fixed, pointed, closed, convex cone in Y.

Definition 3.1. Let $\mathcal{A}$ be a family of nonempty subsets of Y. Let $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map satisfying (2.2).

(a) A set $\bar{A} \in \mathcal{A}$ is called a minimal element of $\mathcal{A}$ w.r.t. $\preceq_l^{\mathcal{K}}$, if

$$A \in \mathcal{A}, A \preceq_l^{\mathcal{K}} \bar{A} \quad \Longrightarrow \quad \bar{A} \preceq_l^{\mathcal{K}} A.$$

(b) A set $\bar{A} \in \mathcal{A}$ is called a strictly minimal element of $\mathcal{A}$ w.r.t. $\preceq_l^{\mathcal{K}}$, if

$$A \in \mathcal{A}, A \preceq_l^{\mathcal{K}} \bar{A} \quad \Longrightarrow \quad \bar{A} = A.$$

The set of all minimal and strictly minimal elements of $\mathcal{A}$ w.r.t. $\preceq_l^{\mathcal{K}}$ is denoted by $\mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ and $\mathrm{SMin}(\mathcal{A}, \preceq_l^{\mathcal{K}})$, respectively.

Remark 3.2. Obviously, if $\bar{A} \in \mathrm{SMin}(\mathcal{A}, \preceq_l^{\mathcal{K}})$, then $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$, but the inverse relation is not true in general, see [32] for more details. In addition, when $\mathcal{K}$ is a constant pointed cone, both of these notions reduce to the classical concept of minimality in vector optimization if we consider $\mathcal{A} = \{\{y\}, y \in M\}$ for some $M \subseteq Y$.
Observe that if $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ and if $\preceq_l^{\mathcal{K}}$ is transitive, then for all $A' \in \mathcal{A}$ with $A' \sim \bar{A}$, we have $A' \in \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$. Moreover, if $\bar{A} \in \mathrm{SMin}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ and $\bar{B} \sim \bar{A}$, it holds that $\bar{A} = \bar{B}$.

The following proposition is generated directly from Definition 3.1 and its proof is therefore skipped.

Proposition 3.3. *Let $\mathcal{A}$ be a family of nonempty subsets of Y and $\bar{A}, A \in \mathcal{A}$ be given. Then, the following statements are equivalent:*

(i) $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$.
(ii) $A \in \mathcal{A},\ \bar{A} \not\preceq_l^{\mathcal{K}} A \quad \Longrightarrow \quad A \not\preceq_l^{\mathcal{K}} \bar{A}$.
(iii) $A \in \mathcal{A},\ A \preceq_l^{\mathcal{K}} \bar{A} \quad \Longrightarrow \quad A \sim \bar{A}$.

Let us now introduce a set-valued optimization problem equipped with $\preceq_l^{\mathcal{K}}$. Assume that X, Y are linear topological spaces and $S \subset X$. We consider a set-valued map $F : X \rightrightarrows Y$ with $F(x) \neq \emptyset$ for all $x \in S$ and a set-valued map $\mathcal{K} : Y \rightrightarrows Y$ such that the relation $\preceq_l^{\mathcal{K}}$ is reflexive. We denote by (P) the minimization problem

$$\text{(P)} \qquad \mathcal{K} - \min_{x \in S} F(x)$$

and define the optimal solutions of (P) in the following way.

Definition 3.4. (i) A point $x^0 \in S$ is called a minimal solution of the set-valued problem (P) w.r.t. $\preceq_l^{\mathcal{K}}$, if $F(\bar{x})$ is a minimal element of the family $\{F(x)\}_{x \in S}$ i.e.,

$$x \in S, F(x) \preceq_l^{\mathcal{K}} F(\bar{x}) \quad \Longrightarrow \quad F(\bar{x}) \preceq_l^{\mathcal{K}} F(x).$$

(ii) A point $x^0 \in S$ is called a strictly minimal solution of the set-valued problem (P) w.r.t. $\preceq_l^{\mathcal{K}}$, if

$$x \in S, F(x) \preceq_l^{\mathcal{K}} F(\bar{x}) \quad \Longrightarrow \quad x = \bar{x}.$$

Remark 3.5. (i) If for all $y \in Y$, $\mathcal{K}(y) \equiv K$, where K is a convex cone in Y, Definition 3.4 reduces to the definition of minimal and strictly minimal solutions of a set-valued problem which are widely used in the literature, see for instance [18, 27, 28, 35].

(ii) Observe that if $\preceq_l^{\mathcal{K}}$ is transitive and x^0 is a minimal solution of (P) w.r.t. $\preceq_l^{\mathcal{K}}$, then so is any $x' \in S$ such that $F(x) \sim F(x^0)$.
If for all $x,\ x' \in S,\ x \neq x'$, it holds that $F(x) \neq F(x')$, then x^0 is a strictly

minimal solution of (P) if and only if $F(x^0)$ is a strictly minimal element of the family set $\{F(x)\}_{x\in S}$ w.r.t. $\preceq_l^{\mathcal{K}}$.

The two above concepts of solutions for (P) have the following relationship.

Proposition 3.6. *If x^0 is a strictly minimal solution of (P) w.r.t. $\preceq_l^{\mathcal{K}}$, then x^0 is a minimal solution of (P) w.r.t. $\preceq_l^{\mathcal{K}}$.*

Proof. The result follows from Definition 3.4. □

3.2. **Scalarization in set optimization w.r.t variable domination structures.** The aim of this section is to apply the scalarization technique for deriving some properties of solutions of a set optimization problem equipped with variable domination structures. We first introduce a nonlinear scalarizing functional of the set-valued map and then we study the characterization of solutions of a set-valued optimization problem w.r.t. a variable domination structure.

Let $Q \subset Y$ be a proper, closed set and $k^0 \in Y \setminus \{0\}$ satisfying the condition

$$Q + [0, +\infty)k^0 \subseteq Q. \tag{3.1}$$

The following nonlinear scalarization functional (see [16] and [26] for an overview) has been widely applied in vector optimization.
For Q and k^0 satisfying (3.1), let $z_{Q,k^0} : Y \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ be defined by

$$\forall\, y \in Y : \quad z_{Q,k^0}(y) = \inf\{t \in \mathbb{R} | y \in tk^0 - Q\}. \tag{3.2}$$

Important properties of the functional z_{Q,k^0} can be found in [15] and [26]. In the literature, several authors have extended this scalarizing functional to set optimization equipped with a constant cone where the objective space $\mathcal{P}(Y)$ is ordered by the lower set less order relation (see, for instance, [17, 18] and references therein). There are also many applications of these extensions in investigating well-posedness for set optimization w.r.t. a fixed cone, see [18, 35, 40]. Dealing with the case that the domination structure is variable, recently, Bouza and Tammer in [3] also have introduced a scalarizing functional to characterize and compute minimal points of a subset of a Banach space where the domination structure is given by a set-valued mapping. In addition, the authors in [32] have used the functional (3.2) for set optimization w.r.t. domination structures for several set relations. As indicated in the previous part, it is necessary to introduce a new scalarizing functional to study the well-posedness property of set-valued optimization problems equipped with variable domination structures such that this property is equivalent to that of a scalar problem in which the new functional is involved. This situation leads to introduce a new scalarizing functional in the following way.

Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map. For each $k^0 \in Y \setminus \{0\}$ satisfying

$$\forall y \in Y : [0, +\infty)k^0 + \mathcal{K}(y) \subseteq \mathcal{K}(y), \tag{3.3}$$

we consider a scalarizing functional $\varphi_{k^0} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi_{k^0}(A, B) = \inf\{t \geq 0 \mid A \preceq_l^{\mathcal{K}} tk^0 + B\}, \tag{3.4}$$

where $\inf(\emptyset) = +\infty$.

If there is no confusion, from now, for k^0 satisfying (3.3) and $B \in \mathcal{P}(Y)$ fixed, we write

$$\varphi_{k^0,B}(A) := \varphi_{k^0}(A,B) = \inf\{t \geq 0 \mid A \preceq_l^{\mathcal{K}} tk^0 + B\}. \tag{3.5}$$

Remark 3.7. Obviously, when $A = \{y\}$, $B = \{0\}$ and $\mathcal{K}(z) = Q$ for all $z \in Y$, instead of taking the infimum over $t \in \mathbb{R}_+$ in (3.5), we take the infimum over $t \in \mathbb{R}$ of the set $\{A \preceq_l^{\mathcal{K}} tk^0 + B\}$ to receive the value $z_{Q,k^0}(y)$ determined by (3.2).

It is important to mention that, if $A \in \mathcal{P}(Y)$, $B = \{y\}$, and for every $z \in Y$, $\mathcal{K}(z) = \{0\}$, then the scalarizing functional given by (3.4) becomes the directional minimal time function

$$T_{k^0}(A,y) := \varphi_{k^0}(A,y) = \inf\{t \geq 0 \mid tk^0 + y \in A\}, \tag{3.6}$$

which is introduced by Nam and Zălinescu in [38]. The functional (3.6) is called **directional minimal time function**. In addition, functional (3.6) has an interesting application in locational analysis, see [38] for more details. Recently, Durea, Pantiruc and Strugariu [8] have generalized the functional (3.6) to the case of a set of directions. As for the functional (3.4), we illustrate in the following another application in location problems of the functional $\varphi_{k^0,B}(A)$, where B is a fixed singleton set, $B = \{y\}$, and some uncertain conditions are involved.
Suppose that $A_1, ..., A_n$ be n concerned destinations to which the producer, which is denoted by the vector $y \in Y$, wants to deliver some products (clothes, food, furniture,...). Each destination A_i has its direction k^i, where $i \in \{1, 2, ..., n\}$. Assume that $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued mapping which describes the changes acting on each point $z \in Y$ during the considered time. These changes often appear in many practical problems, for instance, traffic jams, renovation plans, weather conditions and so on. We suppose that the relation $y + tk^i \in \bigcup\limits_{a \in A_i} (a + \mathcal{K}(a))$, i.e., $A_i \preceq_l^{\mathcal{K}} y + tk^i$, means that the producer y delivers the products to the target A_i successfully, where $i \in \{1, 2, ..., n\}$. Then the problem of finding the point $y \in \Omega$ such that the total time for the vector y to deliver products to the target sets $\{A_1, .., A_n\}$ can be modeled as follows

$$\text{Minimize} \sum_{i=1,...,n} \varphi_{k^i}(A_i, y) \text{ subject to } y \in \Omega.$$

We call $\varphi_{k^0,B}$ $\preceq_l^{\mathcal{K}}$-monotone if

$$A_1, A_2 \in \mathcal{P}(Y), A_1 \preceq_l^{\mathcal{K}} A_2 \quad \Longrightarrow \quad \varphi_{k^0,B}(A_1) \leq \varphi_{k^0,B}(A_2).$$

In the following theorem, we present several properties of the functional $\varphi_{k^0,B}$ given by (3.5).

Theorem 3.8. *Let $A, A_1, A_2, B \in \mathcal{P}(Y)$ and the set-valued map $\mathcal{K} : Y \rightrightarrows Y$ be given. Suppose that $k^0 \in Y \backslash \{0\}$ such that* (3.3) *holds. Then, the following properties of the functional $\varphi_{k^0,B}$ are satisfied.*

(a) *If $\mathcal{K}(\cdot)$ satisfies the conditions (2.3) and (2.4), then*

$$\varphi_{k^0,B} \text{ is } \preceq_l^{\mathcal{K}} \text{-monotone.}$$

In addition,

$$A_1 \sim A_2 \quad \Longrightarrow \quad \varphi_{k^0,B}(A_1) = \varphi_{k^0,B}(A_2).$$

(b) *If* $\mathcal{K}(y + tk^0) = \mathcal{K}(y)$ *for all* $y \in Y$ *and* $t \in \mathbb{R}$, *then* $\varphi_{k^0,B}(A + rk^0) = \varphi_{k^0,B}(A) + r$ *for all* $r \in \mathbb{R}_+$.

(c) *For all* $r \in \mathbb{R}_+$, *it holds that*

$$\varphi_{k^0,B}(A) \leq r \quad \Longleftrightarrow \quad \cup_{t>r}(tk^0 + B) \subseteq \cup_{a\in A}(a + \mathcal{K}(a)).$$

(d) *If* $\mathcal{K}(\cdot)$ *satisfies (2.2), then* $\varphi_{k^0,B}(B)$=0.

(e) *Suppose that for all* $A \in \mathcal{P}(Y)$ *the set* $\cup_{a\in A}(a + \mathcal{K}(a))$ *is closed and* $\mathcal{K}(\cdot)$ *satisfies (2.2), (2.3) and (2.4). Then*

$$\varphi_{k^0,B}(A)=0 \quad \Longleftrightarrow \quad A \preceq_l^{\mathcal{K}} B.$$

(f) *Let* $A, B \in \mathcal{K}(Y)$. *Suppose that* $\mathcal{K}(\cdot)$ *satisfies (2.2), (2.3) and (2.4). Then*

$$A \sim B \quad \Longleftrightarrow \quad \cup_{a\in A}(a + \mathcal{K}(a)) = \cup_{b\in B}(b + \mathcal{K}(b)).$$

(g) *If* B *is* $\mathcal{K}(A)$*-bounded and for all* $r > 0$ *it holds that* $r \operatorname{int} \mathcal{K}(A) + \mathcal{K}(A) \subseteq \mathcal{K}(A)$, *then* $\varphi_{k^0,B}(A) < +\infty$ *for all* $k^0 \in \operatorname{int} \mathcal{K}(A)$.

Proof. (a) Let $A_1, A_2 \in \mathcal{P}(Y)$ such that $A_1 \preceq_l^{\mathcal{K}} A_2$. It is sufficient to prove that

$$\{t \in \mathbb{R}_+ | A_1 \preceq_l^{\mathcal{K}} tk^0 + B\} \supseteq \{t \in \mathbb{R}_+ | A_2 \preceq_l^{\mathcal{K}} tk^0 + B\}.$$

The above assertion is obvious if $\{t \in \mathbb{R}_+ | A_2 \preceq_l^{\mathcal{K}} tk^0 + B\} = \emptyset$. Now we consider the case $\{t \in \mathbb{R}_+ | A_2 \preceq_l^{\mathcal{K}} tk^0 + B\} \neq \emptyset$. Let $t \in \mathbb{R}_+$ such that $A_2 \preceq_l^{\mathcal{K}} tk^0 + B$. This implies $tk^0 + B \subseteq \cup_{a\in A_2}(a + \mathcal{K}(a))$, i.e., for arbitrary $b \in B$, there exists $a_b^2 \in A_2$ satisfying $tk^0 + b \in a_b^2 + \mathcal{K}(a_b^2)$. Since $A_1 \preceq_l^{\mathcal{K}} A_2$ and $a_b^2 \in A_2$, we obtain $\exists\ a_b^1 \in A_1$ such that $a_b^2 \in a_b^1 + \mathcal{K}(a_b^1)$, i.e., $\exists\ d_1 \in \mathcal{K}(a_b^1)$ satisfies $a_b^2 = a_b^1 + d_1$. We have that

$$\begin{aligned} tk^0 + b \in a_b^1 + d_1 + \mathcal{K}(a_b^1 + d_1) &\subseteq a_b^1 + \mathcal{K}(a_b^1) + \mathcal{K}(a_b^1 + d_1) \\ &\subseteq a_b^1 + \mathcal{K}(a_b^1) \subseteq \cup_{a\in A_1}(a + \mathcal{K}(a)). \end{aligned}$$

Therefore,

$$\begin{aligned} & tk^0 + B \subseteq \cup_{a\in A_1}(a + \mathcal{K}(a)) \\ \Longrightarrow\ & A_1 \preceq_l^{\mathcal{K}} tk^0 + B \\ \Longleftrightarrow\ & t \in \{t \in \mathbb{R}_+ | A_1 \preceq_l^{\mathcal{K}} tk^0 + B\}. \end{aligned}$$

Taking into account that t be arbitrarily chosen in $\mathbb{R}_+$ and $A_2 \preceq_l^{\mathcal{K}} tk^0 + B$, it holds that

$$\begin{aligned} & \{t \in \mathbb{R}_+ | A_1 \preceq_l^{\mathcal{K}} tk^0 + B\} \supseteq \{t \in \mathbb{R}_+ | A_2 \preceq_l^{\mathcal{K}} tk^0 + B\} \\ \Longrightarrow\ & \inf\{t \in \mathbb{R}_+ | A_1 \preceq_l^{\mathcal{K}} tk^0 + B\} \leq \inf\{t \in \mathbb{R}_+ | A_2 \preceq_l^{\mathcal{K}} tk^0 + B\} \\ \Longleftrightarrow\ & \varphi_{k^0,B}(A_1) \leq \varphi_{k^0,B}(A_2), \end{aligned}$$

i.e., $\varphi_{k^0,B}$ is $\preceq_l^{\mathcal{K}}$-monotone.
Now, we prove the second assertion. Suppose that $A_1 \sim A_2$, by the observation that

$$A_1 \sim A_2 \iff A_1 \preceq_l^{\mathcal{K}} A_2 \text{ and } A_2 \preceq_l^{\mathcal{K}} A_1,$$

and taking into account the $\preceq_l^{\mathcal{K}}$ $-$monotonicity of $\varphi_{k^0,B}$, it holds that

$$\varphi_{k^0,B}(A_1) \leq \varphi_{k^0,B}(A_2) \text{ and } \varphi_{k^0,B}(A_2) \leq \varphi_{k^0,B}(A_1), \text{ respectively.}$$

Hence, $\varphi_{k^0,B}(A_1) = \varphi_{k^0,B}(A_2)$.

(b) We prove that

$$\{t \in \mathbb{R}_+ | A \preceq_l^{\mathcal{K}} tk^0 + B\} + r = \{t \in \mathbb{R}_+ | A + rk^0 \preceq_l^{\mathcal{K}} tk^0 + B\}.$$

Let $\hat{t} \in \mathbb{R}_+$ such that $A \preceq_l^{\mathcal{K}} \hat{t}k^0 + B$. It holds that

$$\begin{aligned} & \hat{t}k^0 + B \subseteq \cup_{a\in A}(a + \mathcal{K}(a)) \\ \iff & (\hat{t} + r)k^0 + B \subseteq \cup_{a\in A}(a + rk^0 + \mathcal{K}(a)) \\ \iff & (\hat{t} + r)k^0 + B \subseteq \cup_{a\in A}(a + rk^0 + \mathcal{K}(a + rk^0)) \\ \iff & (\hat{t} + r) \in \{t \in \mathbb{R}_+ | A + rk^0 \preceq_l^{\mathcal{K}} tk^0 + B\}. \end{aligned}$$

Therefore,

$$\{t \in \mathbb{R}_+ | A \preceq_l^{\mathcal{K}} tk^0 + B\} + r = \{t \in \mathbb{R}_+ | A + rk^0 \preceq_l^{\mathcal{K}} tk^0 + B\}.$$

Taking the infimum over $t \in \mathbb{R}_+$, we get

$$\inf\{\{t \in \mathbb{R}_+ | A \preceq_l^{\mathcal{K}} tk^0 + B\} + r\} = \inf\{t \in \mathbb{R}_+ | A + rk^0 \preceq_l^{\mathcal{K}} tk^0 + B\}.$$

This yields

$$\varphi_{k^0,B}(A) + r = \varphi_{k^0,B}(A + rk^0).$$

(c) Suppose that $\varphi_{k^0,B}(A) = u$ and $r \in \mathbb{R}_+$ such that $u \leq r$.
We prove the following assertion

$$\text{for all } t > u : \ tk^0 + B \subseteq \cup_{a\in A}(a + \mathcal{K}(a)).$$

By the definition of infimum and $\varphi_{k^0,B}(A)$, there is $\bar{t}$, $u \leq \bar{t} < t$ such that $A \preceq_l^{\mathcal{K}} \bar{t}k^0 + B$ i.e., $\bar{t}k^0 + B \subseteq \cup_{a\in A}(a + \mathcal{K}(a))$. Therefore,

$$tk^0 + B = \bar{t}k^0 + B + (t - \bar{t})k^0 \subseteq \cup_{a\in A}(a + \mathcal{K}(a)) + (t - \bar{t})k^0.$$

Taking into account (3.3) we get that $\cup_{a\in A}(a+\mathcal{K}(a))+(t-\bar{t})k^0 \subseteq \cup_{a\in A}(a+\mathcal{K}(a))$. This implies $tk^0 + B \subseteq \cup_{a\in A}(a + \mathcal{K}(a))$, i.e., $A \preceq_l^{\mathcal{K}} tk^0 + B$.
Now let $t > r$ arbitrary. Since $r \geq u$, we have that $t > u$ and thus $tk^0 + B \subseteq \cup_{a\in A}(a + \mathcal{K}(a))$. This implies $\cup_{t>r}(tk^0 + B) \subseteq \cup_{a\in A}(a + \mathcal{K}(a))$, which finishes the proof of the necessary condition.
Now we prove the sufficient condition. Assume by contradiction that

$$\cup_{t>r}(tk^0 + B) \subseteq \cup_{a\in A}(a + \mathcal{K}(a)) \text{ and } \varphi_{k^0,B}(A) = v > r.$$

Let $\varepsilon := v - r > 0$ and $v' := r + \frac{\varepsilon}{2}$. We have that

$$v > v' > r \text{ and } v'k^0 + B \subseteq \cup_{a\in A}(a + \mathcal{K}(a)), \ i.e., \ A \preceq_l^{\mathcal{K}} v'k^0 + B.$$

Taking into account the definition of $\preceq_l^{\mathcal{K}}$, it holds that

$$\varphi_{k^0,B}(A) = \inf\{t \in \mathbb{R}_+ \mid A \preceq_l^{\mathcal{K}} tk^0 + B\} \leq v'.$$

Therefore, $\varphi_{k^0,B}(A) \leq v' < v$, a contradiction, and the proof of the sufficient condition is complete.

(d) Obviously, the following relations hold true for all $t > 0$

$$tk^0 + B = \cup_{b\in B}(b + 0 + tk^0) \subseteq \cup_{b\in B}(b + \mathcal{K}(b) + tk^0) \subseteq \cup_{b\in B}(b + \mathcal{K}(b)).$$

Then

$$\cup_{t>0}(tk^0 + B) \subseteq \cup_{b\in B}(b + \mathcal{K}(b)).$$

Taking into account part (c), we get that $\varphi_{k^0,B}(B) \leq 0$. In addition, since the definition of $\varphi_{k^0,B}(B)$, $\varphi_{k^0,B}(B) \geq 0$. Therefore, $\varphi_{k^0,B}(B) = 0$.

(e) The sufficient condition is a consequence of part (a) and part (d). Conversely, if $\varphi_{k^0,B}(A)$=0, by part (c) it holds that

$$\cup_{t>0}(tk^0 + B) \subseteq \cup_{a\in A}(a + \mathcal{K}(a)).$$

Take $b \in B$ arbitrary, it is clear that for all $n > 0$ we have

$$\frac{1}{n}k^0 + b \subseteq \cup_{a\in A}(a + \mathcal{K}(a)).$$

Therefore, taking the limit when $n \to +\infty$ we obtain

$$b \in \mathrm{cl}(\cup_{a\in A}(a + \mathcal{K}(a))) = \cup_{a\in A}(a + \mathcal{K}(a)).$$

Thus, $B \subseteq \cup_{a\in A}(a + \mathcal{K}(a))$, i.e., $A \preceq_l^{\mathcal{K}} B$.

(f) $A \sim B$ implies that $A \preceq_l^{\mathcal{K}} B$, *i.e.*, $B \subseteq \cup_{a\in A}(a+\mathcal{K}(a))$. Let $b \in B$ arbitrary. There exist $a_b \in A$ and $d_b \in \mathcal{K}(a_b)$ such that $b = a_b + d_b$. Since $\mathcal{K}$ satisfies (2.3), $\mathcal{K}(b) = \mathcal{K}(a_b+d_b) \subseteq \mathcal{K}(a_b)$. Taking into account that $\mathcal{K}$ satisfies (2.4), we have

$$\begin{aligned} b + \mathcal{K}(b) &= a_b + d_b + \mathcal{K}(b) \\ &\subseteq a_b + \mathcal{K}(a_b) + \mathcal{K}(a_b) \\ &\subseteq a_b + \mathcal{K}(a_b). \end{aligned}$$

Therefore, $b + \mathcal{K}(b) \subseteq \underset{a\in A}{\cup}(a + \mathcal{K}(a))$. Because b is taken arbitrarily, it holds that $\underset{b\in B}{\cup}(b + \mathcal{K}(b)) \subseteq \underset{a\in A}{\cup}(a + \mathcal{K}(a))$. Similarly, we get $\underset{a\in A}{\cup}(a + \mathcal{K}(a)) \subseteq \underset{b\in B}{\cup}(b + \mathcal{K}(b))$. Therefore, $\underset{a\in A}{\cup}(a + \mathcal{K}(a)) = \underset{b\in B}{\cup}(b + \mathcal{K}(b))$.

Conversely, suppose that $\underset{a\in A}{\cup}(a+\mathcal{K}(a)) = \underset{b\in B}{\cup}(b+\mathcal{K}(b))$. We will prove that $A \sim B$. Since $0 \in \mathcal{K}(y)$ for all $y \in Y$, we have that

$$\begin{aligned} &A \subseteq \underset{a\in A}{\cup}(a + \mathcal{K}(a)) = \underset{b\in B}{\cup}(b + \mathcal{K}(b)) \\ \Rightarrow &A \subseteq \underset{b\in B}{\cup}(b + \mathcal{K}(b)) \\ \Rightarrow &B \preceq_l^{\mathcal{K}} A. \end{aligned}$$

Similarly, $A \preceq_l^{\mathcal{K}} B$, and thus $A \sim B$.

(g) Since B is $\mathcal{K}(A)$-bounded and $\operatorname{int}\mathcal{K}(A) - k^0$ is a neighborhood of 0, there is $r > 0$ such that

$$\begin{aligned} & B \subseteq r(\operatorname{int}\mathcal{K}(A) - k^0) + \mathcal{K}(A) \subseteq -rk^0 + \mathcal{K}(A) \\ \Rightarrow\ & B + rk^0 \subseteq \mathcal{K}(A) = \bigcup_{a\in A}(a + \mathcal{K}(a)) \\ \Rightarrow\ & B + rk^0 \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)) \\ \Rightarrow\ & \varphi_{k^0,B}(A) \leq r, \text{ i.e., } \varphi_{k^0,B}(A) < +\infty. \end{aligned}$$

□

Remark 3.9. (i) Theorem 3.8 (a)-(f) extends [18, Theorem 4.2], where $\mathcal{K}(y)$ is a constant convex cone $K \subset Y$ for all $y \in Y$. Note that even if B is not a K-proper set, i.e., $B+K = Y$, the assertion (d) holds true. However, $B+K \neq Y$ is needed in the proof of [18, Theorem 4.2] to obtain $\varphi_{k^0,B}(B) = 0$.

(ii) Let $A, B \in \mathcal{P}(Y)$ such that $A \sim B$, $\bigcup_{a\in A}(a+\mathcal{K}(a))$ is closed and $\mathcal{K}(\cdot)$ satisfies (2.2), (2.3) and (2.4). Then it holds from Theorem 3.8(e) that $\varphi_{k^0,B}(A)= 0$. In addition, by using the same lines in the proof of Theorem 3.8(e), we get the following assertion for all $\gamma \geq 0$ and $A, B \in \mathcal{P}(Y)$ under the assumption that $\bigcup_{a\in A}(a + \mathcal{K}(a))$ is closed:

$$\varphi_{k^0,B}(A) \leq \gamma \quad \Longleftrightarrow \quad \gamma k^0 + B \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)), \text{ i.e., } A \preceq_l^{\mathcal{K}} \gamma k^0 + B.$$

(iii) If $\mathcal{K}(y) = K$ where K is a convex cone with nonempty interior, $\varphi_{k^0,B}(A) < +\infty$ for all K- bounded set B and $k^0 \in \operatorname{int} K$, see [35, Proposition 3.2]

Remark 3.10. The assumptions (2.3) and (2.4) of $\mathcal{K}(\cdot)$ can be fulfilled when $\mathcal{K}(y)$ is not necessarily given by a cone for all $y \in Y$. For instance, the mapping $\mathcal{K}(\cdot)$ given by

$$\begin{aligned} \mathcal{K} : Y &\rightrightarrows Y \\ \mathcal{K}(y) &= \mathbb{N}y, \end{aligned}$$

where $\mathbb{N}y := \{ny | n \in \mathbb{N}\}$ is not a cone.

In addition, an example for a set-valued map satisfying the condition in Theorem 3.8 (b), which is neither a constant map nor a cone-valued map, can be given as

$$\begin{aligned} \mathcal{K} : Y &\rightrightarrows Y \\ \mathcal{K}(y) &= \mathbb{N}y + \mathbb{R}k^0. \end{aligned}$$

Indeed, we have

$$\begin{aligned} \forall\ t \in \mathbb{R} : \mathcal{K}(y + tk^0) &= \mathbb{N}(y + tk^0) + \mathbb{R}k^0 \\ &= \mathbb{N}y + \mathbb{R}k^0 \\ &= \mathcal{K}(y). \end{aligned}$$

Therefore, $\mathcal{K}(y + tk^0) = \mathcal{K}(y)$ for all $y \in Y, t \in \mathbb{R}$.

Now we briefly make a comparison between our scalarizing functional (3.5) and the scalarizing functional $g^{\preceq_l^{\mathcal{K}}}$ used in [32] for set optimization equipped with the relation $\preceq_l^{\mathcal{K}}$. The functional $g^{\preceq_l^{\mathcal{K}}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}}$ is defined as

$$A, B \in \mathcal{P}(Y), \qquad g^{\preceq_l^{\mathcal{K}}}(A, B) := \sup_{b \in B} \inf_{a \in A} z_{a+\mathcal{K}(a),k^0}(-b). \tag{3.7}$$

where $k^0 \in Y \setminus \{0\}$ is taken such that (3.3) is fulfilled.

The following proposition shows the relationship between $\varphi_{k^0,B}(A)$ and $g^{\preceq_l^{\mathcal{K}}}(A, B)$, where $A, B \in \mathcal{P}(Y)$

Proposition 3.11. *Let $A,\ B \in \mathcal{P}(Y)$ and suppose that $g^{\preceq_l^{\mathcal{K}}}(A, B) \in \mathbb{R}_+$. Then the following statement holds true:*

$$\varphi_{k^0,B}(A) = g^{\preceq_l^{\mathcal{K}}}(A, B).$$

Proof. Suppose that $g^{\preceq_l^{\mathcal{K}}}(A, B) = u \in \mathbb{R}_+$. By [32, Theorem 4(a)], it holds that $\cup_{t>u}(tk^0 + B) \subseteq \cup_{a\in A}(a + \mathcal{K}(a))$. Taking into account Theorem 3.8 (c), $\varphi_{k^0,B}(A) \leq u$. Assume by contradiction that $0 \leq \varphi_{k^0,B}(A) = v < u$. Therefore, there exists $w \in \mathbb{R}$ such that $v < w < u$. By Theorem 3.8 (c), it holds that

$$wk^0 + B \subseteq \cup_{a\in A}(a + \mathcal{K}(a),\ i.e.,\ A \preceq_l^{\mathcal{K}} wk^0 + B.$$

Taking into account [32, Theorem 4(b)], we get that $g^{\preceq_l^{\mathcal{K}}}(A, B) \leq w < u$, a contradiction. Therefore, $\varphi_{k^0,B}(A) = u = g^{\preceq_l^{\mathcal{K}}}(A, B)$. □

4. Characterizations for solutions of set optimization w.r.t. variable domination structures via scalarization

This section is devoted to characterizations of minimal and strictly minimal solutions of set optimization w.r.t. variable domination structures by using the scalarizing functional given by (3.5). We assume in this part that $\mathcal{K}(\cdot)$ satisfies the condition (2.2), (2.3) and (2.4). Let $\mathcal{A}$ be a nonempty subset of $\mathcal{P}(Y)$. We begin this section with the following theorem, where we are using the function (3.5) with $B = \bar{A}$, and $k^0 \in Y \setminus \{0\}$ such that (3.3) holds true.

Theorem 4.1. *The following assertions are satisfied.*

(a) *Assume that $\cup_{a\in A}(a+\mathcal{K}(a))$ is closed for all $A \in \mathcal{A}$. Then $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ if and only if $\varphi_{k^0,\bar{A}}(A) > 0$ for all $A \in \mathcal{A}, A \not\sim \bar{A}$.*

(b) *Assume that $\cup_{a\in A}(a+\mathcal{K}(a))$ is closed for all $A \in \mathcal{A}$. Then $\bar{A} \in \mathrm{SMin}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ if and only if $\varphi_{k^0,\bar{A}}(A) > 0$ for all $A \in \mathcal{A} \setminus \{\bar{A}\}$.*

Proof. (a) Consider $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ and suppose that there exists $A \in \mathcal{A}, A \not\sim \bar{A}$ satisfying $\varphi_{k^0,\bar{A}}(A) = 0$. Taking into account Theorem 3.8(e), it holds that $A \preceq_l^{\mathcal{K}} \bar{A}$. Since $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$, $\bar{A} \preceq_l^{\mathcal{K}} A$ and thus $A \sim \bar{A}$. This is a contradiction.

Conversely, assume that $\varphi_{k^0,\bar{A}}(A) > 0$ for all $A \in \mathcal{A}, A \not\sim \bar{A}$ and $\bar{A}$ is not a minimal element of $\mathcal{A}$. Then from the definition of minimal elements of $\mathcal{A}$ there exists a set $A \in \mathcal{A}, A \preceq_l^{\mathcal{K}} \bar{A}$ and $\bar{A} \not\preceq_l^{\mathcal{K}} A$. Using Theorem 3.8(a) it holds that $\varphi_{k^0,\bar{A}}(A) \leq \varphi_{k^0,\bar{A}}(\bar{A})$. In addition, by Theorem 3.8(d)

we get $\varphi_{k^0,\bar{A}}(\bar{A})=0$. Therefore, $\varphi_{k^0,\bar{A}}(A) \leq 0$, a contradiction. Thus, the assumption $\bar{A} \notin \operatorname{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ is false and the proof of the sufficient condition is complete.

(b) Suppose that $\bar{A} \in \operatorname{SMin}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ and there is $A \in \mathcal{A} \setminus \{\bar{A}\}$ satisfying $\varphi_{k^0,\bar{A}}(A) = 0$. By Theorem 3.8(e), we have that $A \preceq_l^{\mathcal{K}} \bar{A}$. Since $\bar{A} \in \operatorname{SMin}(\mathcal{A}, \preceq_l^{\mathcal{K}})$, it yields $A = \bar{A}$, which is a contradiction.
Let us prove the sufficient condition. By contradiction, assume that $\varphi_{k^0,\bar{A}}(A) > 0$ for all $A \in \mathcal{A}\setminus\{\bar{A}\}$ and $\bar{A} \notin \operatorname{SMin}(\mathcal{A}, \preceq_l^{\mathcal{K}})$. Using the definition of strictly minimal elements of $\mathcal{A}$, there exists $A \in \mathcal{A}$ such that $A \preceq_l^{\mathcal{K}} \bar{A}$ and $A \neq \bar{A}$. Taking into account part (d) and (e) of Theorem 3.8, it holds that

$$\varphi_{k^0,\bar{A}}(A) \leq \varphi_{k^0,\bar{A}}(\bar{A})=0.$$

This implies $\varphi_{k^0,\bar{A}}(A)=0$, which is a contradiction.

□

Remark 4.2. A similar result as Theorem 4.1 is generated in [32] where the authors used the scalarizing functional $g^{\preceq_l^{\mathcal{K}}}$, compare [32, Theorem 17].

In the following theorem, we present characterizations for minimal and strictly minimal solutions of a set-valued optimization problem w.r.t. variable domination structures. When $\mathcal{K}(\cdot) = K$, where K is a convex cone in Y, a similar result is given in [18].

Theorem 4.3. *Let $F : X \rightrightarrows Y$ and $\mathcal{K} : Y \rightrightarrows Y$ be set-valued maps such that $\cup_{y\in F(x)}(y+\mathcal{K}(y))$ is closed for each $x \in X$ and the conditions (2.2), (2.3) and (2.4) are fulfilled . Consider problem* (P) *and $\bar{x} \in X$. Then the following assertions hold true.*

(a) *$\bar{x}$ is a minimal solution of (P) if and only if there is a functional $G : \operatorname{Im} F \to \mathbb{R}_+ \cup \{+\infty\}$ being $\preceq_l^{\mathcal{K}}$-monotone such that*

$$x \in S, \quad F(x) \sim F(\bar{x}) \quad \Longleftrightarrow \quad G(F(x))=0. \tag{4.1}$$

(b) *$\bar{x}$ is a strictly minimal solution of (P) if and only if there is a functional $G : \operatorname{Im} F \to \mathbb{R}_+ \cup \{+\infty\}$ being $\preceq_l^{\mathcal{K}}$-monotone such that*

$$x \in S, \quad G(F(x))=0 \quad \Longleftrightarrow \quad x = \bar{x}. \tag{4.2}$$

Proof. The idea of this proof is as similar as that in [18, Theorem 4.4], where $\mathcal{K}(\cdot) = K$, K is a convex cone in Y. We illustrate in the following for the case the domination structure is variable and the scalarizing functional is given by (3.5).

(a) Suppose that $\bar{x}$ is a minimal solution of (P). Let $k^0 \in Y$ such that for all $y \in Y$, $\mathcal{K}(y) + [0, +\infty)k^0 \subseteq \mathcal{K}(y)$ and define the following functional as

$$G : \operatorname{Im} F \to \mathbb{R} \cup \{+\infty\}$$
$$G(F(x)) := \varphi_{k^0,F(\bar{x})}(F(x)),$$

where $\varphi_{k^0,F(\bar{x})}$ given by (3.5) with $B = F(\bar{x})$ is involved.
From Theorem 3.8(a), we get that G is $\preceq_l^{\mathcal{K}}$-monotone. Let us now prove that

$$x \in S, F(x) \sim F(\bar{x}) \quad \Longleftrightarrow \quad G(F(x))=0.$$

Taking into account Theorem 3.8(a) and (d), it holds that

$$\begin{aligned} F(x) \sim F(\bar{x}) &\Longrightarrow F(x) \preceq_l^{\mathcal{K}} F(\bar{x}) \\ &\Longrightarrow G(F(x)) \leq G(F(\bar{x})) = \varphi_{F(\bar{x})}F(\bar{x})=0 \\ &\Longrightarrow G(F(x))=0. \end{aligned}$$

Now if we suppose that $F(x) \not\sim F(\bar{x})$, by Theorem 4.1 (b), it holds that $G(F(x)) > 0$. Therefore, if $G(F(x))=0$, we have that $F(x) \sim F(\bar{x})$.
Reciprocally, suppose that there exists a functional $G : \operatorname{Im} F \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying (4.1) and G is $\preceq_l^{\mathcal{K}}$-monotone. Let $x \in S$ such that $F(x) \preceq_l^{\mathcal{K}} F(\bar{x})$. It is sufficient to prove that $F(\bar{x}) \preceq_l^{\mathcal{K}} F(x)$. Since $\mathcal{K}$ satisfies (2.2), the relation $\preceq_l^{\mathcal{K}}$ is reflexive and thus $F(\bar{x}) \sim F(\bar{x})$. Taking into account (4.1), we get that $G(F(\bar{x}))=0$. Since G is $\preceq_l^{\mathcal{K}}$ monotone and $F(x) \preceq_l^{\mathcal{K}} F(\bar{x})$, it yields

$$\begin{aligned} 0 \leq &G(F(x)) \leq G(F(\bar{x}))=0 \\ &\Rightarrow G(F(x))=0 \end{aligned}$$

Taking into (4.1) we get that $F(x) \sim F(\bar{x}) \Rightarrow F(\bar{x}) \preceq_l^{\mathcal{K}} F(x)$, which is the desired conclusion.

(b) Let $\bar{x}$ be a strictly minimal solution of problem (P) and the functional G defined as in part (a), that is $G(F(x)) = \varphi_{k^0,F(\bar{x})}(F(x))$. Because $\bar{x}$ is a strictly minimal solution of (P), it yields that

$$\forall\, x \neq \bar{x} :\ F(x) \not\preceq_l^{\mathcal{K}} F(\bar{x}).$$

Now we suppose that $G(F(x))=0$. Taking into account Theorem 3.8(e), it holds that $F(x) \preceq_l^{\mathcal{K}} F(\bar{x})$. This implies $x = \bar{x}$. Therefore, if $G(F(x))=0$ then $x = \bar{x}$. On the other hand,

$$x = \bar{x} \quad \Longrightarrow \quad G(F(x)) = G(F(\bar{x})) = \varphi_{k^0,F(\bar{x})}F(\bar{x})=0.$$

Thus, the conclusion (4.2) holds true.
Now we prove the sufficient condition. Suppose that there exists a functional $G : \operatorname{Im} F \to \mathbb{R}_+ \cup \{+\infty\}$ satisfying (4.2) and G is $\preceq_l^{\mathcal{K}}$-monotone. Let $x \in S$ such that $F(x) \preceq_l^{\mathcal{K}} F(\bar{x})$. Since (4.2) holds true, it yields

$$\begin{aligned} F(x) \preceq_l^{\mathcal{K}} F(\bar{x}) &\Rightarrow 0 \leq G(F(x)) \leq G(F(\bar{x})) \leq 0 \\ &\Rightarrow G(F(x))=0 \\ &\Rightarrow x = \bar{x}. \end{aligned}$$

The last equation states that $\bar{x}$ is a strictly minimal solution of (P).

□

Remark 4.4. • Since $G : X \to \mathbb{R}_+ \cup \{+\infty\}$, we can rewrite (4.1) and (4.2) respectively by

$$\operatorname{argmin}(G \circ F, S) = \{x \in S|\ F(x) \sim F(\bar{x})\}$$

and

$$\operatorname{argmin}(G \circ F, S) = \{\bar{x}\}.$$

- If for all $y \in Y,\ \mathcal{K}(y) = K$, where K is a convex cone in Y and $F(x) + K$ is closed for all $x \in S$, Theorem 4.3 reduces to [18, Theorem 4.4].

5. Pointwise well-posedness for set optimizations with respect to variable domination structures

Investigating well-posedness properties for vector as well as set optimization has attracted many authors in the literature. Usually, one proves the equivalence between the well-posedness property of the concerned problem and the Tykhonov well-posedness property of a scalar problem in which the objective function of the original problem is involved. Then by using many classical results related to this property of the scalar problems, one can derive some classes of well-posed vector (set) optimization problems for the concerned problem. There are many publications investigating the equivalence between the well-posedness property of a vector optimization problem and the Tykhonov well-posedness property of a scalar problem, see for example, [6, 36, 37]. A similar result for set optimization problems w.r.t. fixed cones was first introduced in [40] and recently studied in [18, 35] and the references therein.

In this section, we will show that under some appropriate conditions, we also obtain this equivalence for set-valued optimization using the set relation equipped with a variable domination structure. Moreover, we will find two sets of points at which a set-valued optimization problem is well-posed. Throughout this part, we suppose that the following assumption is fulfilled.

Assumption (A):

- $\mathcal{K} : Y \rightrightarrows Y$ is a set-valued map such that for all $y \in Y$, $\mathcal{K}(y)$ is a proper, closed, convex cone in Y and $\operatorname{int} \bigcap\limits_{y \in Y} \mathcal{K}(y) \neq \emptyset$.
- $F : X \rightrightarrows Y$ is a set-valued map between two real topological vector spaces, $S \subset X$ and for all $x \in S$, $\bigcup\limits_{y \in F(x)} (y + \mathcal{K}(y))$ is closed.
- k^0 is taken in Y such that $k^0 \in \operatorname{int} \bigcap\limits_{y \in Y} \mathcal{K}(y)$.

We begin this section by recalling the notion of well-posedness property of an extended real-valued function (see [5]).

Definition 5.1. Let $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be an extended real-valued function and consider problem

$$\text{(P')} \qquad \operatorname*{Min}_{x \in S} f(x).$$

We say that problem (P') is:

(i) Tykhonov well-posed, if it has a unique solution $\bar{x} \in S$ and

$$\{x_n\} \subset S,\ f(x_n) \to f(\bar{x}) \text{ implies } \{x_n\} \to \bar{x}.$$

(ii) generalized well-posed, if $\arg\min(f, S) \neq \emptyset$ and

$$\{x_n\} \subset S,\ f(x_n) \to f(\bar{x}) \text{ implies } \exists \{x_{n_k}\} \subseteq \{x_n\} :\ \{x_{n_k}\} \to \bar{x}.$$

Remark 5.2. (P') is Tykhonov well-posed if and only if it is generalized well-posed and the set $\arg\min(f, S)$ is a singleton.

Now we will present the well-posedness property for the set-valued problem (P) given in Section 3.1 under Assumption (A). Recall that for $F : X \rightrightarrows Y$ and $\mathcal{K} : Y \rightrightarrows Y$, (P) has the following formula

$$\mathcal{K} - \min_{x \in S} F(x).$$

The following definition extends Definition 5.1 in [18] for a set-valued problem (P) equipped with a variable domination structure.

Definition 5.3. Let $k^0 \in \operatorname{int} \bigcap\limits_{y \in Y} \mathcal{K}(y)$ and $\bar{x}$ be a minimal solution of problem (P).

(a) A sequence $\{x_n\} \subset S$ is said to be k^0-minimizing for (P) at $\bar{x}$, if

$$\exists \{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}, \{\varepsilon_n\} \to 0 : \ F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \varepsilon_n k^0, \forall n.$$

(b) (P) is said to be k^0-well-posed at $\bar{x}$, if every k^0-minimizing sequence at $\bar{x}$ converges to $\bar{x}$.

(c) $\{x_n\} \subset S$ is said to be minimizing at $\bar{x}$, if

$$\exists \{d_n\} \subset \bigcap_{y \in Y} \mathcal{K}(y) \setminus \{0\}, \ \{d_n\} \to 0 : \ F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + d_n, \forall n.$$

(d) (P) is said to be well-posed at $\bar{x}$, if $\bar{x}$ is a strictly minimal solution and for all minimizing $\{x_n\}$ at $\bar{x}$ it holds that $\{x_n\} \to \bar{x}$.

The following lemma, given by Durea [6], will be used in the next proposition which states that Definition 5.3(a) and Definition 5.3(c) are equivalent.

Lemma 5.4 ([6, Lemma 2.2])**.** *Let $K \subseteq Y$ be a proper, closed, convex cone with nonempty interior and $\{k_n\}$ be a sequence of elements from Y that converges to 0. Then for every $k \in \operatorname{int} K$ there exists a sequence $\{\alpha_n\}$ of positive real numbers s.t. $\{\alpha_n\} \to 0$ and $\alpha_n k - k_n \in \operatorname{int} K$ for every natural number n.*

Proposition 5.5. *Let $\{x_n\} \subset S$, $k^0 \in \operatorname{int} \bigcap\limits_{y \in Y} \mathcal{K}(y)$ and $\bar{x}$ be a minimal solution of problem (P). Then the two following assertions are equivalent:*

(i) *$\{x_n\}$ is k^0-minimizing for (P) at $\bar{x}$.*

(ii) *$\{x_n\}$ is minimizing for (P) at $\bar{x}$.*

Proof. $[(i) \to (ii)]$: Since $\{x_n\}$ is k^0-minimizing for (P) at $\bar{x}$, we have that

$$\exists \{\varepsilon_n\} \subset \mathbb{R}_+ \setminus \{0\}, \{\varepsilon_n\} \to 0 : \ F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \varepsilon_n k^0, \forall n.$$

Let $d_n := \varepsilon_n k^0, \ \forall n$. It holds that

$$\{d_n\} \subset \bigcap_{y \in Y} \mathcal{K}(y) \setminus \{0\}, \{d_n\} \to 0 \text{ and } F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + d_n.$$

Taking into account definition of minimizing property, we get that $\{x_n\}$ is minimizing for (P) at $\bar{x}$, i.e., (ii) holds true.
$[(ii) \to (i)]$: Suppose that $\{x_n\}$ is minimizing for (P) at $\bar{x}$, i.e.,

$$\exists \{d_n\} \subset \bigcap_{y \in Y} \mathcal{K}(y) \setminus \{0\}, \ \{d_n\} \to 0 : \ F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + d_n, \ \forall n.$$

We will prove that

$$\exists\{\alpha_n\} \subset \mathbb{R}_+ \setminus \{0\}, \{\alpha_n\} \to 0 : \ F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \alpha_n k^0, \ \forall n.$$

We have that

$$\begin{aligned} F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + d_n &\Leftrightarrow F(\bar{x}) + d_n \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) \\ &\Leftrightarrow F(\bar{x}) \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) + (-d_n) \end{aligned} \tag{5.1}$$

Let $K := \bigcap_{y \in Y} \mathcal{K}(y)$. Since for all $y \in Y$, $\mathcal{K}(y)$ is a conex cone, we have that $\mathcal{K}(y) + K \subseteq \mathcal{K}(y)$. Therefore, for all $n \in \mathbb{N}$, it holds that

$$\bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) + \operatorname{int} K \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) + K \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)). \tag{5.2}$$

By Assumption (A), K is a proper, closed, convex cone with $\operatorname{int} K \neq \emptyset$. Taking into account $k^0 \in \operatorname{int} K$, $\{d_n\} \xrightarrow{Y} 0$ and applying Lemma 5.4, we obtain that

$$\exists\{\alpha_n\} \subseteq \mathbb{R}_+ \setminus \{0\}, \{\alpha_n\} \to 0 : \ \alpha_n k^0 - d_n \in \operatorname{int} K, \ \forall n \in \mathbb{N}.$$

This implies that $-d_n \in -\alpha_n k^0 + \operatorname{int} K$. Taking into account (5.1), it holds for all $n \in \mathbb{N}$ that

$$\begin{aligned} & F(\bar{x}) \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) - \alpha_n k^0 + \operatorname{int} K \\ \Longleftrightarrow \ & F(\bar{x}) + \alpha_n k^0 \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)) + \operatorname{int} K. \end{aligned}$$

Taking into account (5.2), we get that

$$\begin{aligned} & F(\bar{x}) + \alpha_n k^0 \subseteq \bigcup_{y_n \in F(x_n)} (y_n + \mathcal{K}(y_n)), \ \forall n \in \mathbb{N} \\ \Longleftrightarrow \ & F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \alpha_n k^0, \ \forall n \in \mathbb{N}. \end{aligned} \tag{5.3}$$

The relation (5.3) ensures that $\{x_n\}$ is k^0-minimizing for the problem (P) at $\bar{x}$. The proof is complete. □

The following theorem states that there exists a class of scalar problems whose Tykhonov well-posedness property is equivalent to the well-posedness of the original set optimization problem (P).

Theorem 5.6. *Suppose that $\mathcal{K} : Y \rightrightarrows Y$ satisfies (2.3). Furthermore, let $\bar{x}$ be a strictly minimal solution of problem (P) such that $\varphi_{k^0, F(\bar{x})}(F(\bar{x})) \in \mathbb{R}$. Consider the scalar problem*

$$(P_{\varphi_{k^0, F(\bar{x})}}) \qquad \operatorname{Min}\{\varphi_{k^0, F(\bar{x})}(F(x)) \mid x \in S\}.$$

Then the following statements are equivalent:

(a) *Problem (P) is well-posed at $\bar{x}$.*
(b) *For every $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$, problem $(P_{\varphi_{k^0, F(\bar{x})}})$ is Tykhonov well-posed.*
(c) *There is $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$ such that problem $(P_{\varphi_{k^0, F(\bar{x})}})$ is Tykhonov well-posed.*

$(a) \Rightarrow (b)$. : Let $k^0 \in \operatorname{int} \bigcap\limits_{y\in Y} \mathcal{K}(y)$ arbitrary. Taking into account Theorem 4.3(b), we have that

$$\varphi_{k^0,F(\bar{x})}(F(\bar{x}))=0 \text{ and for all } x \neq \bar{x} : \varphi_{k^0,F(\bar{x})}(F(x)) > 0.$$

Thus $\operatorname{argmin}_{x\in S} \varphi_{k^0,F(\bar{x})}(F(x)) = \{\bar{x}\}$, i.e, $\bar{x}$ is a unique solution of $(P_{\varphi_{k^0,F(\bar{x})}})$. Now take $\{x_n\} \subseteq S$ such that $\varphi_{k^0,F(\bar{x})}(F(x_n)) \to \varphi_{k^0,F(\bar{x})}(F(\bar{x}))$. It is sufficient to prove that $\{x_n\} \to \bar{x}$.
Let $\bar{t}_n := \varphi_{k^0,F(\bar{x})}(F(x_n))$, and $\varepsilon_n := \varphi_{k^0,F(\bar{x})}(F(x_n)) + \frac{1}{n}$. It holds that

$$\{\varepsilon_n\} \to 0,\ \varepsilon_n > \bar{t}_n \text{ and } F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \varepsilon_n k^0.$$

By the last relation, we get that $\{x_n\}$ is k^0-minimizing and thus, a minimizing sequence for (P). Since (P) is well-posed, $\{x_n\} \to \bar{x}$.
$[(b) \Rightarrow (c)]$ This implication is obvious.
$[(c) \Rightarrow (a)]$: Suppose that (c) holds true, we will prove that (a) is fulfilled. Let $\{x_n\}$ is a minimal solution sequence for problem (P) at $\bar{x}$. By Proposition 5.5, there is a sequence $\{\varepsilon_n\} \to 0^+$ and

$$\forall\ n : F(x_n) \preceq_l^{\mathcal{K}} F(\bar{x}) + \varepsilon_n k^0 \quad \Longrightarrow \quad \varphi_{k^0,F(\bar{x})}(F(x_n)) \leq \varepsilon_n.$$

Taking into account $\bar{x}$ is a strictly minimal solution of (P), it holds that

$$\forall x_n \neq \bar{x} : \varphi_{k^0,F(\bar{x})}(F(x_n)) > 0.$$

Thus, we get $\{\varphi_{k^0,F(\bar{x})}(F(x_n))\} \to 0 = \varphi_{k^0,F(\bar{x})}(F(\bar{x}))$. Since $(P_{\varphi_{k^0,F(\bar{x})}})$ is Tykhonov well-posed, it holds that $\{x_n\} \to \bar{x}$, i.e., problem (P) is well-posed at $\bar{x}$. □

Now, we are finding some classes of well-posed set optimization problems. We recall the two following classical results of well-posed scalar optimization problems, which will be used in the sequel.

Theorem 5.7 ([2, Theorem 2.1])**.** *Let X be a locally compact metric space. Suppose $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a proper lower semicontinuous and quasiconvex function on X. The following conditions are equivalent:*

(a) *Problem* (P') *is generalized well-posed;*
(b) $\operatorname{argmin}(f, X)$ *is nonempty and compact.*

Proposition 5.8 ([5, Example 6])**.** *Let X be a normed vector space, $S \subset X$ be a compact set and $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a proper and lower semicontinuous function on X. Suppose that* $\operatorname{argmin}(f, S)$ *has a unique element. Then problem* (P') *is Tykhonov well-posed.*

In the following proposition, we show the sufficient conditions which ensure the lower-semicontinuous property of the composition function $\varphi_{k^0,B} \circ F$, where $k^0 \in \operatorname{int} \bigcap\limits_{y\in Y} \mathcal{K}(y)$ and $B \in \mathcal{P}(Y)$.

Proposition 5.9. *Suppose that $F : X \rightrightarrows Y$ satisfies that $S(F, \preceq_l^{\mathcal{K}}, rk^0 + A) := \{x \in X |\ F(x) \preceq_l^{\mathcal{K}} rk^0 + A\}$ is closed for all $A \in \mathcal{P}(Y)$ and $r \geq 0$. In addition, assume that $\mathcal{K}(\cdot)$ satisfies (2.3). Then $\varphi_{k^0,B} \circ F : X \to \mathbb{R}_+ \cup \{+\infty\}$ is lower semicontinuous on S for all $k^0 \in \operatorname{int} \bigcap\limits_{y\in Y} \mathcal{K}(y)$.*

Proof. We prove that for all $\gamma \in \mathbb{R}$, the set $S(\varphi_{k^0,B} \circ F, \gamma)$ is closed. This assertion holds true when $\gamma < 0$ since $S(\varphi_{k^0,B} \circ F, \gamma) = \emptyset$. If $\gamma \geq 0$, we prove that $S(\varphi_{k^0,B} \circ F, \gamma) = S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B)$.
Let $x \in S(\varphi_{k^0,B} \circ F, \gamma)$. Taking into account Remark 3.9 (ii), we have that

$$\varphi_{k^0,B}F(x) \leq \gamma \Longrightarrow F(x) \preceq_l^{\mathcal{K}} \gamma k^0 + B \Rightarrow x \in S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B).$$

Therefore,

$$S(\varphi_{k^0,B} \circ F, \gamma) \subseteq S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B). \tag{5.4}$$

Conversely, let $x \in S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B)$, i.e., $F(x) \preceq_l^{\mathcal{K}} \gamma k^0 + B$. By the definition (3.5), it holds that

$$\varphi_{k^0,B}F(x) \leq \gamma \Rightarrow x \in S(\varphi_{k^0,B} \circ F, \gamma).$$

Therefore,

$$S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B) \subseteq S(\varphi_{k^0,B} \circ F, \gamma) \tag{5.5}$$

(5.4) together with (5.5) imply that $S(\varphi_{k^0,B} \circ F, \gamma) = S(F, \preceq_l^{\mathcal{K}}, \gamma k^0 + B)$. □

Now we present the first class of well-posed set-valued optimization problems w.r.t. variable domination structures.

Theorem 5.10. *Let X be a normed vector space and Y be a linear topological space. Consider problem* (P) *with the mappings $F : X \rightrightarrows Y$ and $\mathcal{K} : Y \rightrightarrows Y$ satisfy all the assumptions given in Proposition 5.9. Let $\bar{x}$ be a strictly minimal solution of problem* (P) *and S be a compact subset of X. Then* (P) *is well-posed at $\bar{x}$.*

Proof. Let $k^0 \in \operatorname{int} \bigcap_{y \in Y} \mathcal{K}(y)$. By Proposition 5.9, $\varphi_{k^0,F(\bar{x})} \circ F$ is lower semicontinuous. Furthermore, by Theorem 4.3 (b), it holds that $\operatorname{argmin}(\varphi_{k^0,F(\bar{x})} \circ F, S) = \{\bar{x}\}$. Therefore, according to Proposition 5.8, problem $(P_{\varphi_{k^0,F(\bar{x})}})$ is Tykhonov well-posed. Applying Theorem 5.6, we have that problem (P) is well-posed at $\bar{x}$. □

Before deriving the second class of well-posed set optimization problems w.r.t. variable domination structures, we introduce a $\mathcal{K}$-quasiconvex map. Recall that when $\mathcal{K}(\cdot) = K$, where K is a convex cone in Y with nonempty interior, a K-quasiconvex set valued map is defined in [35, Definition 2.2] for $F : X \rightrightarrows Y$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \preceq_l^K (F(x_1) + K) \cap (F(x_2) + K), \ \forall \lambda \in [0,1], x_1, x_2 \in S,$$

where S is a convex subset of Y. We now extend this definition to the case the domination is variable as follows.

Definition 5.11. The set-valued mapping $F : X \rightrightarrows Y$ is said to be $\mathcal{K}$-quasiconvex w.r.t. $\preceq_l^{\mathcal{K}}$ on a nonempty, convex set $S \subseteq X$ if for all $x_1, x_2 \in S$ and $\lambda \in [0,1]$ it holds that $F(\lambda x_1 + (1-\lambda)x_2) \preceq_l^{\mathcal{K}} \bigcup_{y \in F(x_1)} (y + \mathcal{K}(y)) \cap \bigcup_{y \in F(x_2)} (y + \mathcal{K}(y))$.

In the following, we show that the quasiconvex property can be inherited via scalarizing functional given by (3.5).

Proposition 5.12. *If $F : X \rightrightarrows Y$ is $\mathcal{K}$-quasiconvex w.r.t. $\preceq_l^{\mathcal{K}}$ on a nonempty convex set $S \subseteq X$ then $\varphi_{k^0,B} \circ F$ is a quasiconvex function on S for all $k^0 \in \operatorname{int} \bigcap_{y\in Y} \mathcal{K}(y)$ and $B \in \mathcal{P}(Y)$. Furthermore, the converse statement is true if $\mathcal{K}(\cdot)$ satisfies (2.3).*

Proof. [$\Rightarrow$] : Let $x_1, x_2 \in S$ be two arbitrary elements. We have to show that for all $\lambda \in [0,1]$ it holds that

$$\varphi_{k^0,B} \circ F(\lambda x_1 + (1-\lambda)x_2) \leq \max\{\varphi_{k^0,B} \circ F(x_1), \varphi_{k^0,B} \circ F(x_2)\}.$$

Obviously, this assertion holds true for the case either $\varphi_{k^0,B} \circ F(x_1) = +\infty$ or $\varphi_{k^0,B} \circ F(x_2) = +\infty$. We now suppose that both $\varphi_{k^0,B} \circ F(x_1)$ and $\varphi_{k^0,B} \circ F(x_2)$ are real numbers. We will prove that the set $S(\varphi_{k^0,B} \circ F, \gamma)$ is convex for all $\gamma \in \mathbb{R}$. This assertion is trivial when $\gamma < 0$ since $S(\varphi_{k^0,B} \circ F, \gamma) = \emptyset$. Now we suppose that $\gamma \geq 0$ and $\varphi_{k^0,B}F(x_1) \leq \gamma$ and $\varphi_{k^0,B}F(x_2) \leq \gamma$. Let $\alpha_1 := \varphi_{k^0,B}F(x_1)$ and $\alpha_2 := \varphi_{k^0,B}F(x_2)$. Take $\bar{\alpha} := \max\{\alpha_1, \alpha_2\} \leq \gamma$ and $\varepsilon > 0$ arbitrary.
Since Theorem 3.8 (c), it holds that

$$F(x_1) \preceq_l^{\mathcal{K}} (\bar{\alpha} + \varepsilon)k^0 + B$$

and

$$F(x_2) \preceq_l^{\mathcal{K}} (\bar{\alpha} + \varepsilon)k^0 + B.$$

Therefore,

$$(\bar{\alpha} + \varepsilon)k^0 + B \subseteq \bigcup_{y\in F(x_1)} (y + \mathcal{K}(y)) \cap \bigcup_{y\in F(x_2)} (y + \mathcal{K}(y)).$$

Taking into account Definition 5.11, we get that

$$(\bar{\alpha} + \varepsilon)k^0 + B \subseteq \bigcup_{y\in F(\lambda x_1+(1-\lambda)x_2)} (y + \mathcal{K}(y)).$$

Therefore, $\varphi_{k^0,B}F(\lambda x_1 + (1-\lambda)x_2) \leq \bar{\alpha} + \varepsilon$, for all $\varepsilon > 0$.
Thus, $\varphi_{k^0,B}F(\lambda x_1 + (1-\lambda)x_2) \leq \bar{\alpha} \leq \gamma$, i.e., $\lambda x_1 + (1-\lambda)x_2 \in S(\varphi_{k^0,B} \circ F, \gamma)$ or $S(\varphi_{k^0,B} \circ F, \gamma)$ is convex.

[$\Leftarrow$] : Conversely, suppose that $\varphi_{k^0,B} \circ F$ is quasiconvex, we prove that for all $x_1, x_2 \in S$ and $\lambda \in [0,1]$ it holds that $F(\lambda x_1 + (1-\lambda)x_2) \preceq_l^{\mathcal{K}} \bigcup_{y\in F(x_1)} (y + \mathcal{K}(y)) \cap \bigcup_{y\in F(x_2)} (y + \mathcal{K}(y))$.
Take $z \in \bigcup_{y\in F(x_1)} (y + \mathcal{K}(y)) \cap \bigcup_{y\in F(x_2)} (y + \mathcal{K}(y))$, arbitrarily. This is equivalent to $F(x_i) \preceq_l^{\mathcal{K}} \{z\}$, for $i = 1, 2$. Therefore, by Theorem (3.8)(e), $\varphi_{k^0,\{z\}}(F(x_i)) = 0$. Since $\varphi_{k^0,\{z\}} \circ F$ is quasiconvex,

$$\varphi_{k^0,\{z\}} \circ F(\lambda x_1 + (1-\lambda)x_2) \leq 0.$$

By Theorem (3.8)(e),

$$F(\lambda x_1 + (1-\lambda)x_2) \preceq_l^{\mathcal{K}} \{z\},$$

that is, $z \in \bigcup_{y\in F(\lambda x_1+(1-\lambda)x_2)}(y+\mathcal{K}(y))$ for all $z \in \bigcup_{y\in F(x_1)}(y+\mathcal{K}(y))\cap \bigcup_{y\in F(x_2)}(y+\mathcal{K}(y))$, i.e.,

$$F(\lambda x_1 + (1-\lambda)x_2) \preceq_l^{\mathcal{K}} \bigcup_{y\in F(x_1)}(y+\mathcal{K}(y)) \cap \bigcup_{y\in F(x_2)}(y+\mathcal{K}(y)).$$

The proof is complete. □

In the following, we present the second class of well-posed set optimization problems whose the objective map is $\mathcal{K}$-quasiconvex.

Theorem 5.13. *Let X be a locally compact metric space, S be a convex subset of X. Suppose that $F : X \rightrightarrows Y$ and $\mathcal{K} : Y \rightrightarrows Y$ satisfy all the assumptions given in Proposition 5.9 and F is $\mathcal{K}$-quasiconvex w.r.t. $\preceq_l^{\mathcal{K}}$ on S. Let $\bar{x}$ be a strictly minimal solution of problem* (P). *Then* (P) *is well-posed at $\bar{x}$.*

Proof. Let $k^0 \in \operatorname{int} \bigcap_{y\in Y}\mathcal{K}(y)$. By Proposition 5.9 and Proposition 5.12, $\varphi_{k^0,F(\bar{x})} \circ F$ is lower semicontinuous and quasiconvex. Taking into account Theorem 5.7 and

$$\operatorname{argmin}(\varphi_{k^0,F(\bar{x})} \circ F, S) = \{\bar{x}\},$$

problem $(P_{\varphi_{k^0,F(\bar{x})}})$ is generalized well-posed and also is Tykhonov well-posed. Applying Theorem 5.6, we have that problem (P) is well-posed at $\bar{x}$. The proof is complete. □

Remark 5.14. Theorem 5.10 and Theorem 5.13 respectively extend [35, Theorem 4.5] and [35, Theorem 4.6], in which the authors used the domination $\mathcal{K}(\cdot) \equiv C$, where $C \subseteq Y$ is a convex cone such that $\operatorname{int} C \neq \emptyset$. Note that in this case (2.3) holds true and thus one can get [35, Theorem 4.5] and [35, Theorem 4.6] without the fulfilment of this condition.

6. Application to uncertain optimization

Robust Optimization has been of great interest in the optimization community since the groundbreaking work by Ben-Tal, El Ghaoui, and Nemirovski in the 1990ies (see, for instance, [1]). However, the field of robust optimization dates back to the 1940ies, where Wald [39] investigated worst case analysis in decision theory. Uncertain data contaminate most optimization problems in various applications ranging from science and engineering to industry and thus represent an essential component in optimization. From a mathematical point of view, many problems can be modeled as an optimization problem and be solved, but in real life, having exact data is very rare and seems almost impossible. Due to a lack of complete information, uncertain data can highly affect solutions and thus influence the decision making process. Hence, it is crucial to address this important issue in optimization theory. Potential applications of uncertain optimization include supply and inventory management, since demand and tools needed for the production process can easily be exposed to uncertain changes. Further examples for uncertain

data in optimization problems can be found in the field of market analysis, share prices, transportation science, timetabling and location theory.

In order to gain realistic insights into a problem in a complex surrounding, contrary objectives play an important role and are thus intensely studied in optimization. In this section, we study such multi-objective problems that are contaminated with uncertain data in a general setting.

The first robust concepts for uncertain multicriteria optimization problems was introduced by Deb and Gupta [9]. Using an idea by Branke [4], the authors define robustness as some sensitivity against disturbances in the decision space. They call a solution to a problem robust if small perturbations in the decision space result in only small disturbances in the objective space. Kuroiwa and Lee [30] presented the first scenario-based approach by directly transferring the main idea of robust scalar optimization to multicriteria optimization. This concept was generalized by Ehrgott et al. [14] who implicitly used a set-order relation to define robust solutions for uncertain multicriteria optimization problems. As was recently observed in [20, 21], robust multi-objective optimization is an important application of set optimization. Different approaches to robust multi-objective optimization with a fixed domination structure were examined in [20, 21].

In this section, we will introduce a concept for obtaining *optimistic* solutions of an uncertain multi-objective optimization problem, where the domination structure is equipped with a variable ordering. Moreover, we develop optimality conditions for optimistic solutions of uncertain vector optimization problems based the the results derived in the preceding sections. Our approach enables the decision maker to specify his preferences with regard to the domination structure rather than relying on a given optimality concept.

Now we recall some notation of uncertain multi-objective optimization introduced in Ehrgott et al. [14] (see also [21]) which will be used throughout this section. Let Y be a linear topological space, X is a linear space, $S \subseteq X$ a nonempty set, and let an uncertainty set $\emptyset \neq \mathcal{U} \subseteq \mathbb{R}^N$ be given. The uncertainty set $\mathcal{U}$ contains all possible parameter values that the uncertain parameter may attain. Let $f : S \times \mathcal{U} \to Y$ be the function that is to be minimized. Our goal is to obtain solutions that are *optimistic*, i.e., that perform well in the best-case scenario. For the scalar case $Y = \mathbb{R}$, this would mean to minimize the functional $\inf_{\xi \in \mathcal{U}} f(x, \xi)$ on $\mathcal{X}$. Of course, if f is vector-valued, this scalar approach cannot be easily transferred to vector optimization. Due to the absence of a total order on Y, we need to define the meaning of an optimal solution.

We define for $x \in \mathcal{X}$ the outcome set

$$f_{\mathcal{U}}(x) := \{f(x, \xi) |\ \xi \in \mathcal{U}\},$$

i.e., the image of f under $\mathcal{U}$. For a fixed $\xi \in \mathcal{U}$, the vector optimization problem is denoted by

$$(P(\xi)) \qquad \min_{x \in \mathcal{X}} f(x, \xi).$$

The family of all problems $\bigcup_{\xi \in \mathcal{U}} (P(\xi))$, is called **uncertain optimization problem**, and is denoted by $P(\mathcal{U})$. Furthermore, the family of all sets $f_{\mathcal{U}}(x)$, $x \in S$, is

denoted by $\mathcal{A}$. In contrast to the original robustness concepts, our "optimistic" concept uses the lower set less order relation equipped with a variable domination structure according to Definition 3.1. This kind of optimality focuses on the lower bound of a set $f_{\mathcal{U}}(x)$. Contrary to the traditional robustness approach, we are therefore not interested in a worst-case concept but a best-case concept. Thus, this approach is suitable for a decision maker who is not considered to be risk averse but rather risk affine and has positive expectations about the future.

Definition 6.1. Let an uncertain optimization problem $P(\mathcal{U})$ be given and let $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map satisfying (2.2). $\bar{x} \in S$ is called an *optimistic* solution of problem $P(\mathcal{U})$ if $f_{\mathcal{U}}(\bar{x})$ is a minimal element of $\mathcal{A}$ in terms of Definition 3.1 (a). $\bar{x} \in S$ is called a *strictly optimistic* solution of problem $P(\mathcal{U})$ if $f_{\mathcal{U}}(\bar{x})$ is a strictly minimal element of $\mathcal{A}$ in terms of Definition 3.1 (b).

Now we discuss the role of the variable domination structure. For simplicity, we consider the case $Y = \mathbb{R}^2$, i.e., we consider an uncertain bicriteria optimization problem. Assume that the data of a vector $a \in \mathbb{R}^2$ is perturbed by uncertain data and only an approximation $A \subset \mathbb{R}^2$ is known (see Figure 1 (a)). Similarly, the data of a vector $\widetilde{b}$ is disturbed and only an estimated set $\widetilde{B}$ can be generated. In order to compare the set A to the set $\widetilde{B}$, the lower set less order relation $\preceq_l^Q$ with the fixed ordering cone $Q = \mathbb{R}^2_+$ shall be used, such that $\widetilde{B} \subseteq A + Q \Longleftrightarrow A \preceq_l^Q \widetilde{B}$. This relation ensures that the lower bounds of $\widetilde{B}$ are not "worse" than those of A. Since the data are uncertain, it seems likely that there exist undesired elements located far from where most uncertain data is found. If there exists such an element $\bar{b} \notin \widetilde{B}$ which is located far away from $\widetilde{B}$, then the relation $A \preceq_l^Q B$, where $B := \widetilde{B} \cup \{\bar{b}\}$, may not hold anymore (see Figure 1 (b)). In order to still include $\bar{b}$ in the analysis but to obtain the result that the set A is, for the most part, preferred to B, a planner can introduce a variable domination structure in the following way: Let $\underline{a} \in A$ and $\mathcal{K} : Y \rightrightarrows Y$ with

$$\mathcal{K}(y) := \begin{cases} K & \text{if } y = \underline{a}, \\ \mathbb{R}^2_+ & \text{else,} \end{cases}$$

where K is a cone which fulfills $\bar{b} \in \{\underline{a}\} + K$ ($K := \mathcal{K}(\underline{a})$, see Figure 1, (b)). Then we have $A \preceq_l^{\mathcal{K}} B$. This ensures that all estimated elements are taken into account, as nondesired elements can be handled by using variable domination structures.

Now we are ready to apply the characterizations of solutions of set optimization problems w.r.t. variable domination structures, which were derived in Section 4, to the uncertain optimization problem $P(\mathcal{U})$.

Corollary 6.2. *Let $k^0 \in Y \setminus \{0\}$ be given such that the inclusion* (3.3) *is satisfied. Then the following assertions hold.*

(a) *Assume that $\cup_{y \in f_{\mathcal{U}}(x)}(y + \mathcal{K}(y))$ is closed for all $f_{\mathcal{U}}(x) \in \mathcal{A}$. Then $\bar{x} \in S$ is an optimistic solution of problem $P(\mathcal{U})$ if and only if $\varphi_{k^0, f_{\mathcal{U}}(\bar{x})}(f_{\mathcal{U}}(x)) > 0$ for all $f_{\mathcal{U}}(x) \in \mathcal{A}$, $f_{\mathcal{U}}(x) \not\preceq f_{\mathcal{U}}(\bar{x})$.*

(b) *Assume that $\cup_{y \in f_{\mathcal{U}}(x)}(y + \mathcal{K}(y))$ is closed for all $f_{\mathcal{U}}(x) \in \mathcal{A}$. Then $\bar{x} \in S$ is a strictly optimistic solution of problem $P(\mathcal{U})$ if and only if $\varphi_{k^0 f_{\mathcal{U}}(\bar{x})}(f_{\mathcal{U}}(x)) > 0$ for all $f_{\mathcal{U}}(x) \in \mathcal{A} \setminus \{f_{\mathcal{U}}(\bar{x})\}$.*

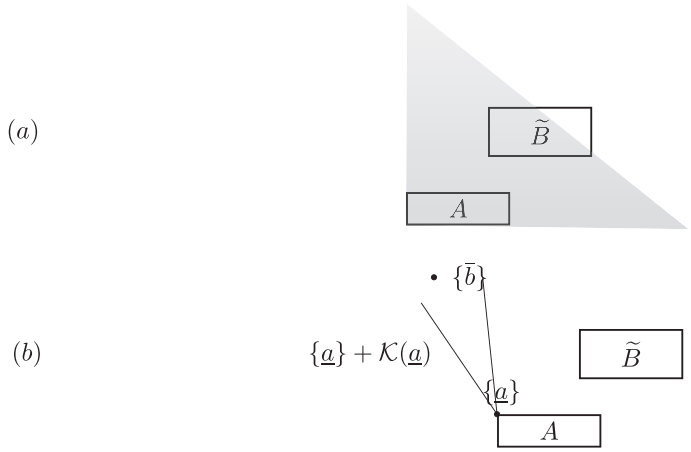

FIGURE 1. Visualization of two outcome sets $A, \widetilde{B} \subset \mathbb{R}^2$ of a uncertain bicriteria optimization problem with undesired elements.

In the next corollary, we denote $\operatorname{Im} f_{\mathcal{U}} := \{f_{\mathcal{U}}(x) \mid x \in S \text{ and } f_{\mathcal{U}}(x) \neq \emptyset\}$.

Corollary 6.3. *Let $k^0 \in Y \setminus \{0\}$ be given such that the inclusion* (3.3) *is satisfied and let $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map such that $\cup_{y \in F(x)}(y + \mathcal{K}(y))$ is closed for each $x \in S$ and the conditions (2.2), (2.3) and (2.4) are fulfilled. Consider $\bar{x} \in S$. The following assertions hold true.*

(a) *$\bar{x}$ is an optimistic solution of problem $P(\mathcal{U})$ if and only if there is a functional $G : \operatorname{Im} f_{\mathcal{U}} \to \mathbb{R}_+ \cup \{+\infty\}$ being $\preceq_l^{\mathcal{K}}$-monotone such that*

$$x \in S, \quad f_{\mathcal{U}}(x) \sim f_{\mathcal{U}}(\bar{x}) \quad \Longleftrightarrow \quad G(f_{\mathcal{U}}(x)) = 0.$$

(b) *$\bar{x}$ is a strictly optimistic solution of problem $P(\mathcal{U})$ if and only if there is a functional $G : \operatorname{Im} f_{\mathcal{U}} \to \mathbb{R}_+ \cup \{+\infty\}$ being $\preceq_l^{\mathcal{K}}$-monotone such that*

$$x \in S, \quad G(f_{\mathcal{U}}(x)) = 0 \quad \Longleftrightarrow \quad x = \bar{x}.$$

7. Conclusion

This paper introduces a new scalarizing functional and investigates its properties to characterize solutions of a set-valued problem equipped with variable domination structures. This functional is also used to study the well-posedness properties of a set-valued problem w.r.t. variable domination structures. In addition, we highlight that this functional has many applications not only in location problems but also in uncertain problems. Our future research is studying some numerical methods to calculate solutions of set-valued optimization problems with respect to variable domination structures.

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