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THE INFLUENCE FUNCTION PROPERTIES FOR A PROBLEM WITH DISCONTINUOUS SOLUTIONS

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Abstract. In the present paper we consider the boundary value problem describing deformations of a discontinuous Stieltjes string. Properties of the influence function (Green function) are investigated. The analysis is based on a refined Stieltjes integral.

1. Introduction

By the middle of the 20-th century, a topical issue in mathematical physics had concerned with the properties of the spectrum of the Schrödinger equation

$$-(u')' + qu = \lambda u \tag{1.1}$$

where q is a singular potential containing singularities of the delta function type and stronger singularities generated by discontinuities of solutions. This direction was stirred to activity by problems in theoretical physics (quantum mechanics) and attracted the attention of a rather broad circle of researches, which led to the writing of fundamental monographs like [1]–[8] and subsequently of many hundreds of papers (see, for example, [9]–[18]).

It should be noted that cases involving diverse impulsive perturbations (singular potentials) of Sturm — Liouville (Schrödinger) differential operators have been quite actively studied during the past decade. The deepest recent results are connected with the names of A.A. Shkalikov, A. M. Savchuk [19]–[21], P. Djakov, B.S. Mityagin [22], V.A. Mikhailets [23] and others.

Physicists explain the presence of the delta function in the coefficient q in equation (1.1) by the so-called delta interaction, in which case the equality

$$u'(\xi + 0) - u'(\xi - 0) = \gamma u(\xi)$$

holds at a singular point ξ (see [2]). Mathematically, the symbol δ' means that q contains the derivative of the delta function, while for physicists $\delta'(x - \xi)$ means that $u'(\xi - 0) = \Delta u(\xi)$ and $u'(\xi + 0) = \Delta u(\xi)$, where $\Delta u(\xi) = u(\xi + 0) - u(\xi - 0)$. Moreover, in the case of the δ' -interaction, the solution may be discontinuous at the corresponding singular point ξ.

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In mathematical physics, such singularities are traditionally described within the theory of generalized functions (Sobolev-Schwartz distributions). Due to the generality of this approach, the solutions (1.1) are deprived of a qualitative analysis. A pointwise analysis of solutions (1) in the case when q has simple singularities and can be assumed to be the distributional derivative of a bounded variation function Q was proposed in [24], where, in the development of Feller [25] and M. Krein's ideas (see the comments in [1]), the equation with generalized coefficients

$$-(pu')' + Q'u = F' \tag{1.2}$$

was replaced with the integrodifferential equation

$$-(pu')(x) + \int_0^x u dQ = F(x) - F(0) - pu'(0),$$

where p, Q and F are the functions of bounded variation on the interval $[0, \ell]$, the integral is understood in the Stiltjes sense, and the solutions belong to the class of absolutely continuous functions whose derivatives have a bounded variation on $[0, \ell]$.

In this paper we consider equation (1.2) in the case of considerably stronger singularities in the potential (of the δ' – interaction type). Here we replace equation (1.2) with the integrodifferential equation

$$-(pu'_\mu)(x) + \int_0^x u d[Q] = F(x) - F(0). \tag{1.3}$$

Our advance is associated with the concept of a generalized Stieltjes integral. To emphasize that we are considering this integral, we write a function in the differential in square brackets.

Based on physical, more precisely, variational justification of equation (1.3) we give an exact description of the influence function $K(x, s)$ for (1.3), which allows us to express the solution of equation (1.3) in traditional for the influence function form

$$u(x) = \int_0^\ell K(x, s) d[F(s)].$$

Initial physical boundary conditions, that we use while the determination of the Green function, are associated with the values of jumps of functions p, Q and F. Equation (1.3) is the base of all classical properties of a boundary value problem with an equation of the second order. The influence function acts as the traditional Greenfunction, although in this case traditional description in terms of the axiomatic for the Green function is very difficult.

2. Preliminaries

In this section we consider some notions, facts and definitions which we will need in the sequel.

The generalized Stieltjes inregral $\int_\alpha^\beta ud[v]$ (which is referred in the sequel as the π -integral) was first introduced by Yu. V. Pokornyi in [26]. Following [26], recall that the π - integral $\int_\alpha^\beta ud[v]$ for functions $u(x)$ and $v(x)$ of bounded variation can be represented as

$$\int_\alpha^\beta ud[v] = \int_\alpha^\beta udv_0 + \sum_{\alpha<s\leq\beta} u(s-0)\Delta^- v(s) + \sum_{\alpha\leq s<\beta} u(s+0)\Delta^+ v(s),$$

where v_0 is the continuous part of v and the integral $\int_\alpha^\beta udv_0$ is understood in the Lebesgue – Stieltjes sense. Here $\Delta^+ v(x)$ and $\Delta^- v(x)$ denote the right and left jumps of v at the point x, respectively; i.e., $\Delta^+ v(x) = v(x+0) - v(x)$ and $\Delta^- v(x) = v(x) - v(x-0)$. In view of the general nature of the π - integral, the integrating function $v(s)$ in this integral defines splitting measures (left and right) at singular points. Notice that the square brackets in the integral mean that the sum is extended over measures splitting at singular points. If $u(x)$ or $v(x)$ is continuous, then the π - integral coincides with the usual Stieltjes integral.

Following [26], if $v(x)$ and $u(x)$ are functions of bounded variation then for the π -integral $\int_\alpha^\beta u\, d[v]$ we have

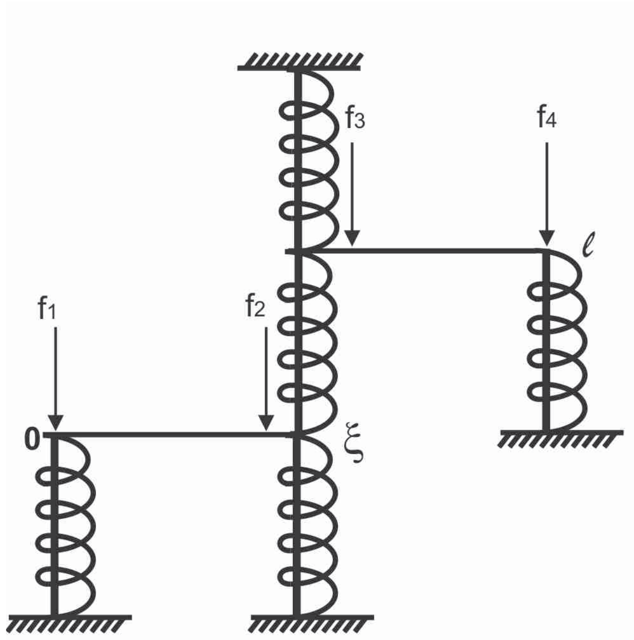
$$\int_\alpha^\beta u\, d[v] = u(\beta)v(\beta) - u(\alpha)v(\alpha) - \int_\alpha^\beta v\, du, \tag{2.1}$$

where the integral $\int_\alpha^\beta v\, du$ is understood in the Lebesgue-Stieltjes sense.

The main object in this paper is the integrodifferential equation (1.3), i.e.,

$$-pu'_\mu(x) + \int_0^x ud[Q] = F(x) - F(0).$$

Consider the variational motivation for equation (1.3). Let us consider a discontinuous Stieltjes string (a chain of strings, which are elastically connected by springs), which is located along the segment $[0, \ell]$. At the left end (at the point $x = 0$) and at the right end (at the point $x = \ell$) it has elastic supports (springs) with elasticities γ_1 and γ_4, respectively. For determinacy, we assume that only at the point $x = \xi$ we have a singularity, which is generated by the discontinuity of the string.



Let us investigate the case of small deformations. Let $u(x)$ denote a deviation of the considered physical system at the point x from the equilibrium state under the action of the external force of the intensity $F(x)$. Let us remark that at the point $x = \xi$ the function $u(x)$ is not defined, but the limit values $u(\xi - 0)$, $u(\xi + 0)$ (deviations of the respective ends of the string fastened by a spring) are defined and have a physical sense. If we denote by $F(x)$ the sum of all external forces applied to the segment $[0, x)$, then the total energy expended by the external force on giving the form $u(x)$ can be written as the following integral

$$\int_{[0,\ell]} u(x)\, d[F(x)].$$

In the particular case, when the force f_1 acts on the left end of the chain, f_2 and f_3 at the point of the junction of the strings, where the force f_2 acts on the right end of the first string and the force f_3 on the left end of the second string, f_4 acts on the right end of the chain, then

$$F(x) = \begin{cases} 0, & x = 0, \\ f_1, & 0 < x < \xi, \\ f_1 + f_2, & x = \xi, \\ f_1 + f_2 + f_3, & \xi < x < \ell, \\ f_1 + f_2 + f_3 + f_4, & x = \ell, \end{cases}$$

so that

$$\int_{[0,\ell]} u(x)\, d[F(x)] = u(0)f_1 + u(\xi - 0)f_2 + u(\xi + 0)f_3 + u(\ell)f_4.$$

Assume that $p(x)$ is a function characterizing the local tension of the strings of the considered model. We define the function $p(x)$ at the point of discontinuity as equal to the resistance of the spring that connects the strings. Then the internal energy, which is accumulated by the system due to its own resistance, is equal to

$$\frac{1}{2}\int\limits_{(0,\ell)} p(x)u_\mu'^2(x)\,d\mu(x),$$

where $\mu(x)$ is a strictly increasing function on the segment $[0,\ell]$ and $\mu(x)$ is discontinuous at the points of discontinuity of the function $u(x)$. Since the function $\mu(x)$ is continuous at the points $x = 0$ and $x = \ell$, we can assume that $p(0)u_\mu'(0) = p(\ell)u_\mu'(\ell) = 0$. So the following equality holds true

$$\int\limits_{(0,\ell)} p(x)u_\mu'^2(x)\,d\mu(x) = \int\limits_{[0,\ell]} p(x)u_\mu'^2(x)\,d\mu(x).$$

In the case when the model consists of the chain of 2 strings, which are connected by a spring with an elasticity γ at the point ξ, we have

$$\int\limits_0^\ell \frac{p(x)u_\mu'^2(x)}{2}\,d\mu(x) = \int\limits_0^{\xi-0} \frac{p(x)u_x'^2(x)}{2}\,dx + \frac{\gamma(\Delta u(\xi))^2}{2} + \int\limits_{\xi+0}^{\ell} \frac{p(x)u_x'^2(x)}{2}\,dx,$$

where $\mu(x) = x + \theta(x-\xi)$, $\theta(x)$ is the Heaviside function. Let the function $Q(x)$ characterize the elasticity of the external medium. As the considered model contains elastic supports of the spring type, we assume that the function $Q(x)$ is discontinuous at the points of the supports concentrating, and jumps of the function Q at these points must coincide with the elasticities of the corresponding springs. Then the elasticity of the external medium can be taken into account by means of the integral

$$\int\limits_{[0,\ell]} \frac{u^2(x)}{2}\,d[Q(x)].$$

In the particular case, when the right end of the first string has the spring with the elasticity γ_2 and the left end of the second string has the spring with the elasticity γ_3, function $Q(x)$ can be represented as

$$Q(x) = \begin{cases} 0, & x = 0, \\ \gamma_1, & 0 < x < \xi, \\ \gamma_1 + \gamma_2, & x = \xi, \\ \gamma_1 + \gamma_2 + \gamma_3, & \xi < x < \ell, \\ \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4, & x = \ell. \end{cases}$$

So

$$\int\limits_{[0,\ell]} \frac{u^2(x)}{2}\,d[Q(x)] = \frac{u^2(0)}{2}\gamma_1 + \frac{u^2(\xi-0)}{2}\gamma_2 + \frac{u^2(\xi+0)}{2}\gamma_3 + \frac{u^2(\ell)}{2}\gamma_4.$$

In the general case the potential energy can be expressed in the form

$$\Phi(u) = \frac{1}{2}\int\limits_{[0,\ell]} p(x)(u'_\mu(x))^2 d\mu(x) + \frac{1}{2}\int\limits_{[0,\ell]} u^2(x)\, d[Q(x)] - \int\limits_{[0,\ell]} u(x)\, d[F(x)]. \tag{2.2}$$

Notice that the functional $\Phi(u)$ can express the energy of the chain of the described type, that contains any number of strings. Suppose that $p(x)$, $Q(x)$, and $F(x)$ are the functions of bounded variation on the interval $[0,\ell]$, where $\inf\limits_{(0,\ell)} p(x) > 0$. The function $\mu(x)$ strictly increases on the interval $[0,\ell]$. We consider the functional (2.2) on the set E of μ - absolutely continuous functions whose derivatives $v'_\mu(x)$ are functions of bounded variation on the interval $[0,\ell]$. We emphasize that the considered function $u(x)$ is a hypothetical (virtual) deformation. According to Hamilton-Ostrogradski principle the real deformation $u_0(x)$ should give a minimum to the functional (2.2). If $u_0(x)$ gives a minimum of $\Phi(u)$ to E, then the first variation $\delta\Phi(u_0)h = \frac{d}{d\lambda}\Phi(u_0+\lambda h)\Big|_{\lambda=0}$ must equal to zero, i.e.,

$$\delta\Phi(u_0)h = \int\limits_0^\ell p(x)(u'_{0\mu}h'_\mu)(x)d\mu(x) + \int\limits_0^\ell u_0(x)h(x)\, d[Q(x)] - \int\limits_0^\ell h(x)\, d[F(x)] = 0.$$

Then with respect to (2.1) we obtain the equality

$$-\int\limits_0^\ell h(x)\, d[p(x)u'_{0\mu}(x)] + \int\limits_0^\ell h(x)\, d[g(x)] - \int\limits_0^\ell h(x)\, d[F(x)] = 0,$$

where $d[g(x)] = u_0(x)d[Q(x)]$. Hence,

$$\int\limits_0^\ell h(x)\, d[-p(x)u'_{0\mu}(x) + g(x) - F(x)] = 0$$

and

$$-p(x)u'_{0\mu}(x) + \int\limits_0^x u_0(s)\, d[Q(s)] - F(x) \equiv const = c,$$

or

$$-p(x)u'_{0\mu}(x) + \int\limits_0^x u_0(s)\, d[Q(s)] = F(x) - F(0). \tag{2.3}$$

Thus, we get that the real form of the considered system is a solution of equation (1.3). Moreover, from equality (2.3) it follows that at the points $x = 0$ and $x = \ell$ we have the boundary conditions

$$p(\ell-0)u'_{0\mu}(\ell-0) + u_0(\ell)\gamma_4 = f_4,$$

$$-p(+0)u'_{0\mu}(+0) + u_0(0)\gamma_1 = f_1.$$

Let us consider in more detail equation (1.3).

We assume that there exists a strictly increasing function $\mu(x)$ scaling the interval $[0,\ell]$ such that solutions of (1.3) are μ - absolutely continuous. We assume that p, Q and F are functions of bounded variation on $[0,\ell]$; $\inf\limits_{(0,\ell)} p>0$ and the integral is understood in the π-integral sense.The μ - derivative introduced above obeys the law

$$u(\beta)-u(\alpha)=\int\limits_{\alpha}^{\beta}u'_{\mu}d\mu,$$

so that $u'_{\mu}(\xi)=\dfrac{\Delta u(\xi)}{\Delta\mu(\xi)}$ at the discontinuity point ξ of μ.

To extend correctly the classical calculus methods to the situation under study, we have to replace conflict points with their extensions (more exactly, separations). Let $S(\mu)\in(0,\ell)$ be the set of discontinuity points of $\mu(x)$. Notice that $S(\mu)$ may be countable or finite. Solutions $u(x)$ of equation (1.3) belong to the class E_{μ} of μ-absolutely continuous functions whose derivatives u'_{μ} are functions of bounded variation on $[0,\ell]$. Thus, any solution $u(x)$ of equation (1.3) is a function of bounded variation on $[0,\ell]$, that can be discontinuous only at points from $S(\mu)$. The values of $u(\xi_i)$, where $\xi_i\in S(\mu)$, are not defined: only the limit values $u(\xi_i-0)$ and $u(\xi_i+0)$ are of importance in the π - integral.

We introduce $J_{\mu}=[0,\ell]\backslash S(\mu)$ with the metric $\rho(x,y)=|\mu(x)-\mu(y)|$. Clearly, the metric space (J_{μ},ρ) is not complete. Let $\overline{[0,\ell]}_{\mu}$ denote its completion with respect to ρ. Given an arbitrary discontinuity point ξ of $\mu(x)$, the set $\overline{[0,\ell]}_{\mu}$ contains a pair of elements, denoted by $\xi-0$ and $\xi+0$ respectively. Thus, any solution of equation (1.3) is defined on $\overline{[0,\ell]}_{\mu}$.

Let $R_{\mu}=\overline{[0,\ell]}_{\mu}\bigcup S(\mu)$. Define the function $\sigma(x)=x+p_1+p_2+Q_1+Q_2+F_1+F_2$, where p_i, Q_i and F_i are the increasing functions from the Jordan representation of the boundary variation functions $p(x)=p_1(x)-p_2(x)$, $Q(x)=Q_1(x)-Q_2(x)$, $F(x)=F_1(x)-F_2(x)$. It can be assumed that $\sigma(x)$ contains only the discontinuity points of p, Q and F. Denote by S the set of discontinuity points of $\sigma(x)$ that don't lie in $S(\mu)$. Let $JR_{\mu}=R_{\mu}\setminus S$. Let $\overline{JR_{\mu}}$ denote its completion with respect to the metric $\rho(x,y)=|\sigma(x)-\sigma(y)|$. We obtain that any point $s\in S$ is replaced with the pair $\{s-0,s+0\}$, 0 is replaced with $+0$, ℓ is replaced with $\ell-0$. Let $\overline{[0,\ell]}_S=\overline{JR_{\mu}}\cup 0\cup\ell$. Notice that $\overline{[0,\ell]}_S$ together with any discontinuity point ξ of $\mu(x)$, contains the pair $\{\xi-0,\xi+0\}$, while any point $s\in S$ is replaced by the pair $\{s-0,s+0\}$, and also this set contains the pair $\{0,+0\}$ and the pair $\{\ell,\ell-0\}$.

It follows from (1.3) that the derivative $u'_{\mu}(x)$ exists at any point x at which μ, p, Q and F are all continuous. At the other points, there are left and right derivatives $u'_{\mu}(\xi-0)$ and $u'_{\mu}(\xi+0)$ coinciding with one-sided limits. It follows from (1.3) that

$$-p(\xi)\frac{\Delta u(\xi)}{\Delta\mu(\xi)}+p(\xi-0)u'_{\mu}(\xi-0)+u(\xi-0)\Delta^{-}Q(\xi)=\Delta^{-}F(\xi),$$

$$p(\xi)\frac{\Delta u(\xi)}{\Delta\mu(\xi)}-p(\xi+0)u'_{\mu}(\xi+0)+u(\xi+0)\Delta^{+}Q(\xi)=\Delta^{+}F(\xi),$$

for discontinuity points ξ of $\mu(x)$ and

$$-p(s+0)u'_\mu(s+0)+p(s-0)u'_\mu(s-0)+u(s)\Delta Q(s)=\Delta F(s)$$

for points $s\in S$. Here, $\Delta F(\xi)=F(\xi+0)-F(\xi-0)$, $\Delta Q(\xi)=Q(\xi+0)-Q(\xi-0)$.

Also it follows from (1.3) that for points $x=0$ and $x=\ell$ we obtain boundary conditions

$$p(\ell-0)u'_\mu(\ell-0)+u(\ell)\Delta^-Q(\ell)=\Delta^-F(\ell),$$

$$-p(+0)u'_\mu(+0)+u(0)\Delta^+Q(0)=\Delta^+F(0).$$

A case when the functions p, Q, F, are continuous at points $x=0$ and $x=\ell$ was considered in [27]. According to [27], the equation

$$-pu'_\mu(x)+\int_0^x ud[Q]=F(x)-F(0)-pu'(0)$$

is similar in properties to a second order ordinary differential equation. Some of this properties will be used later.

Consider a homogeneous equation

$$-pu'_\mu(x)+\int_0^x ud[Q]=-pu'_\mu(0).$$

We assume that p, Q are continuous at points $x=0$ and $x=\ell$. A point $s\in(0,\ell)$ is called a zero point of a solution of the homogeneous equation if $u(s-0)u(s+0)\le 0$. Following [27], any nontrivial solution of the homogeneous equation can have only a finite number of zero points.

Theorem 2.1. *Suppose that the function $Q(x)$ does not decrease on the interval $[0,\ell]$ and $Q(x)$ is not a constant; $p(x)$, $F(x)$ are functions of boundary variation on $[0,\ell]$; $\mu(x)$ strictly increases on $[0,\ell]$; $\inf\limits_{(0,\ell)} p(x)>0$. Then equation (1.3) has a unique solution.*

Proof. Notice, that equation (1.3) can be rewritten in the form of a boundary value problem

$$\begin{cases} -p(x)u'_\mu(x)+\int\limits_{+0}^x u(s)\,d[Q(s)]=F(x)-F(+0)-p(+0)u'_\mu(+0),\\ p(\ell-0)u'_\mu(\ell-0)+u(\ell)\Delta^-Q(\ell)=\Delta^-F(\ell),\\ -p(+0)u'_\mu(+0)+u(0)\Delta^+Q(0)=\Delta^+F(0). \end{cases} \tag{2.4}$$

Indeed, from (1.3) it follows that

$$-p(x)u'_\mu(x)+\int_0^{+0} u(x)\,d[Q(x)]+\int_{+0}^x u(s)\,d[Q(s)]=F(x)-F(0).$$

We have

$$\int_{0}^{+0} u(x)\, d[Q(x)] = u(0)\Delta^{+}Q(0) = F(+0) - F(0) + p(+0)u'_{\mu}(+0).$$

Then

$$-p(x)u'_{\mu}(x) + F(+0) - F(0) + p(+0)u'_{\mu}(+0) + \int_{+0}^{x} u(s)\, d[Q(s)] = F(x) - F(0)$$

and

$$-p(x)u'_{\mu}(x) + \int_{+0}^{x} u(s)\, d[Q(s)] = F(x) - F(+0) - p(+0)u'_{\mu}(+0),$$

$x \in \overline{[0,\ell]}_s \backslash \{0 \cup \ell\}$. Define the functions p, Q, F at the points $x = 0$ and $x = \ell$ by limit values. Following [26], a solution of the equation

$$-p(x)u'_{\mu}(x) + \int_{+0}^{x} u(s)\, d[Q(s)] = F(x) - F(+0) - p(+0)u'_{\mu}(+0)$$

can be represented in the form

$$u(x) = c_1\varphi_1(x) + c_2\varphi_2(x) + z(x),$$

where $\varphi_1(x)$, $\varphi_2(x)$ is a fundamental system of solutions of the homogeneous equation and z is a solution of the inhomogeneous equation. Thus, it remains to prove that there exist c_1 and c_2 such that $u(x)$ satisfies the boundary conditions. The last question is equivalent to the problem of a trivial solution existence for the corresponding homogeneous problem

$$\begin{cases} -p(x)u'_{\mu}(x) + \displaystyle\int_{+0}^{x} u(s)\, d[Q(s)] = -p(+0)u'_{\mu}(+0), \\ p(\ell - 0)u'_{\mu}(\ell - 0) + u(\ell)\Delta^{-}Q(\ell) = 0, \\ -p(+0)u'_{\mu}(+0) + u(0)\Delta^{+}Q(0) = 0. \end{cases} \tag{2.5}$$

Following [26], for the case when the function Q does not decrease, any nontrivial solution of the equation

$$-p(x)u'_{\mu}(x) + \int_{+0}^{x} u(s)\, d[Q(s)] = -p(+0)u'_{\mu}(+0) \tag{2.6}$$

can have no more than one zero point. Suppose that problem (2.5) has a nontrivial solution $u(x)$. Consider a case when $u(x) > 0$. Then $p(+0)u'_{\mu}(+0) > 0$ and $u'_{\mu}(x) > 0$. But from the last fact it follows that $u'_{\mu}(\ell - 0) > 0$. This contradicts to the equality

$$p(\ell - 0)u'_{\mu}(\ell - 0) + u(\ell - 0)\Delta^{-}Q(\ell) = 0.$$

Similarly, the case $u(x) < 0$ is impossible. Let τ be a zero point of the function $u(x)$. If $u(0) = 0$, then $u'_\mu(+0) = 0$. According to [26], the Cauchy problem

$$\begin{cases} -p(x)u'_\mu(x) + \int\limits_{+0}^{x} u(s)\, d[Q(s)] = -p(+0)u'_\mu(+0), \\ u(0) = 0, \\ u'_\mu(+0) = 0 \end{cases}$$

has only the trivial solution. Similarly we can consider the case $u(\ell) = 0$. Suppose that the zero point τ belongs to $(0, \ell)$. Let $u(\tau - 0) < 0$, $u(\tau + 0) \geq 0$. Then $\Delta u(\tau) = u(\tau + 0) - u(\tau - 0) > 0$, i.e. $u'_\mu(\tau) = \dfrac{\Delta u(\tau)}{\Delta \mu(\tau)} > 0$. Let us consider (2.6), when $x \in [\tau + 0, \ell - 0]$. Then

$$-p(x)u'_\mu(x) + \int\limits_{\tau+0}^{x} u(s)\, d[Q(s)] = -p(\tau + 0)u'_\mu(\tau + 0).$$

Since the zero point τ can be only unique, we get $u(x) > 0$ for all $x > \tau$. Hence

$$\int\limits_{\tau+0}^{x} u(s)\, d[Q(s)] > 0.$$

On the other hand, we note that at the point τ we have the equality

$$-p(\tau + 0)u'_\mu(\tau + 0) + p(\tau)u'_\mu(\tau) + u(\tau + 0)\Delta^+ Q(\tau) = 0,$$

from which it follows that $p(\tau + 0)u'_\mu(\tau + 0) > 0$. From

$$p(x)u'_\mu(x) = \int\limits_{\tau+0}^{x} u(s)\, d[Q(s)] + p(\tau + 0)u'_\mu(\tau + 0),$$

it follows that $u'_\mu(x) > 0$, in particular, $u'_\mu(\ell - 0) > 0$. This contradicts to the equality

$$p(\ell - 0)u'_\mu(\ell - 0) + u(\ell)\Delta^- Q(\ell) = 0.$$

□

3. The influence function and its properties

In this section we give an exact description of the influence function, starting from the physical or, more precisely, the variational motivation for equation (1.3). At the physical level, the influence function $K(x, s)$ is defined as the deformation of the original system under the action of a unit force applied at the point $x = s$.

As it was shown above, the potential energy corresponding to a virtual deformation $u(x)$ arising under the influence of an external force $d[F(x)]$ is expressed by the functional

$$\Phi(u) = \int\limits_{[0,\ell]} p\frac{(u'_\mu)^2}{2} d\mu + \int\limits_{[0,\ell]} \frac{u^2}{2} d[Q] - \int\limits_{[0,\ell]} u d[F].$$

If the external load has a unit value and is applied only at a point $x = s$, where $u(x)$ is continuous at $x = s$, then the work of this force is equal to $u(s)$, and the last term in the representation of Φ becomes $\int_0^\ell u d[\Theta(x-s)]$, where $\Theta(x-s)$ is the Heaviside function. If the function $u(x)$ is discontinuous at a point ξ, then we consider two cases, when the work of this force is equal to $u(\xi - 0)$ or to $u(\xi + 0)$ respectively.

Thus, by the influence function $K(x, s)$ of the original problem (1.3) we mean the solution of the equation

$$-p(x)v'_\mu(x) + \int_0^x v(t)\, d[Q(t)] = \Theta(x-s), \tag{3.1}$$

where $s \in \overline{[0,\ell]}_\mu$ is fixed.

If the function $\mu(x)$ is continuous at a point ξ, then we define

$$\Theta(x-s) = \begin{cases} 1, & x > s, \\ 0, & x < s. \end{cases}$$

If the function $\mu(x)$ is discontinuous at a point ξ, then we define

$$\Theta(x-(\xi-0)) = \begin{cases} 0, & x < \xi, \\ 1, & x \geq \xi, \end{cases}$$

$$\Theta(x-(\xi+0)) = \begin{cases} 0, & x \leq \xi, \\ 1, & x > \xi. \end{cases}$$

If $s = 0$ then

$$\Theta(x) = \begin{cases} 1, & 0 < x < \ell, \\ 0, & x = 0. \end{cases}$$

If $s = \ell$ then

$$\Theta(x-\ell) = \begin{cases} 0, & 0 < x < \ell, \\ 1, & x = \ell. \end{cases}$$

The properties listed below can be obtained immediately from (3.1).

If the function $\mu(x)$ is continuous at a point $s \in (0, \ell)$, then

$$-p(s+0)K'_\mu(s+0,s) + p(s-0)K'_\mu(s-0,s) + K(s,s)\Delta Q(s) = 1.$$

If $s = 0$, then

$$-p(+0)K'_\mu(+0,0) + K(0,0)\Delta^+ Q(0) = 1.$$

If $s = \ell$, then

$$p(\ell-0)K'_\mu(\ell-0,\ell) + K(\ell,\ell)\Delta^- Q(\ell) = 1.$$

If the function $\mu(x)$ is discontinuous at a point ξ, then

$$-p(\xi)\frac{K(\xi+0,\xi-0) - K(\xi-0,\xi-0)}{\Delta\mu(\xi)} +$$

$$+p(\xi-0)K'_\mu(\xi-0,\xi-0) + K(\xi-0,\xi-0)\Delta^- Q(\xi) = 1,$$

$$p(\xi)\frac{K(\xi+0,\xi+0) - K(\xi-0,\xi+0)}{\Delta\mu(\xi)} -$$

$$-p(\xi+0)K'_\mu(\xi+0,\xi+0)+K(\xi+0,\xi+0)\Delta^+Q(\xi)=1.$$

Theorem 3.1. *The following equality holds:*

$$\max_{\overline{[0,\ell]}_\mu} K(x,s)=K(s,s).$$

Proof. Assume that $s=0$. The another cases can be considered similarly. We denote $v(x)=K(x,0)$. Then

$$-p(x)v'_\mu(x)+\int\limits_0^x v(t)\,d[Q(t)]=\Theta(x).$$

If $0<x<\ell$, then

$$-p(x)v'_\mu(x)+\int\limits_{+0}^x v(t)\,d[Q(t)]=-p(+0)v'_\mu(+0),$$

and $-p(+0)v'_\mu(+0)+v(0)\Delta^+Q(0)=1$.

As in Theorem 2.1, we can prove that the function $v(x)$ preserves the sign on the interval $[0,\ell]$ and moreover $v(x)>0$. Consider the equality

$$p(\ell-0)v'_\mu(\ell-0)+v(\ell-0)\Delta^-Q(\ell)=0.$$

Since $v(\ell-0)>0$, we have $p(\ell-0)v'_\mu(\ell-0)<0$. Notice that for $x\in[+0,\ell-0]$ we have

$$-p(x)v'_\mu(x)+\int\limits_{+0}^x v(t)\,d[Q(t)]=-p(+0)v'_\mu(+0),$$

$$-p(\ell-0)v'_\mu(\ell-0)+\int\limits_{+0}^{\ell-0} v(t)\,d[Q(t)]=-p(+0)v'_\mu(+0).$$

We obtain that

$$-p(\ell-0)v'_\mu(\ell-0)+p(x)v'_\mu(x)+\int\limits_x^{\ell-0} v(t)\,d[Q(t)]=0,$$

and it follows that

$$p(x)v'_\mu(x)=p(\ell-0)v'_\mu(\ell-0)-\int\limits_x^{\ell-0} v(t)\,d[Q(t)].$$

Hence, $v'_\mu(x)<0$ for all $x\in[+0,\ell-0]$.

Then the function $v(x)$ decreases on the interval $[+0,\ell-0]$. Hence, $\max_{\overline{[0,\ell]}_\mu} v(x)=v(+0)=v(0)$. □

Theorem 3.2. *Suppose that the function $Q(x)$ does not decrease on the interval $[0,\ell]$ and $Q(x)$ does not equal to a constant. Let $\varphi_1(x)$, $\varphi_2(x)$ be the solutions of the equations*

$$-p(x)v'_\mu(x)+\int\limits_0^x v(t)\,d[Q(t)]=\Theta(x),$$

$$-p(x)v'_\mu(x)+\int\limits_0^x v(t)\,d[Q(t)]=\Theta(x-\ell).$$

Then the influence function $K(x,s)$ can be represented as

$$K(x,s)=\frac{1}{\varphi_2(0)}\begin{cases}\varphi_1(x)\varphi_2(s), & 0\le s\le x\le\ell,\\ \varphi_2(x)\varphi_1(s), & 0\le x\le s\le\ell.\end{cases}\tag{3.2}$$

Proof. Let us prove that the function $K(x,s)$ from (3.2) is a solution of equation (3.1). Notice that $\varphi_2(0)\ne 0$. Indeed, the function $\varphi_2(x)$ is a solution of the problem

$$\begin{cases}-p(x)v'_\mu(x)+\int\limits_{+0}^x v(t)\,d[Q(t)]=-p(+0)v'_\mu(+0),\\ -p(+0)v'_\mu(+0)+v(0)\Delta^+Q(0)=0,\\ p(\ell-0)v'_\mu(\ell-0)+v(\ell)\Delta^-Q(\ell)=1.\end{cases}$$

If $\varphi_2(0)=0$, then $\varphi'_{2\mu}(+0)=0$. Following [26], the homogeneous Cauchy problem can have only the zero solution, hence $\varphi_2(x)\equiv 0$. Then the function $\varphi_2(x)$ does not satisfy the condition at the point ℓ. Hence, $\varphi_2(0)\ne 0$.

The function $K(\cdot,s)$ belongs to the class E when s is fixed, because $\varphi_1(\cdot)\in E$ and $\varphi_2(\cdot)\in E$.

Let us show that

$$-p(+0)K'_\mu(+0,0)+K(0,0)\Delta^+Q(0)=1.$$

We have

$$-\frac{p(+0)\varphi'_{1\mu}(+0)\varphi_2(0)}{\varphi_2(0)}+\frac{\varphi_1(0)\varphi_2(0)}{\varphi_2(0)}\Delta^+Q(0)=1.$$

Let us show that

$$p(\ell-0)K'_\mu(\ell-0,\ell)+K(\ell-0,\ell)\Delta^-Q(\ell)=1.$$

We have

$$\frac{p(\ell-0)\varphi_1(\ell)\varphi'_{2\mu}(\ell-0)}{\varphi_2(0)}+\frac{\varphi_2(\ell)\varphi_1(\ell)}{\varphi_2(0)}\Delta^-Q(\ell)=\frac{\varphi_1(\ell)}{\varphi_2(0)}.\tag{3.3}$$

According to [26] $\widetilde{p}(x)W(x)=const$, where the function $\widetilde{p}(x)$ coincides with $p(x)$ on $(0,\ell)$ and $\widetilde{p}(0)=p(+0)$, $\widetilde{p}(\ell)=p(\ell-0)$,

$$W(x)=\begin{vmatrix}\varphi_1(x) & \varphi_2(x)\\ \varphi'_{1\mu}(x) & \varphi'_{2\mu}(x)\end{vmatrix}.$$

Then

$$\widetilde{p}(\ell-0)W(\ell-0)=p(\ell-0)\begin{vmatrix}\varphi_1(\ell) & \varphi_2(\ell)\\ \varphi'_{1\mu}(\ell-0) & \varphi'_{2\mu}(\ell-0)\end{vmatrix}=$$
$$=p(\ell-0)(\varphi_1(\ell)\varphi'_{2\mu}(\ell-0)-\varphi_2(\ell)\varphi'_{1\mu}(\ell-0)).$$

On the other hand, since

$$p(\ell-0)\varphi'_{2\mu}(\ell-0)+\varphi_2(\ell)\Delta^-Q(\ell)=1,$$
$$p(\ell-0)\varphi'_{1\mu}(\ell-0)+\varphi_1(\ell)\Delta^-Q(\ell)=0,$$

we have

$$\widetilde{p}(\ell-0)W(\ell-0)=\varphi_1(\ell)(1-\varphi_2(\ell)\Delta^-Q(\ell))+\varphi_1(\ell)\varphi_2(\ell)\Delta^-Q(\ell)=\varphi_1(\ell).$$

But

$$\widetilde{p}(x)W(x)=p(+0)W(+0)=p(+0)(\varphi_1(0)\varphi'_{2\mu}(+0)-\varphi_2(0)\varphi'_{1\mu}(+0)).$$

Since

$$p(+0)\varphi'_{2\mu}(+0)=\varphi_2(0)\Delta^+Q(0),$$
$$p(+0)\varphi'_{1\mu}(+0)=\varphi_1(0)\Delta^+Q(0)-1,$$

then

$$p(+0)W(+0)=\varphi_1(0)\varphi_2(0)\Delta^+Q(0)-\varphi_2(0)(\varphi_1(0)\Delta^+Q(0)-1)=\varphi_2(0).$$

Hence, $\varphi_2(0)=\varphi_1(\ell)$ and the right-hand side of (3.3) is equal to 1.

Consider the case when $s=\xi-0$, where $\mu(x)$ is discontinuous at a point ξ. Let us show that

$$-p(x)K'_\mu(x,\xi-0)+\int_0^x K(t,\xi-0)\,d[Q(t)]=\Theta(x-(\xi-0)).$$

Let $x<\xi$. We must prove that

$$-p(x)K'_\mu(x,\xi-0)+\int_0^x K(t,\xi-0)\,d[Q(t)]=0.$$

We have

$$\frac{\varphi_1(\xi-0)}{\varphi_2(0)}\left(-p(x)\varphi'_{2\mu}(x)+\int_0^x \varphi_2(t)\,d[Q(t)]\right)=0.$$

Let us show that

$$-p(\xi)\frac{K(\xi+0,\xi-0)-K(\xi-0,\xi-0)}{\Delta\mu(\xi)}+p(\xi-0)K'_\mu(\xi-0,\xi-0)+$$
$$+K(\xi-0,\xi-0)\Delta^-Q(\xi)=1.$$

We have

$$-p(\xi)\frac{\varphi_1(\xi+0)\varphi_2(\xi-0)-\varphi_1(\xi-0)\varphi_2(\xi-0)}{\varphi_2(0)\Delta\mu(\xi)}+\frac{p(\xi-0)\varphi'_{2\mu}(\xi-0)\varphi_1(\xi-0)}{\varphi_2(0)}+$$
$$+\frac{\varphi_2(\xi-0)\varphi_1(\xi-0)}{\varphi_2(0)}\Delta^-Q(\xi)=$$

$$= -\frac{\varphi_2(\xi-0)p(\xi)\varphi'_{1\mu}(\xi)+p(\xi-0)\varphi'_{2\mu}(\xi-0)\varphi_1(\xi-0)}{\varphi_2(0)} - $$
$$-\frac{\varphi_2(\xi-0)\varphi_1(\xi-0)\Delta^-Q(\xi)}{\varphi_2(0)}.$$

Since

$$p(\xi)\varphi'_{1\mu}(\xi) = p(\xi-0)\varphi'_{1\mu}(\xi-0)+\varphi_1(\xi-0)\Delta^-Q(\xi),$$

we have

$$-\frac{\varphi_2(\xi-0)p(\xi-0)\varphi'_{1\mu}(\xi-0)+p(\xi-0)\varphi'_{2\mu}(\xi-0)\varphi_1(\xi-0)}{\varphi_2(0)} =$$
$$= \frac{p(\xi-0)W(\xi-0)}{\varphi_2(0)} = \frac{\varphi_2(0)}{\varphi_2(0)} = 1.$$

Similarly we can prove that

$$p(\xi)\frac{K(\xi+0,\xi+0)-K(\xi-0,\xi+0)}{\Delta\mu(\xi)} - p(\xi+0)K'_\mu(\xi+0,\xi+0)+$$
$$+K(\xi+0,\xi+0)\Delta^+Q(\xi) = 1.$$

Let $x > \xi$. Let us show that

$$-p(x)K'_\mu(x,\xi-0) + \int_0^x K(t,\xi-0)\, d[Q(t)] = 1.$$

Notice that $K'_\mu(x,\xi-0) = \dfrac{\varphi'_{1\mu}(x)\varphi_2(\xi-0)}{\varphi_2(0)}$,

$$\int_0^x K(t,\xi-0)\, d[Q(t)] = \int_0^{\xi-0} K(t,\xi-0)\, d[Q(t)] + K(\xi+0,\xi-0)\Delta^+Q(\xi)+$$
$$+\int_{\xi+0}^x K(t,\xi-0)\, d[Q(t)] + K(\xi-0,\xi+0)\Delta^-Q(\xi).$$

We have the equality

$$-\frac{p(x)\varphi'_{1\mu}(x)\varphi_2(\xi-0)}{\varphi_2(0)} + \int_0^{\xi-0} \frac{\varphi_2(t)\varphi_1(\xi-0)}{\varphi_2(0)} d[Q(t)]+$$
$$+\frac{\varphi_1(\xi+0)\varphi_2(\xi-0)\Delta^+Q(\xi)+\varphi_1(\xi-0)\varphi_2(\xi-0)\Delta^-Q(\xi)}{\varphi_2(0)}+$$
$$+\int_{\xi+0}^x \frac{\varphi_1(t)\varphi_2(\xi-0)}{\varphi_2(0)} d[Q(t)] = 1.$$

Notice that

$$-p(\xi-0)\varphi'_{2\mu}(\xi-0) + \int_0^{\xi-0} \varphi_2(t)\, d[Q(t)] = 0,$$

$$-p(x)\varphi'_{1\mu}(x)+\int\limits_{\xi+0}^{x}\varphi_1(t)\,d[Q(t)]=-p(\xi+0)\varphi'_{1\mu}(\xi+0).$$

Then the previous equality can be rewritten as

$$-\frac{\varphi_2(0)p(\xi+0)\varphi'_{1\mu}(\xi+0)+\varphi_1(\xi-0)p(\xi-0)\varphi'_{2\mu}(\xi-0)}{\varphi_2(0)}+$$
$$+\frac{\varphi_1(\xi+0)\varphi_2(\xi-0)\Delta^+Q(\xi)+\varphi_1(\xi-0)\varphi_2(\xi-0)\Delta^-Q(\xi)}{\varphi_2(0)}=$$
$$=-\frac{\varphi_2(\xi-0)p(\xi)\varphi'_{1\mu}(\xi)}{\varphi_2(0)}+\frac{\varphi_1(\xi-0)p(\xi-0)\varphi'_{2\mu}(\xi-0)}{\varphi_2(0)}+$$
$$+\frac{\varphi_1(\xi-0)\varphi_2(\xi-0)\Delta^-Q(\xi)}{\varphi_2(0)}=$$
$$=-\frac{\varphi_2(\xi-0)(\varphi_1(\xi-0)\Delta^-Q(\xi)+p(\xi-0)\varphi'_{1\mu}(\xi-0))}{\varphi_2(0)}+$$
$$+\frac{\varphi_1(\xi-0)\varphi_2(\xi-0)\Delta^-Q(\xi)}{\varphi_2(0)}+$$
$$+\frac{\varphi_1(\xi-0)p(\xi-0)\varphi'_{2\mu}(\xi-0)}{\varphi_2(0)}=\frac{p(\xi-0)W(\xi-0)}{\varphi_2(0)}=\frac{\varphi_2(0)}{\varphi_2(0)}=1.$$

The anouther cases can be considered in a similar way. □

Theorem 3.3. *Suppose that the function $Q(x)$ does not decrease on the interval $[0,\ell]$ and $Q(x)$ does not equal to a constant. Let $p(x)$, $F(x)$ be functions of boundary variation; $\inf\limits_{(0,\ell)} p(x)>0$; the function $\mu(x)$ strictly increases on $[0,\ell]$. Then the corresponding solution $u(x)$ of equation (1.3) can be represented in the form*

$$u(x)=\int\limits_0^{\ell}K(x,s)d[F(s)], \tag{3.4}$$

where $K(x,s)$ is the influence functuion.

Proof. We denote the right-hand side of (3.4) by $v(x)$, that is

$$v(x)=\int\limits_0^{\ell}K(x,s)d[F(s)].$$

According to (3.2), the function $v(x)$ can be represented in the form

$$v(x)=\frac{\varphi_1(x)\int\limits_0^x\varphi_2 d[F]}{\varphi_2(0)}+\frac{\varphi_2(x)\int\limits_x^{\ell}\varphi_1 d[F]}{\varphi_2(0)}.$$

To prove that the function $v(x)$ is a solution of equation (1.3), we show first that $v(\cdot)\in E$. For arbitrary $\alpha\le\beta$ we have

$$v(\beta)-v(\alpha)=$$

$$= \frac{1}{\varphi_2(0)} \left((\varphi_1(\beta) - \varphi_1(\alpha)) \int_0^\beta \varphi_2 d[F] + (\varphi_2(\beta) - \varphi_2(\alpha)) \int_\beta^\ell \varphi_1 d[F] \right) +$$

$$+ \frac{1}{\varphi_2(0)} \int_\alpha^\beta ((\varphi_1(\alpha) - \varphi_1(s))\varphi_2(s) + (\varphi_2(s) - \varphi_2(\alpha))\varphi_1(s)) d[F(s)]$$

which implies the μ-absolute continuity of the function $v(x)$.

Let us show that the derivative v'_μ of $v(x)$ is defined by the equality

$$v'_\mu(x) = \frac{\varphi_1{}'_\mu(x) \int_0^x \varphi_2 d[F]}{\varphi_2(0)} + \frac{\varphi_2{}'_\mu(x) \int_x^\ell \varphi_1 d[F]}{\varphi_2(0)}. \tag{3.5}$$

Let $\Delta_\varepsilon z = z(x+\varepsilon) - z(x+0)$, where $\varepsilon > 0$. We carry out the proof for the right-hand derivative (the arguments for the left-hand derivative are similar). We have

$$\frac{\Delta_\varepsilon v}{\Delta_\varepsilon \mu} = \frac{1}{\varphi_2(0)} \frac{\Delta_\varepsilon \varphi_1}{\Delta_\varepsilon \mu} \int_0^{x+\varepsilon} \varphi_2 d[F] + \frac{1}{\varphi_2(0)} \frac{\Delta_\varepsilon \varphi_2}{\Delta_\varepsilon \mu} \int_{x+\varepsilon}^\ell \varphi_1 d[F] +$$

$$+ \frac{1}{\varphi_2(0)} \int_{x+0}^{x+\varepsilon} \frac{\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s)}{\Delta_\varepsilon \mu} d[F(s)].$$

Let us show that

$$\frac{1}{\varphi_2(0)} \lim_{\varepsilon \to 0^+} \left(\frac{\int_{x+0}^{x+\varepsilon} \varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s) d[F(s)]}{\Delta_\varepsilon \mu(x)} \right) = 0.$$

We have

$$\left| \frac{1}{\Delta_\varepsilon \mu(x)} \int_{x+0}^{x+\varepsilon} (\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s)) \, d[F(s)] \right| \le$$

$$\le \frac{\max\limits_{x+0 \le s \le x+\varepsilon} |\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s)|}{\Delta_\varepsilon \mu(x)} Var_{x+0}^{x+\varepsilon}(F),$$

where we denote by Var(F) the total variation of the function F on the corresponding interval. Let τ be point of $\overline{[0,1]}_\mu$, in which the function

$$|\varphi_1(x+0)\varphi_2(s) - \varphi_2(s)\varphi_1(x+0)|$$

reaches a maximum on a compact $[x+0, x+\varepsilon]$. Then

$$\frac{\max\limits_{x+0 \le s \le x+\varepsilon} |\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s)|}{\Delta_\varepsilon \mu(x)} \le$$

$$\leq |\varphi_2(\tau)| \left| \frac{\varphi_1(x+0) - \varphi_1(\tau)}{\Delta_\varepsilon \mu(x)} \right| + |\varphi_1(\tau)| \left| \frac{\varphi_2(x+0) - \varphi_2(\tau)}{\Delta_\varepsilon \mu(x)} \right|,$$

and then the expression

$$\frac{\max\limits_{x+0 \leq s \leq x+\varepsilon} |\varphi_1(x+0)\varphi_2(s) - \varphi_2(x+0)\varphi_1(s)|}{\Delta_\varepsilon \mu(x)}$$

is limited, where $\varepsilon > 0$. Since $Var_{x+0}^{x+\varepsilon}(F) \to 0$ as $\varepsilon \to 0^+$, then the required equality is proved. Let us show that equality (3.5) is true at the point $\xi \in S(\mu)$. We have

$$v'_\mu(\xi) = \frac{v(\xi+0) - v(\xi-0)}{\Delta\mu(\xi)} = \frac{\varphi_1(\xi+0) \int\limits_0^{\xi+0} \varphi_2 d[F]}{\varphi_2(0)} + \frac{\varphi_2(\xi+0) \int\limits_{\xi+0}^{\ell} \varphi_1 d[F]}{\varphi_2(0)} -$$

$$- \frac{\varphi_1(\xi-0) \int\limits_0^{\xi-0} \varphi_2 d[F]}{\varphi_2(0)} + \frac{\varphi_2(\xi-0) \int\limits_{\xi-0}^{\ell} \varphi_1 d[F]}{\varphi_2(0)}.$$

Notice that

$$\int\limits_0^{\xi+0} \varphi_2 d[F] = \int\limits_0^{\xi} \varphi_2 d[F] + \varphi_2(\xi+0)\Delta^+ F(\xi),$$

$$\int\limits_{\xi+0}^{\ell} \varphi_1 d[F] = \int\limits_{\xi}^{\ell} \varphi_1 d[F] - \varphi_1(\xi+0)\Delta^+ F(\xi),$$

$$\int\limits_0^{\xi-0} \varphi_2 d[F] = \int\limits_0^{\xi} \varphi_2 d[F] - \varphi_2(\xi-0)\Delta^- F(\xi),$$

$$\int\limits_{\xi-0}^{\ell} \varphi_1 d[F] = \int\limits_{\xi}^{\ell} \varphi_1 d[F] + \varphi_1(\xi-0)\Delta^- F(\xi),$$

so we obtain the required.

It follows from (3.5) that v'_μ is a function of boundary variation, and thus $v \in E$.

Let us show that the function $v(x)$ is a solution of equation (1.3). First, we have

$$\int\limits_0^x v d[Q] = \frac{1}{\varphi_2(0)} \int\limits_0^x \varphi_1(s) \int\limits_0^s \varphi_2 d[F] d[Q] + \frac{1}{\varphi_2(0)} \int\limits_0^x \varphi_2(s) \int\limits_s^{\ell} \varphi_1 d[F] d[Q].$$

By Fubini's theorem we can interchange the limits of integration and with respect to

$$\int\limits_t^x \varphi_1 d[Q] = (p{\varphi_1}'_\mu)(x) - (p{\varphi_1}'_\mu)(t),$$

we have

$$\frac{1}{\varphi_2(0)}\int\limits_0^x \varphi_1(s)\int\limits_0^s \varphi_2 d[F]d[Q] =$$

$$= \frac{1}{\varphi_2(0)}\int\limits_0^x \varphi_2(t)((p{\varphi_1}'_\mu)(x) - (p{\varphi_1}'_\mu)(t))d[F(t)].$$

Similarly, since

$$\int\limits_0^x \varphi_2 d[Q] = p(x){\varphi_2}'_\mu(x),$$

we have

$$\int\limits_0^x \varphi_2(s)\int\limits_s^\ell \varphi_1 d[F]d[Q] =$$

$$= \int\limits_0^x p(t)\varphi_1(t){\varphi_2}'_\mu(t)d[F(t)] + {\varphi_2}'_\mu(x)p(x)\int\limits_x^\ell \varphi_1(t)d[F(t)].$$

Substituting the expression obtained for $\int\limits_0^x vd[Q]$ into (1.3) and using the equality $\widetilde{p}(x)W(x) = \varphi_2(0)$, we obtain a true equality. Thus, the function $v(x)$ is a solution of problem (1.3). □

Theorem 3.4. *Suppose that the function $Q(x)$ does not decrease on the interval $[0,\ell]$ and $Q(x)$ does not equal to a constant. Then the influence function $K(x,s) > 0$ for all $x, s \in \overline{[0,\ell]}_\mu \times \overline{[0,\ell]}_\mu$.*

Proof. We are going to use representation (3.2) for the influence function. Let us show that $\varphi_1(x) > 0$ for all $x \in \overline{[0,\ell]}_\mu$. Notice that the function $\varphi_1(x)$ is a solution of the problem

$$\begin{cases} -p(x){\varphi_1}'_\mu(x) + \int\limits_{+0}^x \varphi_1(t)\, d[Q(t)] = -p(+0){\varphi_1}'_\mu(+0), \\ -p(+0){\varphi_1}'_\mu(+0) + \varphi_1(0)\Delta^+ Q(0) = 1, \\ p(\ell-0){\varphi_1}'_\mu(\ell-0) + \varphi_1(\ell)\Delta^- Q(\ell) = 0. \end{cases}$$

Since the function $Q(x)$ doesn't decrease we get that the function $\varphi_1(x)$ can have no more than one zero point on $[0,\ell]$. Let ξ be a zero point of $\varphi_1(x)$. If $\xi = \ell$, then $\varphi_1(\ell) = 0$, hence, ${\varphi_1}'_\mu(\ell) = 0$ and we have $\varphi_1 \equiv 0$, because the homogeneous Cauchy problem can have only the zero solution, which contradicts the boundary condition at zero.

If $\xi = 0$, then it follows from the boundary condition that ${\varphi_1}'_\mu(+0) < 0$. Hence $\varphi_1(x) < 0$ and $\varphi_1(\ell) < 0$. From the equation it follows that ${\varphi_1}'_\mu(x) < 0$. So ${\varphi_1}'_\mu(\ell-0) < 0$. But then the boundary condition at the point $x = \ell$ is not satisfied.

Let $\xi \in (0,l)$. We consider the case when $\varphi_1(\xi-0)<0$, $\varphi_1(\xi+0)\geq 0$. The another cases can be considered similarly. We have ${\varphi_1}'_\mu(\xi)=\dfrac{\Delta\varphi_1(\xi)}{\Delta\mu(\xi)}\geq 0$. Since

$$-p(\xi+0){\varphi_1}'_\mu(\xi+0)+p(\xi){\varphi_1}'_\mu(\xi)+\varphi_1(\xi+0)\Delta^+Q(\xi)=0,$$

then ${\varphi_1}'_\mu(\xi+0)\geq 0$. Notice that the equality

$$-p(x){\varphi_1}'_\mu(x)+\int\limits_{\xi+0}^{x}\varphi_1(t)\,d[Q(t)]=-p(\xi+0){\varphi_1}'_\mu(\xi+0)$$

is true on the interval (ξ,ℓ). Hence ${\varphi_1}'_\mu(\ell-0)>0$. But $\varphi_1(\ell)>0$ and it contradicts the condition at the point ℓ.

We obtain that $\varphi_1(x)$ does not have zero points on the interval $[0,\ell]$, hence $\varphi_1(x)$ preserves a sign on the interval $[0,\ell]$. Assume that $\varphi_1(x)<0$. Then

$$-p(+0){\varphi_1}'_\mu(+0)>0$$

and

$$-p(x){\varphi_1}'_\mu(x)>0.$$

We obtain that ${\varphi_1}'_\mu(x)<0$, in particular, ${\varphi_1}'_\mu(\ell-0)<0$. But it contradicts to the condition at the point ℓ. So $\varphi_1(x)>0$. The proof that $\varphi_2(x)>0$ can be carried out similarly. Moreover, from the equalities

$$p(x){\varphi_1}'_\mu(x)=-\int\limits_{x}^{\ell-0}\varphi_1(t)\,d[Q(t)]-\varphi_1(\ell)\Delta^-Q(\ell)$$

and

$$p(x){\varphi_2}'_\mu(x)=\int\limits_{+0}^{x}\varphi_2(t)\,d[Q(t)]+\varphi_2(0)\Delta^+Q(0)$$

it follows, that the function $\varphi_1(x)$ decreases and the function $\varphi_2(x)$ increases on the interval $\overline{[0,\ell]}_\mu$. □

Theorem 3.5. *Let the functions $Q(x)$, $F_1(x)$ be non-decreading on the interval $[0,\ell]$ and $\inf\limits_{(0,\ell)} p>0$. Let $u_1(x)$ be a solution of the problem*

$$-p(x)u'_\mu(x)+\int\limits_{0}^{x}u(t)\,d[Q(t)]=F(x)-F(0) \tag{3.6}$$

where $F(x)=F_1(x)$, and $u_2(x)$ is a solution of problem (3.6) where $F(x)=F_1(x)+\theta(x-s)$, $s\in\overline{[0,\ell]}_\mu$. Then

$$\max_{\overline{[0,\ell]}_\mu}\frac{(u_2-u_1)(x)}{u_1(x)}=\frac{(u_2-u_1)(s)}{u_1(s)}.$$

Proof. Let $s \in \overline{[0,\ell]}_\mu$. Notice that

$$K(x,s) = u_2(x) - u_1(x),$$

and

$$u_1(x) = \int_0^\ell K(x,s)d[F_1(s)].$$

Consider, for determinacy, the case when $s = \xi + 0$, where $\xi \in S(\mu)$. Another cases can be considered similarly. Let us show that

$$\max_{\overline{[0,\ell]}_\mu} \frac{K(x,\xi+0)}{\int_0^\ell K(x,s)d[F_1(s)]} = \frac{K(\xi+0,\xi+0)}{\int_0^\ell K(\xi+0,s)d[F_1(s)]}.$$

Let $0 \le x \le \xi + 0$. Then

$$\frac{K(x,\xi+0)}{\int_0^\ell K(x,s)d[F_1(s)]} = \frac{\varphi_2(x)\varphi_1(\xi+0)}{\varphi_1(x)\int_0^x \varphi_2(s)d[F_1(s)] + \varphi_2(x)\int_x^\ell \varphi_1(s)d[F_1(s)]}.$$

We must show that

$$\frac{\varphi_2(x)}{\varphi_1(\xi)\int_0^x \varphi_2(s)d[F_1(s)] + \varphi_2(x)\int_x^\ell \varphi_1(s)d[F_1(s)]} \le$$

The last inequality is true if inequality

$$\varphi_1(\xi+0)\int_x^{\xi+0} \varphi_2 d[F_1] \le \varphi_2(\xi+0)\int_x^{\xi+0} \varphi_1 d[F_1].$$

holds. Let us consider the function

$$W(x) = \varphi_1(\xi+0)\varphi_2(x) - \varphi_2(\xi+0)\varphi_1(x),$$

where $0 \le x \le \xi + 0$. Since the function $\varphi_2(x)$ monotonically increases, and the function $\varphi_1(x)$ monotonically decreases, then $W'_\mu > 0$. Hence the function $W(x)$ monotonically increases for all $x \le \xi + 0$, and $W(x) \le W(\xi+0) = 0$. The case, when $x \in [\xi+0,\ell]$ can be considered similarly and in this case we have

$$\frac{K(x,\xi+0)}{\int_0^\ell K(x,s)d[F_1(s)]} \le \frac{K(\xi+0,\xi+0)}{\int_0^\ell K(\xi+0,s)d[F_1(s)]}.$$

□

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