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POINCARÉ LEMMA ON HEISENBERG GROUPS

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ABSTRACT. For smooth functions $a_1, b_1, \dots, a_n, b_n$ on a Heisenberg group, a necessary and sufficient condition for the vector field $(a_1, b_1, \dots, a_n, b_n)$ being conservative is proved. In addition, the potential function is solved explicitly for the conservative vector field.

1. INTRODUCTION

There are all variety of vector fields that arise widely in physical system, such as electric fields, magnetic fields, force fields, and gravitational fields. A vector field is *conservative* provided that its line integral along any curve depends only on the endpoints of the curve [13]. To a force field, such a line integral is known as a work done in moving an object along a curve. Besides path independent line integrals, a conservative vector field may be characterized by a gradient of a scalar function. Let $V = (a, b, c)$ be a vector-valued function defined on a simply connected region D in $\mathbb{R}^3$. Then V is conservative if there exists a scalar function f such that $\nabla f = V$, or $\omega = df$ by using the 1-form $\omega = a(x, y, z)dx + b(x, y, z)dy + c(x, y, z)dz$. In which case,

$$\begin{aligned} f(\mathbf{p}) = f(x,y,z) &= \int_{\gamma(t)} \omega = \int_{\gamma(t)} (a,b,c)\cdot(dx,dy,dz) = \int_0^1 (a,b,c)\cdot\dot{\gamma}(t)dt \\ &= \int_0^1 \big(a(t\mathbf{p})x + b(t\mathbf{p})y + c(t\mathbf{p})z\big)dt, \end{aligned}$$

where $\gamma(t) = t(x, y, z) = (tx, ty, tz) = (x(t), y(t), z(t))$ is a curve joining the origin and the point $\mathbf{p} = (x, y, z)$. A straightforward computation gives

$$\begin{aligned} f_x(\mathbf{p}) &= \int_0^1 a(t\mathbf{p}) + tx\partial_x a(t\mathbf{p}) + ty\partial_x b(t\mathbf{p}) + tz\partial_x c(t\mathbf{p})dt \\ &= \int_0^1 a(t\mathbf{p}) + t\left[\frac{d}{dt}a(t\mathbf{p}) - y\partial_y a(t\mathbf{p}) - z\partial_z a(t\mathbf{p})\right] + ty\partial_x b(t\mathbf{p}) + tz\partial_x c(t\mathbf{p})dt \\ &= \int_0^1 \frac{d}{dt}\Big(ta(t\mathbf{p})\Big) + ty(\partial_x b - \partial_y a)(t\mathbf{p}) + tz(\partial_x c - \partial_z a)(t\mathbf{p})dt \\ &= a(\mathbf{p}) + \int_0^1 ty(\partial_x b - \partial_y a)(t\mathbf{p}) + tz(\partial_x c - \partial_z a)(t\mathbf{p})dt. \end{aligned}$$

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Similarly,

$$f_y(\mathbf{p}) = b(\mathbf{p}) + \int_0^1 tx(\partial_y a - \partial_x b)(t\mathbf{p}) + tz(\partial_y c - \partial_z b)(t\mathbf{p})dt,$$

$$f_z(\mathbf{p}) = c(\mathbf{p}) + \int_0^1 tx(\partial_z a - \partial_x c)(t\mathbf{p}) + ty(\partial_z b - \partial_y c)(t\mathbf{p})dt.$$

In vector form,

$$\nabla f(\mathbf{p}) = V(\mathbf{p}) + \int_0^1 tM\mathbf{p}^T dt = V(\mathbf{p}) + \int_0^1 \mathbf{r}(t) \times \operatorname{curl} V dt,$$

where

$$M = \begin{pmatrix} 0 & (\partial_x b - \partial_y a)(t\mathbf{p}) & (\partial_x c - \partial_z a)(t\mathbf{p}) \\ (\partial_y a - \partial_x b)(t\mathbf{p}) & 0 & (\partial_y c - \partial_z b)(t\mathbf{p}) \\ (\partial_z a - \partial_x c)(t\mathbf{p}) & (\partial_z b - \partial_y c)(t\mathbf{p}) & 0 \end{pmatrix},$$

and

$$\operatorname{curl} V = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ a & b & c \end{vmatrix}$$

It is not difficult to verify that $\operatorname{curl} V = 0$ if and only if $\nabla f(\mathbf{p}) = V(\mathbf{p})$. Thus, $\operatorname{curl} V = 0$ also plays as another characterization for V to be conservative.

Now let us turn to a non-commutative case. Here we are interested in the isotropic Heisenberg group since this group and its sub-Laplacian are at the corss-roads of many domains of analysis and geometry: nilpotent Lie groups theory, hypoelliptic second order partial differential equations, strongly pseudoconvex domains in several complex variables, control theory and semiclassical analysis of quantum mechanics, see *e.g.,* [2], [4], [5], [10] and [12].

The Heisenberg group $\mathcal{H}^n$ may be considered as $\mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law [1]

$$\begin{aligned}(\tilde{x}_1, \tilde{y}_1, \dots, \tilde{x}_n, \tilde{y}_n, \tilde{z}) &= (x_1, y_1, \dots, x_n, y_n, z) \cdot (x'_1, y'_1, \dots, x'_n, y'_n, z') \\ &= (x_1 + x'_1, y_1 + y'_1, \dots, x_n + x'_n, y_n + y'_n, z + z' + 2\sum_{j=1}^n (x_j y'_j - y_j x'_j)).\end{aligned} \tag{1.1}$$

The *Heisenberg vector fields* on $\mathcal{H}^n$ are given by

$$X_j = \partial_{x_j} - 2y_j\partial_z, \quad Y_j = \partial_{y_j} + 2x_j\partial_z, \quad j = 1, \dots, n. \tag{1.2}$$

These vector fields are all left-invariant since, by (1.1),

$$\partial_{x'_j} - 2y'_j\partial_{z'} = \left(\frac{\partial \tilde{x}_j}{\partial x'_j}\partial_{\tilde{x}_j} + \frac{\partial \tilde{z}}{\partial x'_j}\partial_{\tilde{z}}\right) - 2(\tilde{y}_j - y_j)\frac{\partial \tilde{z}}{\partial z'}\partial_{\tilde{z}} = \partial_{\tilde{x}_j} - 2\tilde{y}_j\partial_{\tilde{z}},$$

$$\partial_{y'_j} + 2x'_j\partial_{z'} = \left(\frac{\partial \tilde{y}_j}{\partial y'_j}\partial_{\tilde{y}_j} + \frac{\partial \tilde{z}}{\partial y'_j}\partial_{\tilde{z}}\right) + 2(\tilde{x}_j - x_j)\frac{\partial \tilde{z}}{\partial z'}\partial_{\tilde{z}} = \partial_{\tilde{y}_j} + 2\tilde{x}_j\partial_{\tilde{z}}.$$

Given smooth functions $a_j, b_j, j = 1, \ldots, n$. Then $V = (a_1, b_1, \ldots, a_n, b_n)$ is conservative if and only if

$$X_j f = a_j, \quad Y_j f = b_j, \quad j = 1, \ldots, n \tag{1.3}$$

is solvable. To find conditions for the solvability, one first note that the subRiemannian geometry on $\mathcal{H}^n$ is different from that of the Riemannian geometry of $\mathbb{R}^n$. We recorded results on $\mathcal{H}^1$ as follows.

Theorem 1.1 ([3]). Let X_1, Y_1 be the Heisenberg vector fields. The system $X_1 f = a, Y_1 f = b$ has a solution if and only if

$$X_1^2 b = (X_1 Y_1 + [X_1, Y_1])a,$$
$$Y_1^2 a = (Y_1 X_1 + [Y_1, X_1])b.$$

Theorem 1.2 ([6]). Let X_1, Y_1 be the Heisenberg vector fields defined as (1.2) and $\mathbf{p} = (x, y, z)$ in $\mathcal{H}^1$. Given any smooth functions a and b, and set

$$c = X_1 b - Y_1 a, \quad a_1 = a + y\frac{c}{2}, \quad b_1 = b - x\frac{c}{2}, \quad c_1 = \frac{c}{4}.$$

Consider

$$f(\mathbf{p}) = \int_0^1 \big[a_1(t\mathbf{p})x + b_1(t\mathbf{p})y + c_1(t\mathbf{p})z\big]\, dt.$$

Then

$$(X_1 f)(\mathbf{p}) = a(\mathbf{p}) + \int_0^1 \frac{tz}{4}(X_1^2 b - (X_1 Y_1 + [X_1, Y_1])a)(t\mathbf{p})dt,$$
$$(Y_1 f)(\mathbf{p}) = b(\mathbf{p}) - \int_0^1 \frac{tz}{4}(Y_1^2 a - (Y_1 X_1 + [Y_1, X_1])b)(t\mathbf{p})dt.$$

If the conditions

$$X_1^2 b = (X_1 Y_1 + [X_1, Y_1])a, \quad Y_1^2 a = (Y_1 X_1 + [Y_1, X_1])b$$

hold, then $X_1 f = a, Y_1 f = b$ with

$$f(\mathbf{p}) = \int_0^1 \big[a(t\mathbf{p})x + b(t\mathbf{p})y\big]\, dt.$$

Besides $\mathcal{H}^1$, related studies may be consulted from [8, 11]. In the following, we are going to find a necessary and sufficient condition to be given in (2.4), called the *integrability condition*, for the system (1.3) being solvable. The solution of the system will be proved as the form in (3.1).

2. Integrability Condition

Let T be a vector field defined by $T = [X_j, Y_j], 1 \le j \le n$, where $[X, Y] = XY - YX$ is the Lie bracket of vector fields X and Y. Then $\{X_1, Y_1, \ldots, X_n, Y_n,$

$T\}$ forms an orthonormal basis on $\mathcal{H}^n$ with respect to a Riemannian metric g. Observe that $Tf = X_jY_jf - Y_jX_jf, 1 \le j \le n$, and

$$\left\{\begin{array}{c} X_1f = a_1 \\ Y_1f = b_1 \\ \vdots \\ X_nf = a_n \\ Y_nf = b_n \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} X_1f = a_1 \\ Y_1f = b_1 \\ \vdots \\ X_nf = a_n \\ Y_nf = b_n \\ Tf = c, \end{array}\right. \tag{2.1}$$

where $c = X_jb_j - Y_ja_j, 1 \le j \le n$. Consider two vector fields

$$\operatorname{grad} f = \sum_{j=1}^{n}\Big((X_jf)X_j + (Y_jf)Y_j\Big) + (Tf)T, \quad \text{and} \quad U = \sum_{j=1}^{n}(a_jX_j + b_jY_j) + cT.$$

Then (2.1) is equivalent to

$$\left\{\begin{array}{c} X_1f = a_1 \\ Y_1f = b_1 \\ \vdots \\ X_nf = a_n \\ Y_nf = b_n \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} X_1f = a_1 \\ Y_1f = b_1 \\ \vdots \\ X_nf = a_n \\ Y_nf = b_n \\ Tf = c, \end{array}\right. \Leftrightarrow \operatorname{grad} f = U \Leftrightarrow \operatorname{curl} U = 0$$

$$\begin{aligned} &\Leftrightarrow A(X_j, Y_j) = A(X_j, X_l) = A(X_j, Y_l) = A(Y_j, Y_l) \\ &\quad = A(X_j, T) = A(Y_j, T) = 0, \quad 1 \le j \ne l \le n, \end{aligned} \tag{2.2}$$

where $\operatorname{curl} U$ is a 2-covariant antisymmetric tensor A on a pair of vector fields (X, Y) defined by

$$A(X, Y) = Yg(U, X) - Xg(U, Y) + g(U, [X, Y]). \tag{2.3}$$

The proof of $\operatorname{grad} f = U \Leftrightarrow \operatorname{curl} U = 0$ can be found in [9]. Applying (2.3) on $\{X_1, Y_1, \ldots, X_n, Y_n, T\}$ we have

$$\begin{aligned} A(X_j, Y_j) &= Y_ja_j - X_jb_j + c, \quad A(X_j, Y_l) = Y_la_j - X_jb_l, \\ A(X_j, X_l) &= X_la_j - X_ja_l, \quad A(Y_j, Y_l) = Y_lb_j - Y_jb_l, \\ A(X_j, T) &= Ta_j - X_jc = [X_j, Y_j]a_j - X_j(X_jb_j - Y_ja_j) \\ &= ([X_j, Y_j] + X_jY_j)a_j - X_j^2b_j, \\ A(Y_j, T) &= Y_j^2a_j - ([Y_j, X_j] + Y_jX_j)b_j. \end{aligned}$$

Thus (2.2) is equivalent to

$$\begin{gathered} X_jb_j - Y_ja_j = c, \quad X_la_j = X_ja_l, \quad X_jb_l = Y_la_j, \quad Y_lb_j = Y_jb_l, \\ X_j^2b_j = ([X_j, Y_j] + X_jY_j)a_j, \quad Y_j^2a_j = ([Y_j, X_j] + Y_jX_j)b_j. \end{gathered}$$

We proved the following theorem.

Theorem 2.1. Let X_j and Y_j be the Heisenberg vector fields on $\mathcal{H}^n$ defined as in (1.2). For smooth functions a_j and b_j, the system $X_jf = a_j, Y_jf = b_j, j = 1, \dots, n$ is solvable if and only if

$$\begin{cases} X_la_j = X_ja_l, \quad X_jb_l = Y_la_j, \quad Y_lb_j = Y_jb_l, \\ X_j^2b_j = ([X_j, Y_j] + X_jY_j)a_j, \quad Y_j^2a_j = ([Y_j, X_j] + Y_jX_j)b_j, \\ X_1b_1 - Y_1a_1 = \cdots = X_nb_n - Y_na_n, \end{cases} \tag{2.4}$$

where $1 \le j \ne l \le n$.

A similar characterization as of Theorem 2.1 can be obtained for the product space $\mathcal{H}^n \times \mathcal{H}^m$. Geometric analysis on this group has been studied in [7]. In particular, all geodesics connecting any points on $\mathcal{H}^n \times \mathcal{H}^m$ had been calculated explicitly.

Theorem 2.2. Let $X_l^{(n)}, Y_l^{(n)}, X_s^{(m)}, Y_s^{(m)}, 1 \le l \le n, 1 \le s \le m$ be the Heisenberg vector fields on $\mathcal{H}^n \times \mathcal{H}^m$ given by

$$\begin{aligned} X_l^{(n)} &= \partial_{x_l^{(n)}} - 2y_l^{(n)}\partial_{z^{(n)}}, \quad & Y_l^{(n)} &= \partial_{y_l^{(n)}} + 2x_l^{(n)}\partial_{z^{(n)}}, \\ X_s^{(m)} &= \partial_{x_s^{(m)}} - 2y_s^{(m)}\partial_{z^{(m)}}, \quad & Y_s^{(m)} &= \partial_{y_s^{(m)}} + 2x_s^{(m)}\partial_{z^{(m)}}. \end{aligned} \tag{2.5}$$

For smooth functions $a_l^{(n)}, b_l^{(n)}, a_s^{(m)}$, and $b_s^{(m)}$, the system $X_l^{(n)}f = a_l^{(n)}, Y_l^{(n)}f = b_l^{(n)}, X_s^{(m)}f = a_s^{(m)}, Y_s^{(m)}f = b_s^{(m)}, 1 \le l \le n, 1 \le s \le m$ is solvable if and only if

$$\begin{aligned}
&X_l^{(n)}a_j^{(n)} = X_j^{(n)}a_l^{(n)}, \quad X_j^{(n)}b_l^{(n)} = Y_l^{(n)}a_j^{(n)}, \quad Y_l^{(n)}b_j^{(n)} = Y_j^{(n)}b_l^{(n)}, \\
&(X_j^{(n)})^2b_j^{(n)} = ([X_j^{(n)}, Y_j^{(n)}] + X_j^{(n)}Y_j^{(n)})a_j^{(n)}, \\
&(Y_j^{(n)})^2a_j^{(n)} = ([Y_j^{(n)}, X_j^{(n)}] + Y_j^{(n)}X_j^{(n)})b_j^{(n)}, \\
&X_1^{(n)}b_1^{(n)} - Y_1^{(n)}a_1^{(n)} = \cdots = X_n^{(n)}b_n^{(n)} - Y_n^{(n)}a_n^{(n)}, \\
&X_s^{(m)}a_k^{(m)} = X_k^{(m)}a_s^{(m)}, \quad X_k^{(m)}b_s^{(m)} = Y_s^{(m)}a_k^{(m)}, \quad Y_s^{(m)}b_k^{(m)} = Y_k^{(m)}b_s^{(m)}, \\
&(X_k^{(m)})^2b_k^{(m)} = ([X_k^{(m)}, Y_k^{(m)}] + X_k^{(m)}Y_k^{(m)})a_k^{(m)}, \\
&(Y_k^{(m)})^2a_k^{(m)} = ([Y_k^{(m)}, X_k^{(m)}] + Y_k^{(m)}X_k^{(m)})b_k^{(m)}, \\
&X_1^{(m)}b_1^{(m)} - Y_1^{(m)}a_1^{(m)} = \cdots = X_m^{(m)}b_m^{(m)} - Y_m^{(m)}a_m^{(m)}, \\
&X_s^{(m)}a_l^{(n)} = X_l^{(n)}a_s^{(m)}, \quad Y_s^{(m)}a_l^{(n)} = X_l^{(n)}b_s^{(m)}, \\
&X_s^{(m)}b_l^{(n)} = Y_l^{(n)}a_s^{(m)}, \quad Y_s^{(m)}b_l^{(n)} = Y_l^{(n)}b_s^{(m)}, \quad 1 \le j \ne l \le n, 1 \le k \ne s \le m.
\end{aligned}$$

Proof. Let $T^{(n)} = [X_l^{(n)}, Y_l^{(n)}]$ and $T^{(m)} = [X_s^{(m)}, Y_s^{(m)}]$. Consider vector fields

$$\begin{aligned} \operatorname{grad} f = &\sum_{l=1}^{n} \Big((X_l^{(n)}f)X_l^{(n)} + (Y_l^{(n)}f)Y_l^{(n)} \Big) + (T^{(n)}f)T^{(n)} \\ &+ \sum_{s=1}^{m} \Big((X_s^{(m)}f)X_s^{(m)} + (Y_s^{(m)}f)Y_s^{(m)} \Big) + (T^{(m)}f)T^{(m)}, \end{aligned}$$

and

$$U = \sum_{l=1}^{n}\left(a_l^{(n)}X_l^{(n)} + b_l^{(n)}Y_l^{(n)}\right) + c^{(n)}T^{(n)} + \sum_{s=1}^{m}\left(a_s^{(m)}X_s^{(m)} + b_s^{(m)}Y_s^{(m)}\right) + c^{(m)}T^{(m)},$$

where $c^{(n)} = X_l^{(n)}b_l^{(n)} - Y_l^{(n)}a_l^{(n)}$, and $c^{(m)} = X_s^{(m)}b_s^{(m)} - Y_s^{(m)}a_s^{(m)}$. Then,

$$\begin{cases} X_l^{(n)}f = a_l^{(n)} \\ Y_l^{(n)}f = b_l^{(n)} \\ X_s^{(m)}f = a_s^{(m)} \\ Y_s^{(m)}f = b_s^{(m)} \end{cases} \Leftrightarrow \begin{cases} X_l^{(n)}f = a_l^{(n)} \\ Y_l^{(n)}f = b_l^{(n)} \\ X_s^{(m)}f = a_s^{(m)} \\ Y_s^{(m)}f = b_s^{(m)} \\ T^{(n)}f = c^{(n)} \\ T^{(m)}f = c^{(m)} \end{cases} \Leftrightarrow \operatorname{grad} f = U \Leftrightarrow \operatorname{curl} U = 0.$$

Using (2.3), $\operatorname{curl} U = 0$ if and only if the following (2.6), (2.7), and (2.8) hold

$$\text{(2.6)}\quad \begin{cases} A(X_j^{(n)}, Y_j^{(n)}) = A(X_j^{(n)}, X_l^{(n)}) = A(X_j^{(n)}, Y_l^{(n)}) = A(Y_j^{(n)}, Y_l^{(n)}) \\ = A(X_j^{(n)}, T^{(n)}) = A(Y_j^{(n)}, T^{(n)}) = 0, \quad 1 \le j \ne l \le n, \end{cases}$$

$$\text{(2.7)}\quad \begin{cases} A(X_k^{(m)}, Y_k^{(m)}) = A(X_k^{(m)}, X_s^{(m)}) = A(X_k^{(m)}, Y_s^{(m)}) = A(Y_k^{(m)}, Y_s^{(m)}) \\ = A(X_k^{(m)}, T^{(m)}) = A(Y_k^{(m)}, T^{(m)}) = 0, \quad 1 \le k \ne s \le m, \end{cases}$$

$$\text{(2.8)}\quad \begin{cases} A(X_l^{(n)}, X_s^{(m)}) = A(X_l^{(n)}, Y_s^{(m)}) = A(Y_l^{(n)}, X_s^{(m)}) = A(Y_l^{(n)}, Y_s^{(m)}) \\ = A(X_l^{(n)}, T^{(m)}) = A(Y_l^{(n)}, T^{(m)}) = A(X_s^{(m)}, T^{(n)}) = A(Y_s^{(m)}, T^{(n)}) \\ = A(T^{(n)}, T^{(m)}) = 0, \quad 1 \le l \le n, 1 \le s \le m. \end{cases}$$

(2.6) and (2.7) are direct consequences from (2.4). For (2.8), note that

$$A(X_l^{(n)}, X_s^{(m)}) = X_s^{(m)}g(U, X_l^{(n)}) - X_l^{(n)}g(U, X_s^{(m)}) = X_s^{(m)}a_l^{(n)} - X_l^{(n)}a_s^{(m)},$$

$$A(X_l^{(n)}, Y_s^{(m)}) = Y_s^{(m)}a_l^{(n)} - X_l^{(n)}b_s^{(m)},$$

$$A(Y_l^{(n)}, X_s^{(m)}) = X_s^{(m)}b_l^{(n)} - Y_l^{(n)}a_s^{(m)}, \quad A(Y_l^{(n)}, Y_s^{(m)}) = Y_s^{(m)}b_l^{(n)} - Y_l^{(n)}b_s^{(m)},$$

and

$$\begin{aligned} A(X_l^{(n)}, T^{(m)}) &= T^{(m)}a_l^{(n)} - X_l^{(n)}c^{(m)} \\ &= [X_s^{(m)}, Y_s^{(m)}]a_l^{(n)} - X_l^{(n)}(X_s^{(m)}b_s^{(m)} - Y_s^{(m)}a_s^{(m)}) \\ &= X_s^{(m)}(Y_s^{(m)}a_l^{(n)} - X_l^{(n)}b_s^{(m)}) - Y_s^{(m)}(X_s^{(m)}a_l^{(n)} - X_l^{(n)}a_s^{(m)}) \\ &= X_s^{(m)}A(X_l^{(n)}, Y_s^{(m)}) - Y_s^{(m)}A(X_l^{(n)}, X_s^{(m)}), \\ A(Y_l^{(n)}, T^{(m)}) &= X_s^{(m)}A(Y_l^{(n)}, Y_s^{(m)}) - Y_s^{(m)}A(Y_l^{(n)}, X_s^{(m)}), \\ A(X_s^{(m)}, T^{(n)}) &= X_l^{(n)}A(X_s^{(m)}, Y_l^{(n)}) - Y_l^{(n)}A(X_s^{(m)}, X_l^{(n)}), \\ A(Y_s^{(m)}, T^{(n)}) &= X_l^{(n)}A(Y_s^{(m)}, Y_l^{(n)}) - Y_l^{(n)}A(Y_s^{(m)}, X_l^{(n)}), \\ A(T^{(n)}, T^{(m)}) &= [X_s^{(m)}, Y_s^{(m)}](X_l^{(n)}b_l^{(n)} - Y_l^{(n)}a_l^{(n)}) \end{aligned}$$

$$
\begin{aligned}
&- [X_l^{(n)}, Y_l^{(n)}](X_s^{(m)} b_s^{(m)} - Y_s^{(m)} a_s^{(m)}) \\
&= X_l^{(n)} [X_s^{(m)}, Y_s^{(m)}] b_l^{(n)} - Y_l^{(n)} [X_s^{(m)}, Y_s^{(m)}] a_l^{(n)} \\
&\quad - (X_l^{(n)} Y_l^{(n)} - Y_l^{(n)} X_l^{(n)})(X_s^{(m)} b_s^{(m)} - Y_s^{(m)} a_s^{(m)}) \\
&= X_l^{(n)} A(Y_l^{(n)}, T^{(m)}) - Y_l^{(n)} A(X_l^{(n)}, T^{(m)}) \\
&= X_l^{(n)} \big(X_s^{(m)} A(Y_l^{(n)}, Y_s^{(m)}) - Y_s^{(m)} A(Y_l^{(n)}, X_s^{(m)})\big) \\
&\quad - Y_l^{(n)} \big(X_s^{(m)} A(X_l^{(n)}, Y_s^{(m)}) - Y_s^{(m)} A(X_l^{(n)}, X_s^{(m)})\big).
\end{aligned}
$$

Equalities in (2.8) follows. □

3. Poincaré Lemma

If the condition (2.4) is held, then the system $X_j f = a_j, Y_j f = b_j, j = 1, \ldots, n$ is solvable. The function f can solved explicitly as shown in the following theorem.

Theorem 3.1. Let X_j and Y_j be the Heisenberg vector fields on $\mathcal{H}^n$ defined as in (1.2). Given smooth functions $a_1, b_1, \ldots, a_n, b_n$ with

$$X_1 b_1 - Y_1 a_1 = \cdots = X_n b_n - Y_n a_n,$$

and let

$$c^* = \frac{X_j b_j - Y_j a_j}{4}, \quad a_j^* = a_j + 2y_j c^*, \quad b_j^* = b_j - 2x_j c^*, \quad 1 \le j \le n.$$

Consider

$$f(\mathbf{p}) = \int_0^1 \sum_{j=1}^n \big(a_j^*(t\mathbf{p}) x_j + b_j^*(t\mathbf{p}) y_j\big) + c^*(t\mathbf{p}) z dt,$$

where $\mathbf{p} = (x_1, y_1, \ldots, x_n, y_n, z)$ in $\mathcal{H}^n$. Then for $l = 1, \ldots, n$,

$$
\begin{aligned}
X_l f(\mathbf{p}) = a_l(\mathbf{p}) + \int_0^1 t\Big\{ &\sum_{\substack{j=1\\ j\neq l}}^n [x_j (X_l a_j - X_j a_l)(t\mathbf{p}) + y_j (X_l b_j - Y_j a_l)(t\mathbf{p})] \\
&+ \frac{z}{4}\Big(X_l^2 b_l - (X_l Y_l + [X_l, Y_l]) a_l\Big)(t\mathbf{p})\Big\} dt, \\
Y_l f(\mathbf{p}) = b_l(\mathbf{p}) + \int_0^1 t\Big\{ &\sum_{\substack{j=1\\ j\neq l}}^n [x_j (Y_l a_j - X_j b_l)(t\mathbf{p}) + y_j (Y_l b_j - Y_j b_l)(t\mathbf{p})] \\
&+ \frac{z}{4}\Big((Y_l X_l + [Y_l, X_l]) b_l - Y_l^2 a_l\Big)(t\mathbf{p})\Big\} dt.
\end{aligned}
$$

If the conditions

$$
\begin{aligned}
&X_l a_j = X_j a_l, \quad X_j b_l = Y_l a_j, \quad Y_l b_j = Y_j b_l, \\
&X_j^2 b_j = ([X_j, Y_j] + X_j Y_j) a_j, \quad Y_j^2 a_j = ([Y_j, X_j] + Y_j X_j) b_j, \quad 1 \le j \neq l \le n
\end{aligned}
$$

hold, then the system $X_j f = a_j, Y_j f = b_j, j = 1, \ldots, n$ is solvable and

$$f(\mathbf{p}) = \int_0^1 g\big(U(\gamma(t)), \dot{\gamma}(t)\big) dt, \tag{3.1}$$

where γ is a horizontal curve joining the origin and $\mathbf{p}$, $U = \sum_{j=1}^n (a_j X_j + b_j Y_j)$, and $g(\cdot,\cdot)$ is the subRiemannian metric.

Proof. By (1.2),

$$\begin{cases} X_1 f = a_1 \\ Y_1 f = b_1 \\ \quad\vdots \\ X_n f = a_n \\ Y_n f = b_n \end{cases} \Leftrightarrow \begin{cases} \partial_{x_1} f = a_1 + 2y_1 \partial_z f \\ \partial_{y_1} f = b_1 - 2x_1 \partial_z f \\ \quad\vdots \\ \partial_{x_n} f = a_n + 2y_n \partial_z f \\ \partial_{y_n} f = b_n - 2x_n \partial_z f \end{cases} \Leftrightarrow \begin{cases} \partial_{x_1} f = a_1^* \\ \partial_{y_1} f = b_1^* \\ \quad\vdots \\ \partial_{x_n} f = a_n^* \\ \partial_{y_n} f = b_n^* \end{cases} \Leftrightarrow \begin{cases} \partial_{x_1} f = a_1^* \\ \partial_{y_1} f = b_1^* \\ \quad\vdots \\ \partial_{x_n} f = a_n^* \\ \partial_{y_n} f = b_n^* \\ \partial_z f = c^*, \end{cases}$$

where for $1 \le j \le n$,

$$c^* = \frac{c}{4} = \frac{X_j b_j - Y_j a_j}{4}, \quad a_j^* = a_j + 2y_j c^*, \quad b_j^* = b_j - 2x_j c^*.$$

Consider

$$f(\mathbf{p}) = \int_{\gamma(t)} \omega = \int_0^1 \sum_{j=1}^n \big(a_j^*(t\mathbf{p})x_j + b_j^*(t\mathbf{p})y_j\big) + c^*(t\mathbf{p})z dt, \tag{3.2}$$

where $\omega = \sum_{j=1}^n \big(a_j^* dx_j + b_j^* dy_j\big) + c^* dz$ and

$$\begin{aligned} \gamma(t) &= t\mathbf{p} = (tx_1, ty_1, \dots, tx_n, ty_n, tz) \\ &= (x_1(t), x_2(t), \dots, x_n(t), y_n(t), z(t)), \quad t \in [0,1] \end{aligned}$$

is a horizontal curve connecting the origin and $\mathbf{p} = (x_1, y_1, \dots, x_n, y_n, z)$ in $\mathcal{H}^n$. Note that

$$\begin{aligned} \frac{d}{dt} a_l^*(t\mathbf{p}) &= x_l \partial_{x_l} a_l^*(t\mathbf{p}) + \sum_{\substack{j=1 \\ j\neq l}}^n x_j \partial_{x_j} a_l^*(t\mathbf{p}) + \sum_{j=1}^n y_j \partial_{y_j} a_l^*(t\mathbf{p}) + z\partial_z a_l^*(t\mathbf{p}), \\ \frac{d}{dt} b_l^*(t\mathbf{p}) &= y_l \partial_{y_l} b_l^*(t\mathbf{p}) + \sum_{j=1}^n x_j \partial_{x_j} b_l^*(t\mathbf{p}) + \sum_{\substack{j=1 \\ j\neq l}}^n y_j \partial_{y_j} b_l^*(t\mathbf{p}) + z\partial_z b_l^*(t\mathbf{p}), \\ \frac{d}{dt} c^*(t\mathbf{p}) &= \sum_{j=1}^n x_j \partial_{x_j} c^*(t\mathbf{p}) + \sum_{j=1}^n y_j \partial_{y_j} c^*(t\mathbf{p}) + z\partial_z c^*(t\mathbf{p}), \end{aligned}$$

so

$$\begin{aligned} \partial_{x_l} f(\mathbf{p}) = \int_0^1 \Big\{ & t\Big[\frac{d}{dt} a_l^*(t\mathbf{p}) - \sum_{\substack{j=1 \\ j\neq l}}^n x_j \partial_{x_j} a_l^*(t\mathbf{p}) - \sum_{j=1}^n y_j \partial_{y_j} a_l^*(t\mathbf{p}) - z\partial_z a_l^*(t\mathbf{p})\Big] \\ & + a_l^*(t\mathbf{p}) + t\Big[\sum_{\substack{j=1 \\ j\neq l}}^n x_j \partial_{x_l} a_j^*(t\mathbf{p}) + \sum_{j=1}^n y_j \partial_{x_l} b_j^*(t\mathbf{p}) + z\partial_{x_l} c^*(t\mathbf{p})\Big] \Big\} dt \\ = a_l^*(\mathbf{p}) + \int_0^1 t \Big\{ & \sum_{\substack{j=1 \\ j\neq l}}^n x_j [\partial_{x_l} a_j^*(t\mathbf{p}) - \partial_{x_j} a_l^*(t\mathbf{p})] \end{aligned}$$

$$+\sum_{j=1}^{n} y_j[\partial_{x_l} b_j^*(t\mathbf{p}) - \partial_{y_j} a_l^*(t\mathbf{p})] + z[\partial_{x_l} c^*(t\mathbf{p}) - \partial_z a_l^*(t\mathbf{p})]\Big\}dt, \tag{3.3}$$

$$\begin{aligned}
\partial_{y_l} f(\mathbf{p}) &= \int_0^1 \Big\{t[\frac{d}{dt} b_l^*(t\mathbf{p}) - \sum_{j=1}^{n} x_j \partial_{x_j} b_l^*(t\mathbf{p}) - \sum_{\substack{j=1\\ j\neq l}}^{n} y_j \partial_{y_j} b_l^*(t\mathbf{p}) - z\partial_z b_l^*(t\mathbf{p})] \\
&\quad + b_l^*(t\mathbf{p}) + t[\sum_{j=1}^{n} x_j \partial_{y_l} a_j^*(t\mathbf{p}) + \sum_{\substack{j=1\\ j\neq l}}^{n} y_j \partial_{y_l} b_j^*(t\mathbf{p}) + z\partial_{y_l} c^*(t\mathbf{p})]\Big\}dt \\
&= b_l^*(\mathbf{p}) + \int_0^1 t\Big\{\sum_{j=1}^{n} x_j[\partial_{y_l} a_j^*(t\mathbf{p}) - \partial_{x_j} b_l^*(t\mathbf{p})]
\end{aligned}$$

$$+\sum_{\substack{j=1\\ j\neq l}}^{n} y_j[\partial_{y_l} b_j^*(t\mathbf{p}) - \partial_{y_j} b_l^*(t\mathbf{p})] + z[\partial_{y_l} c^*(t\mathbf{p}) - \partial_z b_l^*(t\mathbf{p})]\Big\}dt, \tag{3.4}$$

$$\begin{aligned}
\partial_z f(\mathbf{p}) &= \int_0^1 \Big\{t[\frac{d}{dt} c^*(t\mathbf{p}) - \sum_{j=1}^{n} x_j \partial_{x_j} c^*(t\mathbf{p}) - \sum_{j=1}^{n} y_j \partial_{y_j} c^*(t\mathbf{p})] \\
&\quad + c^*(t\mathbf{p}) + t[\sum_{j=1}^{n} x_j \partial_z a_j^*(t\mathbf{p}) + \sum_{j=1}^{n} y_j \partial_z b_j^*(t\mathbf{p})]\Big\}dt
\end{aligned}$$

$$= c^*(\mathbf{p}) + \int_0^1 t\sum_{j=1}^{n}\Big\{x_j[\partial_z a_j^*(t\mathbf{p}) - \partial_{x_j} c^*(t\mathbf{p})] + y_j[\partial_z b_j^*(t\mathbf{p}) - \partial_{y_j} c^*(t\mathbf{p})]\Big\}dt. \tag{3.5}$$

Let

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & -2y_1 \\ & 1 & \cdots & 0 & 0 & 2x_1 \\ & & \ddots & & & \vdots \\ & & & 1 & 0 & -2y_n \\ & & & & 1 & 2x_n \end{pmatrix}$$

be a $2n \times (2n+1)$ upper-triangular matrix. Then from (1.2), (3.3), (3.4), and (3.5),

$$\begin{pmatrix} X_1 f(\mathbf{p}) \\ Y_1 f(\mathbf{p}) \\ \vdots \\ X_n f(\mathbf{p}) \\ Y_n f(\mathbf{p}) \end{pmatrix} = B \begin{pmatrix} \partial_{x_1} f(\mathbf{p}) \\ \partial_{y_1} f(\mathbf{p}) \\ \vdots \\ \partial_{x_n} f(\mathbf{p}) \\ \partial_{y_n} f(\mathbf{p}) \\ \partial_z f(\mathbf{p}) \end{pmatrix} = B \begin{pmatrix} a_1^*(\mathbf{p}) \\ b_1^*(\mathbf{p}) \\ \vdots \\ a_n^*(\mathbf{p}) \\ b_n^*(\mathbf{p}) \\ c^*(\mathbf{p}) \end{pmatrix} + \int_0^1 (tBM\mathbf{p}^T)(t\mathbf{p})dt$$

$$= \begin{pmatrix} a_1(\mathbf{p}) \\ b_1(\mathbf{p}) \\ \vdots \\ a_n(\mathbf{p}) \\ b_n(\mathbf{p}) \end{pmatrix} + \int_0^1 (tBM\mathbf{p}^T)(t\mathbf{p})dt, \tag{3.6}$$

where $M = (m_{ij})$ is a $(2n+1) \times (2n+1)$ skew-symmetric matrix with entries

$$m_{ij} := \begin{cases} \left.\begin{array}{lcl} m_{(2l-1)(2s-1)} & = & \partial_{x_l} a_s^* - \partial_{x_s} a_l^* \\ m_{(2l)(2s-1)} & = & \partial_{y_l} a_s^* - \partial_{x_s} b_l^* \\ m_{(2l-1)(2s)} & = & \partial_{x_l} b_s^* - \partial_{y_s} a_l^* \\ m_{(2l)(2s)} & = & \partial_{y_l} b_s^* - \partial_{y_s} b_l^* \end{array}\right\} 1 \le l \le s \le n \\ \left.\begin{array}{lcl} m_{(2l-1)(2n+1)} & = & \partial_{x_l} c^* - \partial_z a_l^* \\ m_{(2l)(2n+1)} & = & \partial_{y_l} c^* - \partial_z b_l^* \end{array}\right\} 1 \le l \le n. \end{cases}$$

The integrand $tBM\mathbf{p}^T$ of (3.6) is a $2n \times 1$ vector with entries

$$\begin{aligned} (tBM\mathbf{p}^T)_{2l-1} &= t\Big\{ \sum_{\substack{j=1 \\ j\neq l}}^n x_j(\partial_{x_l} a_j^* - \partial_{x_j} a_l^*) + \sum_{j=1}^n y_j(\partial_{x_l} b_j^* - \partial_{y_j} a_l^*) \\ &\quad + z(\partial_{x_l} c^* - \partial_z a_l^*) - 2y_l \sum_{j=1}^n [x_j(\partial_z a_j^* - \partial_{x_j} c^*) + y_j(\partial_z b_j^* - \partial_{y_j} c^*)] \Big\} \\ &= t\Big\{ \sum_{j=1}^n [x_j(X_l a_j^* - X_j a_l^* - 2y_j \partial_z a_l^*) + y_j(X_l b_j^* - Y_j a_l^* + 2x_j \partial_z a_l^*)] \\ &\quad + z(\partial_{x_l} c^* - \partial_z a_l^*) + 2y_l \sum_{j=1}^n (x_j \partial_{x_j} + y_j \partial_{y_j}) c^* \Big\} \\ &= t\Big\{ \sum_{j=1}^n [x_j(X_l a_j - X_j a_l + 2y_j X_l c^* - 2y_l X_j c^*) \\ &\quad + y_j(X_l b_j - Y_j a_l - 2x_j X_l c^* - 2y_l Y_j c^*)] - 4y_l c^* \\ &\quad + z[(X_l + 2y_l \partial_z) c^* - \partial_z a_l - 2y_l \partial_z c^*] \\ &\quad + 2y_l \sum_{j=1}^n [x_j(X_j + 2y_j \partial_z) + y_j(Y_j - 2x_j \partial_z)] c^* \Big\} \\ &= t\Big\{ \sum_{j=1}^n [x_j(X_l a_j - X_j a_l) + y_j(X_l b_j - Y_j a_l)] - y_l(X_l b_l - Y_l a_l) \\ &\quad + \frac{z}{4}\Big(X_l(X_l b_l - Y_l a_l) - [X_l, Y_l] a_l \Big) \Big\} \\ &= t\Big\{ \sum_{\substack{j=1 \\ j\neq l}}^n [x_j(X_l a_j - X_j a_l) + y_j(X_l b_j - Y_j a_l)] + \frac{z}{4}\Big(X_l^2 b_l - (X_l Y_l + [X_l, Y_l]) a_l \Big) \Big\}, \end{aligned}$$

and, Similarly,

$$\begin{aligned} (tBM\mathbf{p}^T)_{2l} &= t\Big\{ \sum_{j=1}^n x_j(\partial_{y_l} a_j^* - \partial_{x_j} b_l^*) + \sum_{\substack{j=1 \\ j\neq l}}^n y_j(\partial_{y_l} b_j^* - \partial_{y_j} b_l^*) \\ &\quad + z(\partial_{y_l} c^* - \partial_z b_l^*) + 2x_l \sum_{j=1}^n [x_j(\partial_z a_j^* - \partial_{x_j} c^*) + y_j(\partial_z b_j^* - \partial_{y_j} c^*)] \Big\} \end{aligned}$$

$$= t\Big\{\sum_{\substack{j=1\\ j\neq l}}^{n}[x_j(Y_l a_j - X_j b_l) + y_j(Y_l b_j - Y_j b_l)] + \frac{z}{4}\Big((Y_l X_l + [Y_l, X_l])b_l - Y_l^2 a_l\Big)\Big\},$$

for $l = 1, \dots, n$. Under the integrability condition (2.4), the entries of the integrand $tBM\mathbf{p}^T$ are all zero. Thus, $X_j f = a_j$ and $Y_j f = b_j$ for $j = 1, \dots, n$ and its solution f can be deduced from (3.2) as

$$\int_{\gamma(t)} \omega = \int_0^1 \sum_{j=1}^{n} \big(a_j^*(\gamma(t))\dot{x}_j + b_j^*(\gamma(t))\dot{y}_j\big) + c^*(\gamma(t))\dot{z}dt$$

$$= \int_0^1 \sum_{j=1}^{n} \big(a_j(\gamma(t))\dot{x}_j + b_j(\gamma(t))\dot{y}_j\big) + [\dot{z} - 2\sum_{j=1}^{n}(x_j\dot{y}_j - y_j\dot{x}_j)]c^*(\gamma(t))dt. \tag{3.7}$$

Note that

$$\dot{\gamma} = \sum_{j=1}^{n}(\dot{x}_j\partial_{x_j} + \dot{y}_j\partial_{y_j}) + \dot{z}\partial_z$$

$$= \sum_{j=1}^{n}(\dot{x}_j X_j + \dot{y}_j Y_j) + [\dot{z} - 2\sum_{j=1}^{n}(x_j\dot{y}_j - y_j\dot{x}_j)]\partial_z. \tag{3.8}$$

Since γ is horizontal, $\dot{\gamma}$ can be constructed only by X_j's and Y_j's. Hence by (3.8), $\dot{z} = 2\sum_{j=1}^{n}(x_j\dot{y}_j - y_j\dot{x}_j)$ and so (3.7) turns into

$$\int_0^1 \sum_{j=1}^{n} \big(a_j(\gamma(t))\dot{x}_j + b_j(\gamma(t))\dot{y}_j\big)dt = \int_0^1 g\big(U(\gamma(t)), \dot{\gamma}(t)\big)dt,$$

where $U = \sum_{j=1}^{n}(a_j X_j + b_j Y_j)$ and $g(\cdot,\cdot)$ is the subRiemannian metric. We proved the theorem. □

An analogous version as Theorem 3.1 for the product space $\mathcal{H}^n \times \mathcal{H}^m$ is given as follows.

Theorem 3.2. Let $X_l^{(n)}, Y_l^{(n)}, X_s^{(m)}, Y_s^{(m)}, 1 \le l \le n, 1 \le s \le m$ be the Heisenberg vector fields on $\mathcal{H}^n \times \mathcal{H}^m$ defined as in (2.5). Given smooth functions $a_1^{(n)}, b_1^{(n)}, \dots, a_n^{(n)}, b_n^{(n)}, a_1^{(m)}, b_1^{(m)}, \dots, a_m^{(m)}, b_m^{(m)}$ with

$$X_1^{(n)}b_1^{(n)} - Y_1^{(n)}a_1^{(n)} = \cdots = X_n^{(n)}b_n^{(n)} - Y_n^{(n)}a_n^{(n)},$$
$$X_1^{(m)}b_1^{(m)} - Y_1^{(m)}a_1^{(m)} = \cdots = X_m^{(m)}b_m^{(m)} - Y_m^{(m)}a_m^{(m)},$$

and let

$$c^{(n)*} = \frac{X_l^{(n)}b_l^{(n)} - Y_l^{(n)}a_l^{(n)}}{4}, \qquad c^{(m)*} = \frac{X_s^{(m)}b_s^{(m)} - Y_s^{(m)}a_s^{(m)}}{4},$$
$$a_l^{(n)*} = a_l^{(n)} + 2y_l^{(n)}c^{(n)*}, \qquad a_s^{(m)*} = a_s^{(m)} + 2y_s^{(m)}c^{(m)*},$$
$$b_l^{(n)*} = b_l^{(n)} - 2x_l^{(n)}c^{(n)*}, \qquad b_s^{(m)*} = b_s^{(m)} - 2x_s^{(m)}c^{(m)*},$$

for $1 \le l \le n, 1 \le s \le m$. Consider

$$\begin{aligned} f(\mathbf{p}) = \int_0^1 \sum_{l=1}^{n} \big(a_l^{(n)*}(t\mathbf{p})x_l^{(n)} + b_l^{(n)*}(t\mathbf{p})y_l^{(n)}\big) + c^{(n)*}(t\mathbf{p})z^{(n)} \\ + \sum_{s=1}^{m} \big(a_s^{(m)*}(t\mathbf{p})x_s^{(m)} + b_s^{(m)*}(t\mathbf{p})y_s^{(m)}\big) + c^{(m)*}(t\mathbf{p})z^{(m)}dt, \end{aligned} \tag{3.9}$$

where $\mathbf{p} = (x_1^{(n)}, y_1^{(n)}, \dots, x_n^{(n)}, y_n^{(n)}, z^{(n)}, x_1^{(m)}, y_1^{(m)}, \dots, x_m^{(m)}, y_m^{(m)}, z^{(m)})$.Then

$$\begin{aligned} &(X_1^{(n)}f, Y_1^{(n)}f, \dots, X_n^{(n)}f, Y_n^{(n)}f, X_1^{(m)}f, Y_1^{(m)}f, \dots, X_m^{(m)}f, Y_m^{(m)}f)^T(\mathbf{p}) \\ &= (a_1^{(n)}, b_1^{(n)}, \dots, a_n^{(n)}, b_n^{(n)}, a_1^{(m)}, b_1^{(m)}, \dots, a_m^{(m)}, b_m^{(m)})^T(\mathbf{p}) + \int_0^1 (tBM\mathbf{p}^T)(t\mathbf{p})dt, \end{aligned}$$

where the integrand $tBMp^T$ is a $(2n + 2m) \times 1$ vector with entries (3.10), (3.11), (3.12), and (3.13).

Proof. Let $f(\mathbf{p})$ be defined as (3.9). The partial derivatives of f are calculated as follows. Analogous to the proof of Theorem 3.1,

$$\begin{aligned} &\partial_{x_l^{(n)}} f(\mathbf{p}) = a_l^{(n)*}(\mathbf{p}) + \int_0^1 t\Big\{ \sum_{\substack{j=1 \\ j\neq l}}^{n} x_j^{(n)}[\partial_{x_l^{(n)}} a_j^{(n)*}(t\mathbf{p}) - \partial_{x_j^{(n)}} a_l^{(n)*}(t\mathbf{p})] \\ &+ \sum_{j=1}^{n} y_j^{(n)}[\partial_{x_l^{(n)}} b_j^{(n)*}(t\mathbf{p}) - \partial_{y_j^{(n)}} a_l^{(n)*}(t\mathbf{p})] + z^{(n)}[\partial_{x_l^{(n)}} c^{(n)*}(t\mathbf{p}) - \partial_{z^{(n)}} a_l^{(n)*}(t\mathbf{p})] \\ &+ \sum_{k=1}^{m} \{x_k^{(m)}[\partial_{x_l^{(n)}} a_k^{(m)*}(t\mathbf{p}) - \partial_{x_k^{(m)}} a_l^{(n)*}(t\mathbf{p})] + y_k^{(m)}[\partial_{x_l^{(n)}} b_k^{(m)*}(t\mathbf{p}) - \partial_{y_k^{(m)}} a_l^{(n)*}(t\mathbf{p})]\} \\ &+ z^{(m)}[\partial_{x_l^{(n)}} c^{(m)*}(t\mathbf{p}) - \partial_{z^{(m)}} a_l^{(n)*}(t\mathbf{p})]\Big\}dt, \\ &\partial_{y_l^{(n)}} f(\mathbf{p}) = b_l^{(n)*}(\mathbf{p}) + \int_0^1 t\Big\{ \sum_{j=1}^{n} x_j^{(n)}[\partial_{y_l^{(n)}} a_j^{(n)*}(t\mathbf{p}) - \partial_{x_j^{(n)}} b_l^{(n)*}(t\mathbf{p})] \\ &+ \sum_{\substack{j=1 \\ j\neq l}}^{n} y_j^{(n)}[\partial_{y_l^{(n)}} b_j^{(n)*}(t\mathbf{p}) - \partial_{y_j^{(n)}} b_l^{(n)*}(t\mathbf{p})] + z^{(n)}[\partial_{y_l^{(n)}} c^{(n)*}(t\mathbf{p}) - \partial_{z^{(n)}} b_l^{(n)*}(t\mathbf{p})] \\ &+ \sum_{k=1}^{m} \{x_k^{(m)}[\partial_{y_l^{(n)}} a_k^{(m)*}(t\mathbf{p}) - \partial_{x_k^{(m)}} b_l^{(n)*}(t\mathbf{p})] + y_k^{(m)}[\partial_{y_l^{(n)}} b_k^{(m)*}(t\mathbf{p}) - \partial_{y_k^{(m)}} b_l^{(n)*}(t\mathbf{p})]\} \\ &+ z^{(m)}[\partial_{y_l^{(n)}} c^{(m)*}(t\mathbf{p}) - \partial_{z^{(m)}} b_l^{(n)*}(t\mathbf{p})]\Big\}dt, \\ &\partial_{z^{(n)}} f(\mathbf{p}) = c^{(n)*}(\mathbf{p}) + \int_0^1 t \\ &\times \Big\{ \sum_{j=1}^{n} \{x_j^{(n)}[\partial_{z^{(n)}} a_j^{(n)*}(t\mathbf{p}) - \partial_{x_j^{(n)}} c^{(n)*}(t\mathbf{p})] + y_j^{(n)}[\partial_{z^{(n)}} b_j^{(n)*}(t\mathbf{p}) - \partial_{y_j^{(n)}} c^{(n)*}(t\mathbf{p})]\} \\ &+ \sum_{k=1}^{m} \{x_k^{(m)}[\partial_{z^{(n)}} a_k^{(m)*}(t\mathbf{p}) - \partial_{x_k^{(m)}} c^{(n)*}(t\mathbf{p})] + y_k^{(m)}[\partial_{z^{(n)}} b_k^{(m)*}(t\mathbf{p}) - \partial_{y_k^{(m)}} c^{(n)*}(t\mathbf{p})]\} \\ &+ z^{(m)}[\partial_{z^{(n)}} c^{(m)*}(t\mathbf{p}) - \partial_{z^{(m)}} c^{(n)*}(t\mathbf{p})]\Big\}dt, \end{aligned}$$

and

$$\partial_{x_s^{(m)}} f(\mathbf{p}) = a_s^{(m)*}(\mathbf{p}) + \int_0^1 t\Big\{ \sum_{j=1}^n \big\{ x_j^{(n)}[\partial_{x_s^{(m)}} a_j^{(n)*}(t\mathbf{p}) - \partial_{x_j^{(n)}} a_s^{(m)*}(t\mathbf{p})]$$
$$+ y_j^{(n)}[\partial_{x_s^{(m)}} b_j^{(n)*}(t\mathbf{p}) - \partial_{y_j^{(n)}} a_s^{(m)*}(t\mathbf{p})]\big\} + z^{(n)}[\partial_{x_s^{(m)}} c^{(n)*}(t\mathbf{p}) - \partial_{z^{(n)}} a_s^{(m)*}(t\mathbf{p})]$$
$$+ \sum_{\substack{k=1\\k\neq s}}^m x_k^{(m)}[\partial_{x_s^{(m)}} a_k^{(m)*}(t\mathbf{p}) - \partial_{x_k^{(m)}} a_s^{(m)*}(t\mathbf{p})] + \sum_{k=1}^m y_k^{(m)}[\partial_{x_s^{(m)}} b_k^{(m)*}(t\mathbf{p}) - \partial_{y_k^{(m)}} a_s^{(m)*}(t\mathbf{p})]$$
$$+ z^{(m)}[\partial_{x_s^{(m)}} c^{(m)*}(t\mathbf{p}) - \partial_{z^{(m)}} a_s^{(m)*}(t\mathbf{p})]\Big\} dt,$$

$$\partial_{y_s^{(m)}} f(\mathbf{p}) = b_s^{(m)*}(\mathbf{p}) + \int_0^1 t\Big\{ \sum_{j=1}^n \big\{ x_j^{(n)}[\partial_{y_s^{(m)}} a_j^{(n)*}(t\mathbf{p}) - \partial_{x_j^{(n)}} b_s^{(m)*}(t\mathbf{p})]$$
$$+ y_j^{(n)}[\partial_{y_s^{(m)}} b_j^{(n)*}(t\mathbf{p}) - \partial_{y_j^{(n)}} b_s^{(m)*}(t\mathbf{p})]\big\} + z^{(n)}[\partial_{y_s^{(m)}} c^{(n)*}(t\mathbf{p}) - \partial_{z^{(n)}} b_s^{(m)*}(t\mathbf{p})]$$
$$+ \sum_{k=1}^m x_k^{(m)}[\partial_{y_s^{(m)}} a_k^{(m)*}(t\mathbf{p}) - \partial_{x_k^{(m)}} b_s^{(m)*}(t\mathbf{p})] + \sum_{\substack{k=1\\k\neq s}}^m y_k^{(m)}[\partial_{y_s^{(m)}} b_k^{(m)*}(t\mathbf{p}) - \partial_{y_k^{(m)}} b_s^{(m)*}(t\mathbf{p})]$$
$$+ z^{(m)}[\partial_{y_s^{(m)}} c^{(m)*}(t\mathbf{p}) - \partial_{z^{(m)}} b_s^{(m)*}(t\mathbf{p})]\Big\} dt,$$

$$\partial_{z^{(m)}} f(\mathbf{p}) = c^{(m)*}(\mathbf{p}) + \int_0^1 t$$
$$\times \Big\{ \sum_{j=1}^n \big\{ x_j^{(n)}[\partial_{z^{(m)}} a_j^{(n)*}(t\mathbf{p}) - \partial_{x_j^{(n)}} c^{(m)*}(t\mathbf{p})] + y_j^{(n)}[\partial_{z^{(m)}} b_j^{(n)*}(t\mathbf{p}) - \partial_{y_j^{(n)}} c^{(m)*}(t\mathbf{p})]\big\}$$
$$+ z^{(n)}[\partial_{z^{(m)}} c^{(n)*}(t\mathbf{p}) - \partial_{z^{(n)}} c^{(m)*}(t\mathbf{p})]$$
$$+ \sum_{k=1}^m \big\{ x_k^{(m)}[\partial_{z^{(m)}} a_k^{(m)*}(t\mathbf{p}) - \partial_{x_k^{(m)}} c^{(m)*}(t\mathbf{p})] + y_k^{(m)}[\partial_{z^{(m)}} b_k^{(m)*}(t\mathbf{p}) - \partial_{y_k^{(m)}} c^{(m)*}(t\mathbf{p})]\big\}\Big\} dt.$$

Let $B = B^{(n)} \oplus B^{(m)}$ be a $(2n+2m) \times (2n+2m+2)$ matrix, where

$$B^{(n)} = \begin{pmatrix} 1 & & & & -2y_1^{(n)} \\ & 1 & & & 2x_1^{(n)} \\ & & \ddots & & \vdots \\ & & & 1 & -2y_n^{(n)} \\ & & & & 1 \quad 2x_n^{(n)} \end{pmatrix}, \quad B^{(m)} = \begin{pmatrix} 1 & & & & -2y_1^{(m)} \\ & 1 & & & 2x_1^{(m)} \\ & & \ddots & & \vdots \\ & & & 1 & -2y_m^{(m)} \\ & & & & 1 \quad 2x_m^{(m)} \end{pmatrix}.$$

Then we may write

$$\begin{aligned} &(X_1^{(n)}f, Y_1^{(n)}f, \ldots, X_n^{(n)}f, Y_n^{(n)}f, X_1^{(m)}f, Y_1^{(m)}f, \ldots, X_m^{(m)}f, Y_m^{(m)}f)^T(\mathbf{p}) \\ &= B \cdot (\partial_{x_1^{(n)}} f, \partial_{y_1^{(n)}} f, \ldots, \partial_{x_n^{(n)}} f, \partial_{y_n^{(n)}} f, \partial_{z^{(n)}} f, \\ &\qquad \partial_{x_1^{(m)}} f, \partial_{y_1^{(m)}} f, \ldots, \partial_{x_m^{(m)}} f, \partial_{y_m^{(m)}} f, \partial_{z^{(m)}} f)^T(\mathbf{p}) \\ &= B \cdot (a_1^{(n)*}, b_1^{(n)*}, \ldots, a_n^{(n)*}, b_n^{(n)*}, c^{(n)*}, a_1^{(m)*}, b_1^{(m)*}, \ldots, a_m^{(m)*}, b_m^{(m)*}, c^{(m)*})^T(\mathbf{p}) \\ &\quad + \int_0^1 (tBM\mathbf{p}^T)(t\mathbf{p})dt \end{aligned}$$

$$= (a_1^{(n)}, b_1^{(n)}, \dots, a_n^{(n)}, b_n^{(n)}, a_1^{(m)}, b_1^{(m)}, \dots, a_m^{(m)}, b_m^{(m)})^T(\mathbf{p}) + \int_0^1 (tBM\mathbf{p}^T)(t\mathbf{p})dt,$$

where

$$M = \begin{pmatrix} M_{11} & M_{12} \\ -M_{12}^T & M_{22} \end{pmatrix}$$

is a $(2n+2m+2) \times (2n+2m+2)$ skew-symmetric matrix given by

$$(M_{11})_{ij} := \left\{ \begin{array}{l} \left.\begin{array}{lll} (M_{11})_{(2l-1)(2s-1)} & = & \partial_{x_l^{(n)}} a_s^{(n)*} - \partial_{x_s^{(n)}} a_l^{(n)*} \\ (M_{11})_{(2l)(2s-1)} & = & \partial_{y_l^{(n)}} a_s^{(n)*} - \partial_{x_s^{(n)}} b_l^{(n)*} \\ (M_{11})_{(2l-1)(2s)} & = & \partial_{x_l^{(n)}} b_s^{(n)*} - \partial_{y_s^{(n)}} a_l^{(n)*} \\ (M_{11})_{(2l)(2s)} & = & \partial_{y_l^{(n)}} b_s^{(n)*} - \partial_{y_s^{(n)}} b_l^{(n)*} \end{array}\right\} 1 \le l \le s \le n \\ \left.\begin{array}{lll} (M_{11})_{(2l-1)(2n+1)} & = & \partial_{x_l^{(n)}} c^{(n)*} - \partial_{z^{(n)}} a_l^{(n)*} \\ (M_{11})_{(2l)(2n+1)} & = & \partial_{y_l^{(n)}} c^{(n)*} - \partial_{z^{(n)}} b_l^{(n)*} \end{array}\right\} 1 \le l \le n, \end{array} \right.$$

$$(M_{12})_{ij} := \left\{ \begin{array}{l} \left.\begin{array}{lll} (M_{12})_{(2l-1)(2s-1)} & = & \partial_{x_l^{(n)}} a_s^{(m)*} - \partial_{x_s^{(m)}} a_l^{(n)*} \\ (M_{12})_{(2l)(2s-1)} & = & \partial_{y_l^{(n)}} a_s^{(m)*} - \partial_{x_s^{(m)}} b_l^{(n)*} \\ (M_{12})_{(2l-1)(2s)} & = & \partial_{x_l^{(n)}} b_s^{(m)*} - \partial_{y_s^{(m)}} a_l^{(n)*} \\ (M_{12})_{(2l)(2s)} & = & \partial_{y_l^{(n)}} b_s^{(m)*} - \partial_{y_s^{(m)}} b_l^{(n)*} \end{array}\right\} \begin{array}{l} 1 \le l \le n \\ 1 \le s \le m \end{array} \\ \left.\begin{array}{lll} (M_{12})_{(2n+1)(2s-1)} & = & \partial_{z^{(n)}} a_s^{(m)*} - \partial_{x_s^{(m)*}} c^{(n)*} \\ (M_{12})_{(2n+1)(2s)} & = & \partial_{z^{(n)}} b_s^{(m)*} - \partial_{y_s^{(m)}} c^{(n)*} \end{array}\right\} 1 \le s \le m \\ \left.\begin{array}{lll} (M_{12})_{(2l-1)(2m+1)} & = & \partial_{x_l^{(n)}} c^{(m)*} - \partial_{z^{(m)}} a_l^{(n)*} \\ (M_{12})_{(2l)(2m+1)} & = & \partial_{y_l^{(n)}} c^{(m)*} - \partial_{z^{(m)}} b_l^{(n)*} \end{array}\right\} 1 \le l \le n, \\ (M_{12})_{(2n+1)(2m+1)} = \partial_{z^{(n)}} c^{(m)*} - \partial_{z^{(m)}} c^{(n)*} \end{array} \right.$$

$$(M_{22})_{ij} := \left\{ \begin{array}{l} \left.\begin{array}{lll} (M_{22})_{(2l-1)(2s-1)} & = & \partial_{x_l^{(m)}} a_s^{(m)*} - \partial_{x_s^{(m)}} a_l^{(m)*} \\ (M_{22})_{(2l)(2s-1)} & = & \partial_{y_l^{(m)}} a_s^{(m)*} - \partial_{x_s^{(m)}} b_l^{(m)*} \\ (M_{22})_{(2l-1)(2s)} & = & \partial_{x_l^{(m)}} b_s^{(m)*} - \partial_{y_s^{(m)}} a_l^{(m)*} \\ (M_{22})_{(2l)(2s)} & = & \partial_{y_l^{(m)}} b_s^{(m)*} - \partial_{y_s^{(m)}} b_l^{(m)*} \end{array}\right\} 1 \le l \le s \le m \\ \left.\begin{array}{lll} (M_{22})_{(2l-1)(2m+1)} & = & \partial_{x_l^{(m)}} c^{(m)*} - \partial_{z^{(m)}} a_l^{(m)*} \\ (M_{22})_{(2l)(2m+1)} & = & \partial_{y_l^{(m)}} c^{(m)*} - \partial_{z^{(m)}} b_l^{(m)*} \end{array}\right\} 1 \le l \le m. \end{array} \right.$$

By straightforward computation, the integrand $tBMp^T$, a $(2n+2m) \times 1$ vector, is given by

$$(tBMp^T)_{2l-1} = t\Big\{ \sum_{\substack{j=1 \\ j \ne l}}^{n} [x_j^{(n)}(X_l^{(n)} a_j^{(n)} - X_j^{(n)} a_l^{(n)}) + y_j^{(n)}(X_l^{(n)} b_j^{(n)} - Y_j^{(n)} a_l^{(n)})]$$

$$
\begin{aligned}
&+ \frac{z^{(n)}}{4}\Big((X_l^{(n)})^2 b_l^{(n)} - (X_l^{(n)}Y_l^{(n)} + [X_l^{(n)}, Y_l^{(n)}])a_l^{(n)}\Big)\\
&+ \sum_{k=1}^{m}\{x_k^{(m)}[X_l^{(n)}a_k^{(m)} - X_k^{(m)}a_l^{(n)}] + y_k^{(m)}[X_l^{(n)}b_k^{(m)} - Y_k^{(m)}a_l^{(n)}]\}\\
&+ \frac{z^{(m)}}{4}[X_m^{(m)}(X_l^{(n)}b_m^{(m)} - Y_m^{(m)}a_l^{(n)}) - Y_m^{(m)}(X_l^{(n)}a_m^{(m)} - X_m^{(m)}a_l^{(n)})]\Big\}, \qquad (3.10)
\end{aligned}
$$

$$
\begin{aligned}
(tBMp^T)_{2l} = t\Big\{ &\sum_{\substack{j=1\\ j\neq l}}^{n}[x_j^{(n)}(Y_l^{(n)}a_j^{(n)} - X_j^{(n)}b_l^{(n)}) + y_j^{(n)}(Y_l^{(n)}b_j^{(n)} - Y_j^{(n)}b_l^{(n)})]\\
&+ \frac{z^{(n)}}{4}\Big((Y_l^{(n)}X_l^{(n)} + [Y_l^{(n)}, X_l^{(n)}])b_l^{(n)} - (Y_l^{(n)})^2 a_l^{(n)}\Big)\\
&+ \sum_{k=1}^{m}\{x_k^{(m)}[Y_l^{(n)}a_k^{(m)} - X_k^{(m)}b_l^{(n)}] + y_k^{(m)}[Y_l^{(n)}b_k^{(m)} - Y_k^{(m)}b_l^{(n)}]\}\\
&+ \frac{z^{(m)}}{4}[X_m^{(m)}(Y_l^{(n)}b_m^{(m)} - Y_m^{(m)}b_l^{(n)}) - Y_m^{(m)}(Y_l^{(n)}a_m^{(m)} - X_m^{(m)}b_l^{(n)})]\Big\}, \qquad (3.11)
\end{aligned}
$$

for $1 \le l \le n$, and

$$
\begin{aligned}
&(tBMp^T)_{n+2s-1}\\
&= t\Big\{\sum_{\substack{k=1\\ k\neq s}}^{m}[x_k^{(m)}(X_s^{(m)}a_k^{(m)} - X_k^{(m)}a_s^{(m)}) + y_k^{(m)}(X_s^{(m)}b_k^{(m)} - Y_k^{(m)}a_s^{(m)})]\\
&\quad + \frac{z^{(m)}}{4}\Big((X_s^{(m)})^2 b_s^{(m)} - (X_s^{(m)}Y_s^{(m)} + [X_s^{(m)}, Y_s^{(m)}])a_s^{(m)}\Big)\\
&\quad + \sum_{j=1}^{n}\{x_j^{(n)}[X_s^{(m)}a_j^{(n)} - X_j^{(n)}a_s^{(m)}] + y_j^{(n)}[X_s^{(m)}b_j^{(n)} - Y_j^{(n)}a_s^{(m)}]\}\\
&\quad + \frac{z^{(n)}}{4}[X_n^{(n)}(X_s^{(m)}b_n^{(n)} - Y_n^{(n)}a_s^{(m)}) - Y_n^{(n)}(X_s^{(m)}a_n^{(n)} - X_n^{(n)}a_s^{(m)})]\Big\}, \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
&(tBMp^T)_{n+2s}\\
&= t\Big\{\sum_{\substack{k=1\\ k\neq s}}^{m}[x_k^{(m)}(Y_s^{(m)}a_k^{(m)} - X_k^{(m)}b_s^{(m)}) + y_k^{(m)}(Y_s^{(m)}b_k^{(m)} - Y_k^{(m)}b_s^{(m)})]\\
&\quad + \frac{z^{(m)}}{4}\Big((Y_s^{(m)}X_s^{(m)} + [Y_s^{(m)}, X_s^{(m)}])b_s^{(m)} - (Y_s^{(m)})^2 a_s^{(m)}\Big)\\
&\quad + \sum_{j=1}^{n}\{x_j^{(n)}[Y_s^{(m)}a_j^{(n)} - X_j^{(n)}b_s^{(m)}] + y_j^{(n)}[Y_s^{(m)}b_j^{(n)} - Y_j^{(n)}b_s^{(m)}]\}\\
&\quad + \frac{z^{(n)}}{4}[X_n^{(n)}(Y_s^{(m)}b_n^{(n)} - Y_n^{(n)}b_s^{(m)}) - Y_n^{(n)}(Y_s^{(m)}a_n^{(n)} - X_n^{(n)}b_s^{(m)})]\Big\}, \qquad (3.13)
\end{aligned}
$$

for $1 \le s \le m$. □

Combining Theorem 2.2 and 3.2, we have the following theorem.

Theorem 3.3. A solution f of the system $X_l^{(n)}f = a_l^{(n)}, Y_l^{(n)}f = b_l^{(n)}, X_s^{(m)}f = a_s^{(m)}, Y_s^{(m)}f = b_s^{(m)}, 1 \leq l \leq n, 1 \leq s \leq m$ is given by

$$f(\mathbf{p}) = \int_0^1 g\big(U(\gamma(t)), \dot{\gamma}(t)\big)dt = f^{(n)}(\mathbf{p}) + f^{(m)}(\mathbf{p}),$$

where γ is a horizontal curve joining the origin and $\mathbf{p}$, $U = \sum_{l=1}^n (a_l^{(n)}X_l^{(n)} + b_l^{(n)}Y_l^{(n)}) + \sum_{s=1}^m (a_s^{(m)}X_s^{(m)} + b_s^{(m)}Y_s^{(m)})$, $g(\cdot,\cdot)$ is the subRiemannian metric, and $f^{(n)}$ and $f^{(m)}$ are the potential functions in $\mathcal{H}^n$ and $\mathcal{H}^m$, respectively.

Proof. Let $\gamma(t) = (x_1^{(n)}(t), y_1^{(n)}(t), \ldots, x_n^{(n)}(t), y_n^{(n)}(t), z^{(n)}(t), x_1^{(m)}(t), y_1^{(m)}(t), \ldots, x_m^{(m)}(t), y_m^{(m)}(t), z^{(m)}(t))$ be an arbitrary horizontal curve in $\mathcal{H}^n \times \mathcal{H}^m$ with end points $\gamma(0) = \mathbf{0}$ and $\gamma(1) = \mathbf{p}$. Then

$$\begin{aligned}
\dot{\gamma} &= \sum_{l=1}^n (\dot{x}_l^{(n)}\partial_{x_l^{(n)}} + \dot{y}_l^{(n)}\partial_{y_l^{(n)}}) + \dot{z}^{(n)}\partial_{z^{(n)}} + \sum_{s=1}^m (\dot{x}_s^{(m)}\partial_{x_s^{(m)}} + \dot{y}_s^{(m)}\partial_{y_s^{(m)}}) + \dot{z}^{(m)}\partial_{z^{(m)}} \\
&= \sum_{l=1}^n (\dot{x}_l^{(n)}X_l^{(n)} + \dot{y}_l^{(n)}Y_l^{(n)}) + [\dot{z}^{(n)} - 2\sum_{l=1}^n (x_l^{(n)}\dot{y}_l^{(n)} - y_l^{(n)}\dot{x}_l^{(n)})]\partial_{z^{(n)}} \\
&\quad + \sum_{s=1}^m (\dot{x}_s^{(m)}X_s^{(m)} + \dot{y}_s^{(m)}Y_s^{(m)}) + [\dot{z}^{(m)} - 2\sum_{s=1}^m (x_s^{(m)}\dot{y}_s^{(m)} - y_s^{(m)}\dot{x}_s^{(m)})]\partial_{z^{(m)}}.
\end{aligned}$$

Since γ is horizontal, we have $\dot{z}^{(n)} = 2\sum_{l=1}^n (x_l^{(n)}\dot{y}_l^{(n)} - y_l^{(n)}\dot{x}_l^{(n)})$ and $\dot{z}^{(m)} = 2\sum_{s=1}^m (x_s^{(m)}\dot{y}_s^{(m)} - y_s^{(m)}\dot{x}_s^{(m)})$. Hence, by (3.9), the solution f is

$$\begin{aligned}
&\int_0^1 \Big\{ \sum_{l=1}^n \big(a_l^{(n)*}(\gamma(t))\dot{x}_l^{(n)} + b_l^{(n)*}(\gamma(t))\dot{y}_l^{(n)}\big) + c^{(n)*}(\gamma(t))\dot{z}^{(n)} \\
&\quad + \sum_{s=1}^m \big(a_s^{(m)*}(\gamma(t))\dot{x}_s^{(m)} + b_s^{(m)*}(\gamma(t))\dot{y}_s^{(m)}\big) + c^{(m)*}(\gamma(t))\dot{z}^{(m)} \Big\} dt \\
&= \int_0^1 \Big\{ \sum_{l=1}^n \big(a_l^{(n)}(\gamma(t))\dot{x}_l^{(n)} + b_l^{(n)}(\gamma(t))\dot{y}_l^{(n)}\big) \\
&\qquad + [\dot{z}^{(n)} - 2\sum_{l=1}^n (x_l^{(n)}\dot{y}_l^{(n)} - y_l^{(n)}\dot{x}_l^{(n)})]c^{(n)*}(\gamma(t)) \\
&\quad + \sum_{s=1}^m \big(a_s^{(m)}(\gamma(t))\dot{x}_s^{(m)} + b_s^{(m)}(\gamma(t))\dot{y}_s^{(m)}\big) \\
&\qquad + [\dot{z}^{(m)} - 2\sum_{s=1}^m (x_s^{(m)}\dot{y}_s^{(m)} - y_s^{(m)}\dot{x}_s^{(m)})]c^{(m)*}(\gamma(t)) \Big\} dt \\
&= \int_0^1 \Big\{ \sum_{l=1}^n \big(a_l^{(n)}(\gamma(t))\dot{x}_l^{(n)} + b_l^{(n)}(\gamma(t))\dot{y}_l^{(n)}\big)
\end{aligned}$$

$$+\sum_{s=1}^{m}\Big(a_s^{(m)}(\gamma(t))\dot{x}_s^{(m)}+b_s^{(m)}(\gamma(t))\dot{y}_s^{(m)}\Big)\Big\}dt$$

$$=\int_0^1 g\big(U(\gamma(t)),\dot{\gamma}(t)\big)dt=f^{(n)}(\mathbf{p})+f^{(m)}(\mathbf{p}),$$

where $U=\sum_{l=1}^{n}(a_l^{(n)}X_l^{(n)}+b_l^{(n)}Y_l^{(n)})+\sum_{s=1}^{m}(a_s^{(m)}X_s^{(m)}+b_s^{(m)}Y_s^{(m)})$, $g(\cdot,\cdot)$ is the subRiemannian metric, and $f^{(n)}$ and $f^{(m)}$ are the potential functions in $\mathcal{H}^n$ and $\mathcal{H}^m$, respectively. We proved the theorem. □

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